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FINITE COTANGENT SUMS AND THE RIEMANN ZETA FUNCTION

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ABSTRACT. We give a closed-form evaluation of the finite cotangent sums given by

$$\sum_{p=0}^{q-1} \cot^n \left[\frac{(\xi+p)\pi}{q}\right], \qquad \sum_{p=1}^{q-1} \cot^{2n}\left(\frac{p\pi}{q}\right)$$

where n and q are positive integers ($q \geq 2$) and ξ is a non-integer real number. The latter sum enables us to simplify the Apostol formula concerning the Riemann zeta function at integer values of the argument. We demonstrate that, for even integers, this formula involves polynomials with rational arguments.

1. Introduction

Consider the Riemann zeta function $\zeta(z)$ defined by [1; p. 19]

$$\begin{aligned} \zeta(z) &= \sum_{k=1}^{\infty} \frac{1}{k^z} = \frac{1}{1-2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^z} && \operatorname{Re} z > 1 \\ &= \frac{1}{1-2^{1-z}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k_z} && \operatorname{Re} z > 0, \quad z \neq 1. \end{aligned}$$

For $\operatorname{Re} z \leq 1$, $z \neq 1$, $\zeta(z)$ is defined as the analytic continuation of the foregoing series. The Riemann zeta function is analytic over the whole complex plane, except at $z = 1$, where it has a simple pole.

Apostol [2] gave the following formula

$$\zeta(n) = \left(\frac{\pi}{2}\right)^n \lim_{q\to\infty} \frac{\sum_{p=1}^{q} \cot^n\left(\frac{p\pi}{2q+1}\right)}{q^n} \tag{1}$$

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valid for any integer $n \geq 2$. For the case when n is even, he found the asymptotic expansion of the finite sum involved, which readily leads to the Euler relation

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \qquad (n \geq 1) \tag{2}$$

between the even-indexed Bernoulli numbers B_{2n} and $\zeta(2n)$.

Here, we deduce summation formulae for

$$S_n(q;\xi) = \sum_{p=1}^{q-1} \cot^n \left[\frac{(\xi+p)\pi}{q} \right] \qquad (0 < \xi < 1) \tag{3a}$$

and

$$S_n^*(q) = \sum_{p=1}^{q-1} \cot^n \left(\frac{p\pi}{q} \right) \tag{3b}$$

and examine the Apostol formula (1) for even integers.

2. First results

Throughout the text $[x]$ denotes the largest integer not exceeding x, while, as usual, $\binom{n}{m}$ is the binomial coefficient given by

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{m} = \frac{n!}{m!(n-m)!} \qquad (n, m \in \mathbb{N}, \quad 1 \leq m \leq n).$$

Further, the principal values of $\operatorname{arccot} \theta$ are defined for any real θ by $0 < \operatorname{arccot} x < \pi$.

In this section we establish the existence of the polynomials for which all zeros, regardless of the degree, can be given by a simple formula involving the cotangent function. It appears that these polynomials are unknown, and their detailed study is given elsewhere [3].

THEOREM. *Let n be a positive integer, and $C_n(x;\xi)$ a polynomial of degree precisely n, in a variable x with real parameter ξ, which we define as*

$$C_n(x;\xi) = \sum_{m=0}^{n} (-1)^m c_{n,m}(\xi) x^{n-m} \tag{4a}$$

where $c_{n,m}(\xi)$ is

$$c_{n,m}(\xi) = \binom{n}{m} \left[\cos(m\pi/2) + \sin(m\pi/2) \cot(\pi\xi) \right] \qquad (0 < \xi < 1). \tag{4b}$$

Then for any fixed n, *the zeros of* $C_n(x;\xi)$ *are all real, simple and given by*

$$\Gamma_{n,m}(\xi) = \cot\left(\frac{(\xi+m)\pi}{n}\right) \qquad m = 0,1,\dots,n-1\,. \tag{5}$$

Before proving the Theorem we give as an example the first few polynomials

$$\begin{aligned}
C_1(x;\xi) &= x - \cot(\pi\xi)\,,\\
C_2(x;\xi) &= x^2 - 2x\cot(\pi\xi) - 1\,,\\
C_3(x;\xi) &= x^3 - 3x^2\cot(\pi\xi) - 3x + \cot(\pi\xi)\,,\\
C_4(x;\xi) &= x^4 - 4x^3\cot(\pi\xi) - 6x^2 + 4x\cot(\pi\xi) + 1\,,\\
C_5(x;\xi) &= x^5 - 5x^4\cot(\pi\xi) - 10x^3 + 10x^2\cot(\pi\xi) + 5x - \cot(\pi\xi)\,,\\
C_6(x;\xi) &= x^6 - 6x^5\cot(\pi\xi) - 15x^4 + 20x^3\cot(\pi\xi) + 15x^2 - 6x\cot(\pi\xi) - 1\,.
\end{aligned}$$

P r o o f . First of all, we shall prove that

$$\sum_{m=0}^{[n/2]}(-1)^m\binom{n}{2m}x^{n-2m} = \csc^n(\operatorname{arccot} x)\cos(n\operatorname{arccot} x) \tag{6a}$$

and

$$\sum_{m=0}^{[(n-1)/2]}(-1)^m\binom{n}{2m+1}x^{n-2m-1} = \csc^n(\operatorname{arccot} x)\sin(n\operatorname{arccot} x) \tag{6b}$$

where $\operatorname{arccot} x$ takes its principal value, x is real and n is a positive integer.

Starting from the well-known Euler-Bernoulli formulae [4; 4.4.4.15, 4.4.4.16]

$$\begin{aligned}
\cos n\theta &= \sum_{m=0}^{[n/2]}(-1)^m\binom{n}{2m}\cos^{n-2m}\theta\sin^{2m}\theta\\
\sin n\theta &= \sum_{m=0}^{[(n-1)/2]}(-1)^m\binom{n}{2m+1}\cos^{n-2m-1}\theta\sin^{2m+1}\theta
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{m=0}^{[n/2]}(-1)^m\binom{n}{2m}\tan^{2m}\theta &= \sec^n\theta\cos n\theta\\
\sum_{m=0}^{[(n-1)/2]}(-1)^m\binom{n}{2m+1}\tan^{2m+1}\theta &= \sec^n\theta\sin n\theta
\end{aligned}$$

where $\theta \neq (2k+1)\pi/2$ ($k \in Z$). In view of these identities, upon setting $x = \cot\theta$ ($0 < \theta \neq \pi/2 < \pi$) in (6a), (6b), it follows at once that the proposed formulae hold when $x \neq 0$. Moreover, it can be shown by direct verification that they remain valid for $x = 0$.

Next, since the polynomials $C_n(x;\xi)$ may be expressed in the form

$$C_n(x;\xi) = \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{2m} x^{n-2m} - \cot(\pi\xi) \sum_{m=0}^{[(n-1)/2]} (-1)^m \binom{n}{2m+1} x^{n-2m-1} \tag{7}$$

as an immediate consequence of (6) we have

$$C_n(x;\xi) = \csc^n(\operatorname{arccot} x)\big[\cos(n \operatorname{arccot} x) - \cot(\pi\xi)\sin(n \operatorname{arccot} x)\big]. \tag{8}$$

To complete the proof we only need to verify that $C_n\big(\Gamma_{n,m}(\xi);\xi\big) = 0$ for any n and $m = 0, 1, \dots, n-1$, regardless of ξ.

Observe that, in general, if k is an integer and θ is real, then

$$\operatorname{arccot}(\cot\theta) = \theta - k\pi \qquad k\pi < \theta < (k+1)\pi .$$

Thus

$$\operatorname{arccot}\big[\Gamma_{n,m}(\xi)\big] = \frac{(\xi+m)\pi}{n}$$

since

$$0 < \frac{(\xi+m)\pi}{n} < \pi , \qquad m = 0, 1, \dots, n-1 ,$$

provided that $0 < \xi < 1$.

Finally, it follows readily from (8), that

$$C_n\big(\Gamma_{n,m}(\xi);\xi\big) = (-1)^m \csc^n\left[\frac{(\xi+m)\pi}{n}\right]\big[\cos(\pi\xi) - \cot(\pi\xi)\sin(\pi\xi)\big] = 0 .$$

Since $C_n(x;\xi)$ is of degree n, it vanishes at the n points $\Gamma_{n,m}(\xi)$ for $m = 0, 1, \dots, n-1$. In view of the properties of the cotangent functions, these zeros are all real and simple. This completes our proof. □

A similar result is obtained as follows. Let $C_n^*(x)$ denote the polynomial given by the expressions in (6b). Several examples are

$$\begin{aligned}
C_1^*(x) &= 1 ,\\
C_2^*(x) &= 2x ,\\
C_3^*(x) &= 3x^2 - 1 ,\\
C_4^*(x) &= 4x^3 - 4x ,\\
C_5^*(x) &= 5x^4 - 10x^2 + 1 ,\\
C_6^*(x) &= 6x^5 - 20x^3 + 6x ,\\
C_7^*(x) &= 7x^6 - 35x^4 + 21x^2 - 1 ,\\
C_8^*(x) &= 8x^7 - 56x^5 + 56x^3 - 8x .
\end{aligned}$$

The determination of the zeros of $C_n^*(x)$ is clearly equivalent to solving $\sin(n \operatorname{arccot} x) = 0$, which gives rise to the following Proposition.

Proposition 1. *Assume that n is a positive integer. Let $C_n^*(x)$ be the polynomial in a real variable x of degree $n-1$, defined by* (6b). *Then for any fixed $n \geq 2$, the zeros of $C_n^*(x)$ are all real, simple and given by*

$$\Gamma_{n,m}^* = \cot\left(\frac{m\pi}{n}\right), \qquad m = 1, 2, \dots, n-1.$$

3. Finite cotangent sums

As a straightforward consequence of the Theorem, we derive a closed-form determinant formula for finite cotangent sum $S_n(q;\xi)$. We note that the sums S_1, S_2 and S_4 are well known in a slightly different form [4; Section 4.4.7] and [5; Sections 29.1 and 30.1].

We begin by stating some elementary results from the theory of polynomials needed [6; p. 179]. Let x_k $(1 \leq k \leq n)$ be the roots of an algebraic equation of the form

$$\begin{aligned} &x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + \cdots + a_n \\ &\equiv x^n - \sigma_1 x^{n-1} + \cdots + (-1)^{n-1}\sigma_{n-1}x + (-1)^n \sigma_n = 0 \end{aligned} \tag{10}$$

with coefficients $a_1, a_2, \dots, a_n$ $(n \geq 1)$. The elementary symmetric functions $\sigma_1, \sigma_2, \dots, \sigma_n$ of x_k are then given by the Viete formulae

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= \sigma_1, \\ x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n &= \sigma_2, \\ x_1x_2x_3 + x_1x_2x_4 + \cdots + x_{n-2}x_{n-1}x_n &= \sigma_3, \\ &\vdots \\ x_1x_2 + \cdots + x_n &= \sigma_n \end{aligned}$$

while s_k $(k = 1, 2, \dots)$, the kth power sum of zeros, defined as

$$s_k = x_1^k + x_2^k + \cdots + x_n^k$$

is related to σ_k by the Newton identities as follows

$$\begin{aligned} \sigma_1 &= s_1, \\ s_1\sigma_1 - 2\sigma_2 &= s_2, \\ s_2\sigma_1 - s_1\sigma_2 - 3\sigma_3 &= s_3, \\ &\vdots \\ s_k\sigma_1 - s_{k-1}\sigma_2 + \cdots + (-1)^{k+1}k\sigma_k &= s_k. \end{aligned}$$

The Newton identities can clearly be considered as a system of linear equations for $s_1, s_2, \dots, s_n$, which enables us to rewrite them into a more convenient form. Thus by making use of the Cramer rule [6; p. 198] it is easy to see from the first k of the equations, that power sum s_k is given by the following determinant

$$s_k = \begin{vmatrix} \sigma_1 & 1 & 0 & 0 & \dots & 0 \\ 2\sigma_2 & \sigma_1 & 1 & 0 & \dots & 0 \\ 3\sigma_3 & \sigma_2 & \sigma_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (k-1)\sigma_{k-1} & \sigma_{k-2} & \dots & \dots & \dots & 1 \\ k\sigma_k & \sigma_{k-1} & \dots & \dots & \dots & \sigma_1 \end{vmatrix}$$

of the order k.

Now, in view of the Theorem, it is obvious that the finite sum in (3a) is in fact the power sum of the zeros of the polynomial in (4). Thus, we can state the following result.

PROPOSITION 2. *Assume that n is a positive integer, and let $S_n(q;\xi)$ be a finite sum in* (3a). *Then for any fixed n we have*

$$S_n(q;\xi) = \begin{vmatrix} c_{q,1} & 1 & 0 & 0 & \dots & 0 \\ 2c_{q,2} & c_{q,1} & 1 & 0 & \dots & 0 \\ 3c_{q,3} & c_{q,2} & c_{q,1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)c_{q,n-1} & c_{q,n-2} & \dots & \dots & \dots & 1 \\ nc_{q,n} & c_{q,n-1} & \dots & \dots & \dots & c_{q,1} \end{vmatrix}$$

where, for brevity, $c_{q,i}$ stands for $c_{q,i}(\xi)$ defined in (4b), *and the determinant is of the order n.*

In particular

$$\begin{aligned} S_1(q;\xi) &= q\cot(\pi\xi)\,, \\ S_2(q;\xi) &= q^2\big(\cot^2(\pi\xi)+1\big) - q\,, \\ S_3(q;\xi) &= q^3\big(\cot^3(\pi\xi)+\cot(\pi\xi)\big) - q\cot(\pi\xi)\,, \\ S_4(q;\xi) &= q^4\big(\cot^4(\pi\xi)+(4/3)\cot^2(\pi\xi)+1/3\big) - q^2\big((4/3)\cot^2(\pi\xi)+4/3\big) + q\,, \\ S_5(q;\xi) &= q^5\big(\cot^5(\pi\xi)+(5/3)\cot^3(\pi\xi)+(2/3)\cot(\pi\xi)\big) \\ &\quad - q^3\big((5/3)\cot^3(\pi\xi)+(5/3)\cot^3(\pi\xi)\big) + q\cot(\pi\xi)\,, \\ S_6(q;\xi) &= q^6\big(\cot^6(\pi\xi)+2\cot^4(\pi\xi)+(17/15)\cot^2(\pi\xi)+2/15\big) \\ &\quad - q^4\big(2\cot^4(\pi\xi)+(8/3)\cot^2(\pi\xi)+2/3\big) \\ &\quad + q^2\big((23/15)\cot^2(\pi\xi)+23/15\big) - q\,. \end{aligned}$$

We note that the expressions for $S_n(q;1/2)$, $S_n(q;1/4)$, $S_{2n}(q;1/3)$ *and* $S_{2n}(q;1/6)$ *are the polynomials in* q *with rational coefficients. Moreover,* $S_{2n-1}(q;1/2)=0$.

However, it is easier to find the sums for higher n *by using the differential recurrence relation*

$$S_{n+2}(q;\xi) = -S_n(q;\xi) - \frac{q}{(n+1)\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}S_{n+1}(q;\xi) \qquad (n \geq 1)$$

which is readily proved by differentiating the definition (3a) *with respect to* ξ.

4. The Apostol formula

In this section we shall derive the closed-form formula for $S_n^*(q)$ which allows us to investigate the Apostol formula.

The properties of the cotangent function readily lead to

$$\sum_{p=1}^{q-1} \cot^{2n+1}\left(\frac{p\pi}{q}\right) = S_{2n+1}^*(q) = 0$$

and

$$\sum_{p=1}^{q-1} \cot^{2n}\left(\frac{p\pi}{2q}\right) = \frac{1}{2}S_{2n}^*(2q),$$

$$\sum_{p=1}^{q} \cot^{2n}\left(\frac{p\pi}{2q+1}\right) = \frac{1}{2}S_{2n}^*(2q+1).$$

The first indication about S_{2n}^* can be obtained from the expressions found in Section 3 and the relation

$$S_{2n}^*(q) = \lim_{\xi\to 0}\sum_{p=1}^{q-1}\cot^n\left(\frac{(\xi+p)\pi}{q}\right) = \lim_{\xi\to 0}\left[S_n(q;\xi) - \cot^n(\pi\xi/q)\right]$$

resulting in

$$S_2^*(q) = \frac{1}{3}q^2 - q + \frac{2}{3},$$
$$S_4^*(q) = \frac{1}{45}q^4 - \frac{4}{9}q^2 + q - \frac{26}{45},$$
$$S_6^*(q) = \frac{2}{945}q^6 - \frac{2}{45}q^4 + \frac{23}{45}q^2 - q + \frac{502}{945},$$

where our S_2^* and S_4^* are the same as those listed in literature [4; Section 4.4.7].

On the other hand, a straightforward consequence of Proposition 1 is that the sum in (3b) is the power sum of the zeros of $C_n^*(x)$. In this way we shall evaluate S_n^* for any n. Note, however, that the polynomial $C_n^*(x)$ is not monic, that it is of degree $n-1$ and that some of its coefficients are zero (compare with Equation 10).

PROPOSITION 3. *Let n be a positive integer and let $S_n^*(q)$ be a finite sum defined by* (3b). *Then for any fixed n we have*

$$S_n^*(q) = \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 2c_{q,2} & 0 & 1 & 0 & \dots & 0 \\ 0 & c_{q,2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)c_{q,n-1} & c_{q,n-2} & \dots & \dots & \dots & 1 \\ nc_{q,n} & c_{q,n-1} & \dots & \dots & \dots & 0 \end{vmatrix}$$

where $c_{q,i}$ stands for $c_{q,i}(\xi)$, and the determinant is of the order n.

It is not difficult to verify that S_{2n+1}^* is always zero, and it follows directly that, in general, S_{2n}^* is the polynomial of degree $2n$ in q with rational coefficients. Two more special cases are

$$\begin{aligned} S_8^*(q) &= \frac{1}{4725}q^8 - \frac{16}{2835}q^6 + \frac{44}{675}q^4 - \frac{176}{315}q^2 + q - \frac{7102}{14175}, \\ S_{10}^*(q) &= \frac{2}{93555}q^{10} - \frac{2}{2835}q^8 + \frac{86}{8505}q^6 - \frac{718}{8505}q^4 + \frac{563}{945}q^2 - q - \frac{44834}{93555}. \end{aligned}$$

5. Concluding remarks

The Apostol formula implies the existence of a certain function $A(q;n)$, such that

$$\zeta(n) = \left(\frac{\pi}{2}\right)^n \lim_{q\to\infty} \frac{A(q;n)}{q^n} \qquad (n \geq 2).$$

We have demonstrated that, for even integers, we have

$$A(q;2n) = \frac{1}{2}S_{2n}^*(2q+1) \qquad (n \geq 1),$$

where S_{2n}^* is fully determined: it is a polynomial in q with rational coefficients of degree $2n$ and its evaluation is straightforward.

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