ON MAULDIN'S CLASSIFICATION OF REAL FUNCTIONS

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ABSTRACT. In this paper we investigate the Baire system generated by the family of all Darboux quasicontinuous, almost everywhere continuous functions, and prove that every function \( f \) of Mauldin's class \( \alpha > 1 \) is the limit of a sequence of Darboux functions \( f_n \) of Mauldin's class \( \alpha_n < \alpha, n = 1, 2, \ldots \).

Let us establish some terminology to be used later.

A function \( f: \mathbb{R} \to \mathbb{R} \) is said to be quasicontinuous at a point \( x \in \mathbb{R} \) if for all open neighbourhoods \( U \) of \( x \) and \( V \) of \( f(x) \) there exists a nonempty open set \( W \subset U \cap f^{-1}(V) \), ([5]).

Denote by \( Q \) the family of all quasicontinuous functions \( f: \mathbb{R} \to \mathbb{R} \), by \( A \) the family of all almost everywhere continuous functions \( f: \mathbb{R} \to \mathbb{R} \) (with respect to the Lebesgue measure) and by \( D \) the family of all Darboux functions \( f: \mathbb{R} \to \mathbb{R} \).

Given a fixed countable ordinal number \( \alpha > 0 \) and fixed family \( K \) of functions \( f: \mathbb{R} \to \mathbb{R} \) we put

\[
B_0(K) = K, \\
B_\alpha(K) = \left\{ f: \mathbb{R} \to \mathbb{R} : f \text{ is the limit of the sequence of functions } f_n \in \bigcup_{\beta < \alpha} B_\beta(K), \ n = 1, 2, \ldots \right\}.
\]

Let \( P \) denote the family of all functions \( f: \mathbb{R} \to \mathbb{R} \) such that the set \( C(f) \) of its continuity points is dense.

In [3] it is proved that

\[
B_1(D \cap Q) = P.
\]

Key words: continuity, quasicontinuity, Darboux function, Baire system, Mauldin's classification.
In [6] Mauldin proved that for every countable ordinal number $\alpha > 0$,

$$B_\alpha(A) = M_\alpha,$$

where $f \in M_\alpha$ if and only if there exists a function $g: \mathbb{R} \to \mathbb{R}$ of Baire class $\alpha$ and an $F_\sigma$ set $A$ of measure zero such that $\{x \in \mathbb{R}: f(x) \neq g(x)\} \subset A$.

In this paper I prove that $B_1(\mathcal{D} \cap \mathcal{Q} \cap A) = M_1 \cap \mathcal{P}$,

$$B_1(M_1 \cap \mathcal{P} \cap \mathcal{D}) = M_2 \quad \text{and} \quad B_1 \left( \mathcal{D} \cap \bigcup_{\beta < \alpha} M_\beta \right) = M_\alpha.$$

**Theorem 1.** The following equality is true:

$$B_1(\mathcal{D} \cap \mathcal{Q} \cap A) = M_1 \cap \mathcal{P}.$$

**Proof.** Since $B_1(\mathcal{D} \cap \mathcal{Q}) = \mathcal{P}$ and $B_1(A) = M_1$ we have $B_1(\mathcal{D} \cap \mathcal{Q} \cap A) \subset M_1 \cap \mathcal{P}$.

Let $f \in M_1 \cap \mathcal{P}$. There exist a function $g: \mathbb{R} \to \mathbb{R}$ of Baire class 1 and an $F_\sigma$ set $B$ of measure zero such that $\{x \in \mathbb{R}: h(x) \neq g(x)\} \subset B$.

Put $h = f - g$. Evidently $h \in M_1 \cap \mathcal{P}$ and

$$\{x \in \mathbb{R}: h(x) \neq 0\} \subset B.$$

Let

$$F_n = \{x \in \mathbb{R}: \text{osc} h(x) \geq 2^{-n}\}, \quad n = 1, 2, \ldots.$$  \hfill (1)

Since all sets $B \cap F_1$ and $B \cap (F_n \setminus F_{n-1})$, $n = 1, 2, \ldots$ are $F_\sigma$ sets of measure zero, we can write

$$B \cap F_1 = \bigcup_{m} F_{1,m},$$

$$B \cap (F_n \setminus F_{n-1}) = \bigcup_{m} F_{n,m} \quad \text{for} \quad n = 2, 3, \ldots, \hfill (2)$$

where all sets $F_{n,m}$ are closed and pairwise disjoint, $n, m = 1, 2, \ldots$ ([8]).

For a fixed $k \geq 1$ there are pairwise disjoint closed intervals $I_{k,n,m,j} = [a_{k,n,m,j}, b_{k,n,m,j}]$ ($n + m \leq k + 1$, $F_{n,m} \neq \emptyset$ and $j = 1, 2, \ldots$), contained in $\mathbb{R} \setminus F_n \setminus \bigcup_{n+m \leq k+1} F_{n,m}$ such that:

1. if $x \in I_{k,n,m,j}$ there is a point $y \in F_{n,m}$ such that $|x - y| < 1/k$;
2. for each $x \in F_{n,m}$ and for each $r > 0$ there are indices $j_1, j_2$ such that $I_{k,n,m,j_1} \subset (x, x + r)$ and $I_{k,n,m,j_2} \subset (x - r, x)$;
3. if there is the limit $\lim_{i \to \infty} x_i = x$, where $x_i \in I_{k,n,m,j(i)}$ ($j(i_1) > j(i_2)$ for $i_1 > i_2$) then $x \in F_{n,m}$.
For each interval $I_{k,n,m,j}$ ($n + m \leq k + 1$, $j = 1,2,\ldots$) there is a function $h_{k,n,m,j}: I_{k,n,m,j} \to \mathbb{R}$ such that:

1. $h_{k,n,m,j}(a_{k,n,m,j}) = h_{k,n,m,j}(b_{k,n,m,j}) = 0$;
2. $h_{k,1,m,j}(I_{k,1,m,j}) = \mathbb{R}$;
3. $h_{k,n,m,j}(I_{k,n,m,j}) = [-2^{-n+2},2^{-n+2}]$ for $n > 1$;
4. $h_{k,1,m,j}$ is continuous on the interval $(a_{k,1,m,j},b_{k,1,m,j})$ and for $n > 1$ a function $h_{k,n,m,j}$ is continuous on the interval $[a_{k,n,m,j},b_{k,n,m,j}]$.

Let $h_k: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h_k(x) = \begin{cases} h_{k,n,m,j}(x) & \text{for } x \in I_{k,n,m,j}, \\ h(x) & \text{for } x \in F_{n,m}, \\ 0 & \text{otherwise} \end{cases}$$

if $n + m \leq k + 1$ and $j = 1,2,\ldots$.

From (9), (6) and (5) it follows that $h_k$ is continuous at all points of the set $G = \left(\mathbb{R} \setminus \bigcup_{n+m \leq k+1} F_{n,m}\right) \setminus \bigcup_{m \leq k} \{a_{k,1,m,j}\}$.

Since $\mathbb{R} \setminus G$ is of measure zero, $h_k$ is almost everywhere continuous.

By (1), (2), (4), (7), (8) and (9), $h_k$ is quasicontinuous and has the Darboux property.

The function $g$ is the limit of a sequence of continuous functions $g_k$, $k = 1,2,\ldots$. Let $f_k = g_k + h_k$ for $k = 1,2,\ldots$.

The function $f_k$ is quasicontinuous as the sum of the quasicontinuous function $h_k$ and the continuous function $g_k$ ([4]). The same $f_k$ is almost everywhere continuous and continuous at each point of the set $G$.

Now we shall prove that every $f_k$ ($k = 1,2,\ldots$) has the Darboux property. Assume the contrary that $f_k$ does not have the Darboux property. There are real numbers $a$, $b$, $c$ such that $a < b$, $c \in \left(\min(f_k(a),f_k(b)), \max(f_k(a),f_k(b))\right)$ and $c \notin f_k((a,b))$.

For definiteness assume that $f_k(a) < f_k(b)$. Let

$$d = \inf\{x \in (a,b) : f_k(x) > c\}.$$ 

Since $g_k$ is continuous and $f_k = g_k + h_k$ is not continuous at the point $d$, $h_k$ is not continuous at $d$. Consequently, $d \in \mathbb{R} \setminus G$.

If $f_k(d) < c$ and there are indices $n$, $m$, $j$ such that $m \leq k$ and $d = a_{k,1,m,j}$ then we may observe that the restricted function $h_k|_{I_{k,1,m,j}}$ has the Darboux property and it is of Baire class 1. Consequently, $f_k|_{I_{k,1,m,j}}$ has the Darboux property.
property as the sum of continuous function $g_k |_{I_{k,1,m,j}}$ and the Darboux function $h_k |_{I_{k,1,m,j}}$ which is of Baire class 1 ([1]). If $f_k(d) < c$ and $d = \inf \{ x \in (a, b] : f_k(x) > c \}$ then there is a point $z \in (a, b)$ such that $f_k(z) = c$. This contradicts the relation $c \not\in f_k((a, b))$.

If $f_k(d) < c$ and there is an index $m \leq k$ such that $d \in F_{1,m}$ then, by (4), there is an interval $I_{k,1,m,j} \subset (a, b)$. Since the restriction function $f_k |_{I_{k,1,m,j}}$ has the Darboux property, we have, by (7), $f_k((a, b)) = f_k(I_{k,1,m,j}) = \mathbb{R}$ and $c \in f_k((a, b))$. This contradicts the relation $c \not\in f_k((a, b))$.

If $f_k(d) < c$ and there are indices $n, m$ such that $n > 1, n + m \leq k + 1$ and $d \in F_{n,m}$, then $|h_k(d)| < 2^{-n+1}$. Since $f_k(d) = h_k(d) + g_k(d) < c$, it follows from the continuity of $g_k$ at the point $d$ and from (5) that there is an interval $I = [d, e]$ with $e \in (a, b) \setminus \bigcup_j I_{k,1,m,j}$ such that:

\begin{align}
(10) & \quad |g_k(x) - g_k(d)| < 2^{-n+1}; \\
(11) & \quad h_k(d) + g_k(x) < c
\end{align}

for every $x \in (d, e)$.

From the definition of $d$ there is a point $u \in (d, e)$ such that $f_k(u) > c$.

If there is an interval $I_{k,n,m,j}$ with $u \in I_{k,n,m,j}$ then from (8) and (11) there is a point $w \in I_{k,n,m,j}$ such that

$$f_k(w) = g_k(w) + h_k(w) < g_k(w) + h_k(d) < c.$$ 

Since $f_k |_{I_{k,n,m,j}}$ has the Darboux property,

$$c \in f_k(I_{k,n,m,j}) \subset f_k((a, b)),$$

which contradicts the relation $c \not\in f_k((a, b))$.

If $u \not\in \bigcup_j I_{k,n,m,j}$ then $h_k(u) = 0$ or $u \in F_{n,m}$. Let $I_{k,n,m,j} \subset I$. Since $|h_k(u)| < 2^{-n+1}$, it follows from (8) and (10) that there is a point $v \in I_{k,n,m,j}$ such that:

$$f_k(v) = h_k(v) + g_k(v) = 2^{-n+2} + g_k(v) > 2^{-n+2} + g_k(u) - 2^{-n+1} = 2^{-n+1} + g_k(u) > h_k(u) + g_k(u) > c.$$ 

As above, it follows from (11) that there is a point $w \in I_{k,n,m,j}$ such that $f_k(w) < c$ and $c \in f_k((a, b))$, which contradicts the relation $c \not\in f_k((a, b))$.

Similarly, we may consider the case, where $f_k(d) > c$.

So every function $f_k (k = 1, 2, \ldots)$ has the Darboux property.
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Since \( f_k = g_k + h_k \), \( f = g + h \) and \( g = \lim_{k \to \infty} g_k \), it is sufficient for the proof of the equality \( f = \lim_{k \to \infty} f_k \) to prove that \( h = \lim_{k \to \infty} h_k \).

If \( x \in F_{n,m} \) then \( h_k(x) = h(x) \) for \( k > n + m \) and \( h(x) = \lim_{k \to \infty} h_k(x) \).

Suppose that \( h \) is continuous at \( x \). For fixed \( \varepsilon > 0 \) there is an index \( k_0 > 1 \) such that \( 2^{-k_0 + 2} < \varepsilon \). Since \( x \notin F_{k_0} \), there is a positive number \( r \) such that

\[
(x - r, x + r) \cap F_{k_0} = \emptyset.
\]

Let \( k_2 > k_0 \) be an index such that \( 1/k_2 < r \). From (3), (8) and from the definition of \( h_k \) it follows, that for \( k > k_2 \), \(|h_k(x)| \leq 2^{-k_0 + 2} < \varepsilon \). So \( \lim_{k \to \infty} h_k(x) = 0 = h(x) \).

Now, let \( x \in F_n \setminus B \), for some index \( n \). Since \( F_n \subset F_k \) and every \( I_{k,n,m,j} \subset \mathbb{R} \setminus F_k \subset \mathbb{R} \setminus F_n \) for \( k > n \), it follows from (2) and from the definition of \( h_k \) that \( h_k(x) = 0 = h(x) \) for \( k > n \). So \( \lim_{k \to \infty} h_k(x) = h(x) \). This completes the proof.

\[ \Box \]

**Theorem 2.** The following equality is true:

\[
\mathcal{M}_2 = B_1(\mathcal{M}_1 \cap \mathcal{P} \cap \mathcal{D}).
\]

**Proof.** Since \( \mathcal{M}_2 = B_1(\mathcal{M}_1) \), \( \mathcal{M}_2 \supset B_1(\mathcal{M}_1 \cap \mathcal{P} \cap \mathcal{D}) \).

Now, let \( f \in \mathcal{M}_2 \). There exist a function \( g \) of Baire class 2 and an \( F_\sigma \) set \( B \) of measure zero such that:

\[
\{x \in \mathbb{R} : f(x) \neq g(x) \} \subset B.
\]

We can write \( B = \bigcup_{n=1}^{\infty} B_n \), where all the sets \( B_n \) are closed and \( B_n \subset B_{n+1} \) for \( n = 1, 2, \ldots \).

The function \( g \) is the limit of a sequence of functions \( g_n \) of Baire class 1. For \( k = 1, 2, \ldots \) let

\[
h_k(x) = \begin{cases} g_k(x) & \text{for } x \in \mathbb{R} \setminus B, \\
                   f(x) & \text{for } x \in B_k.
\end{cases}
\]

Evidently, every function \( h_k \) \((k = 1, 2, \ldots)\) is pointwise discontinuous. For \( k = 1, 2, \ldots \) there is ([2]) an almost everywhere continuous function \( t_k : \mathbb{R} \to \mathbb{R} \) of Baire class 1 such that:

- \( \{x \in \mathbb{R} : t_k(x) \neq 0 \} \) is \( F_\sigma \) set of measure zero;
- \( \{x \in \mathbb{R} : t_k(x) \neq 0 \} \cap B = \emptyset; \)
- \( \{x \in \mathbb{R} : t_{k_1}(x) \neq 0 \} \cap \{x \in \mathbb{R} : t_{k_2}(x) \neq 0 \} = \emptyset \) if \( k_1 \neq k_2 \) \((k_1, k_2 = 1, 2, \ldots);\)
- \( h_k + t_k \in \mathcal{P} \cap \mathcal{D}. \)
Let $f_k = h_k + t_k$, $k = 1, 2, \ldots$. Since 
\[ \{x \in \mathbb{R} : f_k(x) \neq g_k(x)\} \subset \{x \in \mathbb{R} : t_k(x) \neq 0\} \cup B_k, \]
we have $f_k \in \mathcal{M}_1$. So $f_k \in \mathcal{M}_1 \cap \mathcal{D} \cap \mathcal{P}$ for $k = 1, 2, \ldots$.

If $x \in B$ then there is an index $n$ such that $x \in B_k$ for $k \geq n$ and consequently, $f_k(x) = f(x)$ for $k > n$. So $\lim_{k \to \infty} f_k(x) = f(x)$.

If $x \notin B$ then $h_k(x) = g_k(x)$ for $k = 1, 2, \ldots$. Since $\lim_{k \to \infty} g_k(x) = g(x)$ and $\lim_{k \to \infty} t_k(x) = 0$, we have

\[ \lim_{k \to \infty} f_k(x) = \lim_{k \to \infty} g_k(x) = g(x) = f(x). \]

This completes the proof. \qed

From Theorems 1 and 2 there follows:

**Corollary 1.** For denumerable ordinal numbers $\alpha > 1$ the following equality is true:

\[ B_\alpha(A) = B_\alpha(A \cap Q \cap D). \]

**Theorem 3.** For every denumerable ordinal number $\alpha > 0$ the following equality is true:

\[ B_1(D \cap \bigcup_{\beta < \alpha} \mathcal{M}_\beta) = \mathcal{M}_\alpha. \]

**Proof.** For $\alpha = 2$ this theorem follows from Theorem 2. For $\alpha = 1$ the proof is the same as the proof of Theorem 2, where the $g_k$ are continuous and consequently $h_k \in \mathcal{A}$. (Instead of [2] we need [7].)

Assume that $\alpha > 2$. The inclusion

\[ B_1(D \cap \bigcup_{\beta < \alpha} \mathcal{M}_\beta) \subset \mathcal{M}_\alpha \]

is obvious. If $f \in \mathcal{M}_\alpha$ then there exist a function $g$ of Baire class $\alpha$ and an $F_\sigma$ set $B$ of measure zero such that

\[ \{x \in \mathbb{R} : f(x) \neq g(x)\} \subset B. \]

The function $g$ is the limit of the sequence of functions $g_n$ of Baire class $\beta_n$, where $\beta_n < \alpha$ ($n = 1, 2, \ldots$) and $B = \bigcup_{n=1}^{\infty} B_n$ where $B_n \subset B_{n+1}$ and all the sets $B_n$ are closed ($n = 1, 2, \ldots$).

Let $C_{n,m} \subset \mathbb{R} \setminus B$ $(n, m = 1, 2, \ldots)$ be a family of pairwise disjoint perfect sets of measure zero such that for every open interval $I$ and for every $n = 1, 2, \ldots$ there is $m$ such that $C_{n,m} \subset I$. For all $n, m = 1, 2, \ldots$ let $h_{n,m} : C_{n,m} \to [-m, m]$ be a continuous function.
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For \( k = 1,2,\ldots \) let us put

\[
f_k(x) = \begin{cases} 
h_{k,m}(x) & \text{if } x \in C_{k,m}, \quad m = 1,2,\ldots, \\ f(x) & \text{if } x \in B_k, \\ g_k(x) & \text{otherwise.} \\
\end{cases}
\]

Obviously, \( f_k \) has the Darboux property. Since

\[
\{ x \in \mathbb{R} : f_k(x) \neq g_k(x) \} \subset B_k \cup \bigcup_m C_{k,m}
\]

and the set \( B_k \cup \bigcup_m C_{k,m} \) is an \( F_\sigma \) set of measure zero, the function \( f_k \in \mathcal{M}_{\beta_k} \), where \( \beta_k < \alpha \).

The equality \( f(x) = \lim_{k \to \infty} f_k(x) \) for every \( x \in \mathbb{R} \), is obvious. \( \square \)

PROBLEM 1. Is it true the following equality

\[
\mathcal{M}_{\alpha} \cap \mathcal{P} = B_1 \left( \bigcup_{\beta < \alpha} \mathcal{M}_{\beta} \cap Q \right)
\]

for \( \alpha > 1 \)?

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