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RETRACTS OF ABELIAN MULTILATTICE GROUPS

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ABSTRACT. We study retracts of an abelian m-group which is an internal direct product of a finite number of its m-subgroups. The main result is formulated in 2.6.

Retracts of partially ordered sets were investigated in [3] – [6]. J. Jakubík [8] studied retracts of abelian lattice ordered groups.

Let G be an abelian directed multilattice group (m-group) which is an internal direct product of its m-subgroups A and B and let H be a retract of G. In this paper, it will be shown that there exist retracts R_1 of A, R_2 of B such that H is isomorphic with the external direct product R_1 and R_2. On the other hand, it will be proved that, in general, H need not be an internal product of a retract of A and a retract of B.

This generalizes the results of J. Jakubík [8] concerning retracts of abelian lattice ordered groups.

1. Preliminaries

Let $(P, \leq)$ be a *partially ordered set* and let $x, y \in P$. The set of all lower (upper) bounds of the set $\{x, y\}$ in P will be denoted by $L(x,y)$ $(U(x,y))$.

A subset S of P is called *directed* if $L(x,y) \cap S \neq \emptyset$, $U(x,y) \cap S \neq \emptyset$ for each $x, y \in S$.

Let $x, y \in P$, $x \leq y$. The *interval* $[x,y]$ is the set $\{z \in P : x \leq z \leq y\}$. Let K be a subset of P such that $k_1, k_2 \in K$, $k_1 \leq k_2$ implies $[k_1, k_2] \subseteq K$. Then K is said to be a *convex* subset of P.

The notion of multilattice has been introduced by M. Benado [2] in the following way.

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A partially ordered set $(P,\leq)$ is called a *multilattice* if the following two conditions are fulfilled:

(i) if $x,y \in P$, $z \in L(x,y)$, then there exists $z' \in L(x,y)$ such that $z \leq z'$, and z' is a maximal element of $L(x,y)$,

(ii) the dual condition concerning $U(x,y)$ holds.

If, moreover, $(P,\leq)$ is a directed set, then $(P,\leq)$ is said to be a *directed multilattice.*

Let $(P,\leq)$ be a multilattice. For $x,y \in P$ we denote by $x \wedge y$ and $x \vee y$ the set of all maximal elements of $L(x,y)$ or all minimal elements of $U(x,y)$, respectively. If P is a directed multilattice, then the sets $x \wedge y$ and $x \vee y$ are not empty. We shall write x instead of $\{x\}$.

A partially ordered group $(G,+,\leq)$ will be called a *multilattice group* if the partially ordered set $(G,\leq)$ is a multilattice. If, moreover, $(G,\leq)$ is a directed multilattice, then a partially ordered group $(G,+,\leq)$ is said to be a *directed multilattice group* (*m-group*). For the definitions and properties of partially ordered groups and multilattice groups, see [7] or [1] respectively.

Let $(G,+,\leq)$ be an m-group. In the next, we shall write G instead of $(G,+,\leq)$. A subgroup G' of G is said to be an *m-subgroup* of G if, whenever $x,y \in G'$, then $x \wedge y \subseteq G'$ and $x \vee y \subseteq G'$.

A mapping φ of G onto an m-group $\overline{G}$ is called a *homomorphism* of G onto $\overline{G}$ if the following conditions are satisfied:

(i) φ is a homomorphism of the group $(G,+)$ onto the group $(\overline{G},+)$,

(ii) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for each $x,y \in G$.

An *isomorphism* of m-groups is defined in the obvious way. If G and H are isomorphic m-groups, we shall use the denotation $G \simeq H$.

Let A and B be m-subgroups of G such that the following conditions hold:

(i) for each $g \in G$ there exist uniquely determined elements $a \in A$, $b \in B$ such that $g = a + b$;

(ii) if $g,g' \in G$, $g = a + b$, $g' = a' + b'$, $a,a' \in A$, $b,b' \in B$, then $gtg' = ata' + btb'$, where $t \in \{+,\wedge,\vee\}$.

Under these assumptions, G is said to be an *internal direct product of* A *and* B. It will be expressed by writing $G = (\mathrm{i})\, A \times B$.

It is easily seen that $A \cap B = \{0\}$ and that both A and B are convex subsets of G.

An *internal direct product of m-subgroups* $A_1, A_2, \dots, A_n$ of G is defined analogously, and we write $G = (\mathrm{i})\, A_1 \times A_2 \times \cdots \times A_n$.

Let X and Y be m-groups. We form the (external) direct product G of groups X and Y. Define the operations $\wedge$ and $\vee$ componentwise. Then G is an m-group and G is called the (*external*) *direct product* of X and Y. We shall use the notation $G = A \times B$.

Let H be an m-subgroup of an m-group G. We say that H is a *retract* of G if there is a homomorphism f of G onto H such that $f(h) = h$ for each $h \in H$. The homomorphism f will be said to be a *retract mapping* of G onto H.

2. Direct products and retracts

Throughout this section, we suppose that G is an abelian m-group, A and B are m-subgroups of G, and that the relation

$$G = (\mathrm{i})\, A \times B \tag{1}$$

is valid.

Assume that A_1 is a retract of A with the corresponding retract mapping α and that B_1 is a retract of B with the corresponding retract mapping β. For each $g \in G$, $g = a + b$, $a \in A$, $b \in B$ we put $f(g) = \alpha(a) + \beta(b)$ and denote $f(G) = H$.

2.1. LEMMA. *H is a retract of G with the corresponding retract mapping f, and the relation*

$$H = (\mathrm{i})\, A_1 \times B_1 \tag{2}$$

is valid.

P r o o f. Let $g, g' \in G$, $g = a+b$, $g' = a'+b'$, $a, a' \in A$, $b, b' \in B$. We have $f(g \wedge g') = f\big((a+b) \wedge (a'+b')\big) = f(a \wedge a' + b \wedge b') = \alpha(a \wedge a') + \beta(b \wedge b') = \alpha(a) \wedge \alpha(a') + \beta(b) \wedge \beta(b') = \big(\alpha(a) + \beta(b)\big) \wedge \big(\alpha(a') + \beta(b')\big) = f(g) \wedge f(g')$. In a similar manner, it can be verified that $f(g \vee g') = f(g) \vee f(g')$ and $f(g+g') = f(g) + f(g')$ hold.

Let $h \in H$. Then there is an element $g \in G$, $g = a+b$, $a \in A$, $b \in B$ such that $f(g) = h$. Then $f(h) = f\big(f(g)\big) = f\big(\alpha(a) + \beta(b)\big) = \alpha\big(\alpha(a)\big) + \beta\big(\beta(b)\big) = \alpha(a) + \beta(b) = f(g) = h$.

We have shown that H is a retract of G with the corresponding retract mapping f.

Let $h \in H$, $h = a + b$, $a \in A$, $b \in B$. Then $h = f(h) = \alpha(a) + \beta(b)$, $\alpha(a) \in A_1$, $\beta(b) \in B_1$. Since $A_1 \subseteq A$, $B_1 \subseteq B$, with respect to (1), elements $\alpha(a)$ and $\beta(b)$ are uniquely determined. Again, from (1), it follows that operations $+$, $\wedge$, $\vee$ on H are performed componentwise. Hence (2) holds true. $\square$

Now assume that H is a retract of G, and f is a retract mapping of G onto H. Denote $f(A) = H_1$, $f(B) = H_2$.

R e m a r k . $f(A)$ need not be a retract of A. It can happen that $f(A) \subseteq A$ fails to hold in general (see Example).

2.2. Lemma. *H_1 and H_2 are m-subgroups of H, and $H_1 \cap H_2 = \{0\}$.*

Proof. Since A is an m-subgroup of G and f is a homomorphism of G onto H, H_1 is an m-subgroup of H. Analogously, we obtain that H_2 is an m-subgroup of H.

Denote $H_i^+ = \{h \in H_i : h \geq 0\}$ $(i = 1, 2)$. At first we prove that $H_1^+ \cap H_2^+ = \{0\}$ is fulfilled.

Suppose that there exists an element $h \in H_1^+ \cap H_2^+$, $h > 0$. Then there is $a' \in A$ with $h = f(a')$, and we have $h = h \vee 0 = f(a') \vee 0 = f(a') \vee f(0) = f(a' \vee 0) = f(a)$, where $a > 0$, $a \in a' \vee 0$. Similarly, we can find an element $b \in B$, $b > 0$ with $h = f(b)$. Hence $h = f(a) = f(b) = f(a) \wedge f(b) = f(a \wedge b)$. The relation $A \cap B = \{0\}$ and convexity of A and B in G imply that there is no element $g \in G$ such that $0 < g < a, b$. Therefore $0 \in a \wedge b$, and thus $h = 0$, which is a contradiction.

Let $H_1 \cap H_2 = K \neq \{0\}$. Then there exists an element $k \in K$, $k \neq 0$. Since K is an m-subgroup of H, the relation $k \vee 0 \subseteq K$ holds. Hence there exists an element $k' > 0$, $k' \in k \vee 0$, $k' \in H_1^+ \cap H_2^+$, which is a contradiction. □

2.3. Lemma. *Let H_1 and H_2 be as above. Then*

$$H = (\mathrm{i})\, H_1 \times H_2 . \tag{3}$$

Proof. Let $h \in H$, $h = a+b$, $a \in A$, $b \in B$. Then $h = f(h) = f(a) + f(b)$, $f(a) \in H_1$, $f(b) \in H_2$. Now we intend to show that elements $f(a)$ and $f(b)$ are uniquely determined.

Let $h = h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$. There exist $a' \in A$ and $b' \in B$ such that $h_1 = f(a')$, $h_2 = f(b')$, and we get $h = f(a') + f(b')$. Therefore $f(a) + f(b) = f(a') + f(b')$ and $f(a) - f(a') = f(b') - f(b)$. Since $f(a) - f(a') \in H_1$, $f(b') - f(b) \in H_2$, in view of 2.2, we obtain $f(a) = f(a')$, $f(b) = f(b')$.

It remains to show that the operations $+$, $\wedge$, $\vee$ are performed componentwise. We prove it for the operation $\wedge$. For the operations $\vee$ and $+$ the proofs are similar.

Let $h' \in H$, $h' = a' + b'$, $a' \in A$, $b' \in B$. Hence $h' = f(a') + f(b')$. From (1), it follows that $h \wedge h' = f(h \wedge h') = f(a \wedge a' + b \wedge b') = f(a \wedge a') + f(b \wedge b') = f(a) \wedge f(a') + f(b) \wedge f(b')$. Therefore (3) is satisfied. □

Let $h_1 \in H_1$, $h_1 = a + b$, $a \in A$, $b \in B$. Define the mapping $f_1 \colon H_1 \to A$ by $f_1(h_1) = a$.

2.4. Lemma. *The mapping f_1 is an isomorphism of H_1 into A.*

Proof. It is obvious that f_1 preserves the operations $+$, $\wedge$, $\vee$.

Let $h_1 \in H_1$, $h_1 = a + b$, $a \in A$, $b \in B$. Assume that $f_1(h_1) = 0$. Hence, from $f_1(h_1) = a$, it follows that $a = 0$, and so $h_1 = b \in B$. Since $h_1 \in H$, we get $h_1 = f(h_1) = f(b)$. From this, we infer that $h_1 \in H_2$. According to 2.2, we obtain $h_1 = 0$. This yields that f_1 is one-to-one. We conclude that f_1 is an isomorphism of H_1 into A. □

In an analogous way, we define the mapping $f_2\colon H_2 \to B$, and, similarly, we can verify that f_2 is an isomorphism of H_2 into B.

Let $\varphi\colon A \to A$ be a mapping defined by the rule $\varphi(a) = f_1\big(f(a)\big)$ for each $a \in A$.

2.5. LEMMA. *$f_1(H_1)$ is a retract of A with the corresponding retract mapping φ.*

Proof. The mapping f reduced to A is a homomorphism of A onto H_1. In view of 2.4, f_1 is an isomorphism of H_1 into A. Therefore φ is a homomorphism of A onto $f_1(H_1)$.

Let $a_1 \in f_1(H_1)$. There exist $h_1 \in H_1$, $h_1 = a_1 + b_1$, $a_1 \in A$, $b_1 \in B$ and $a \in A$ such that $f(a) = h_1$. Hence, $f(a) = a_1 + b_1$, $f(a) = f(a_1) + f(b_1)$, $f(a - a_1) = f(b_1)$. Using 2.2 we obtain $f(b_1) = 0$. Therefore $f(a_1) = f(a)$, and thus $\varphi(a_1) = f_1\big(f(a_1)\big) = f_1\big(f(a)\big) = f_1(a_1 + b_1) = a_1$. □

We have proved that $R_1 = f_1(H_1)$ is a retract of A. In a similar manner, it can be shown that $R_2 = f_2(H_2)$ is a retract of B.

Define the mapping $\phi\colon H \to R_1 \times R_2$ as follows: for each $h \in H$, $h = h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$ we put $\phi(h) = \big(f_1(h_1), f_2(h_2)\big)$.

It is easy to prove that the following assertion is valid:

2.6. THEOREM. *The mapping ϕ is an isomorphism of H onto $R_1 \times R_2$,*

$$H \simeq R_1 \times R_2 .$$

By the induction we get

2.7. THEOREM. *Let $A_1, A_2, \dots, A_n$ be m-subgroups of G such that*

$$G = (\mathrm{i})\, A_1 \times A_2 \times \cdots \times A_n ,$$

and let H be a retract of G. Then there exist retracts $R_1, R_2, \dots, R_n$ of $A_1, A_2, \dots, A_n$ such that

$$H \simeq R_1 \times R_2 \times \cdots \times R_n .$$

Again, assume that $G = (\mathrm{i})\, A \times B$. It can happen that there exists a retract H of G with the corresponding retract mapping f such that H cannot be expressed as an internal direct product of a retract of A and a retract of B, and that $f(A) \subseteq A$ is not satisfied.

E x a m p l e . Let $\mathcal{A} = \mathbb{Z}$, where $\mathbb{Z}$ is the additive group of all integers with the natural linear order, $\mathcal{B} = \{(x,y) :\ x, y \in \mathbb{Z},\ \ x - y \text{ is even}\}$. $\mathcal{A}$ is a linearly ordered group. Addition and a partial order on $\mathcal{B}$ are defined componentwise. One verifies easily that $\mathcal{B}$ is an m-group. There are elements in $\mathcal{B}$, e.g., $b_1 = (2,4)$, $b_2 = (1,5)$ possessing no least upper bound in $\mathcal{B}$. Let us consider the direct product $G = \mathcal{A} \times \mathcal{B}$ of m-groups $\mathcal{A}$ and $\mathcal{B}$. Then G is an m-group. Introduce the notation $A = \{(a,b) \in G :\ b = 0\}$, $B = \{(a,b) \in G :\ a = 0\}$, $H = \{(a,(x,y)) \in G :\ x = y = a\}$. Obviously, A, B and H are m-subgroups of G and $G = (\mathrm{i})\, A \times B$ is valid.

Define the mapping $f \colon G \to H$ by $f(g) = (a,(a,a))$ for each element $g \in G$, $g = (a,(x,y))$, $a \in \mathcal{A}$, $(x,y) \in \mathcal{B}$. Then H is a retract of G with the corresponding retract mapping f. We verify only that f preserves the operation $\wedge$. Let $g' \in G$, $g' = (a',(x',y'))$. Assume that $a \le a'$ (if $a > a'$, the result is analogous). Then $f(g \wedge g') = f\big((a,(x,y)) \wedge (a',(x',y'))\big) = f\big((a,(x,y) \wedge (x',y'))\big) = (a,(a,a))$ and $f(g) \wedge f(g') = (a,(a,a)) \wedge (a',(a',a')) = (a,(a,a))$. Hence $f(g \wedge g') = f(g) \wedge f(g')$. Since $f(A) = H$, we have $f(A) \nsubseteq A$. From the fact that H is a diagonal of G, we conclude that H is the internal direct product of no m-subgroups of H.

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