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Rastislav Potocký

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ON THE EXPECTED VALUE OF VECTOR LATTICE — VALUED RANDOM VARIABLES

RASTISLAV POTOCKÝ

The aim of this paper is to present a definition of the expected value of a random variable with values in an Archimedean vector lattice E. The motivation for the study of such variables are possible applications in numerous fields of probability and applied statistics such as stochastic processes, decision theory, estimation and so on. The second reason is that in a number of spaces the convergence in vector lattices (the so-called order convergence) is stronger than the topological one (e. g. L^p-spaces, $1 \leq p < \infty$). The most interesting results concerning the expected value of such random variables have been obtained by Cristescu [1], Kantorovich, Vulich, Pinsker [2]. However, their results are restricted to the so-called regular spaces only. I shall show that the expected value can be defined in more general spaces, too. In addition a convergence theorem (of Beppo-Levi's type) will be proved. My terminology follows [3], [4], [5]. See also [6].

Definition 1. *Let (Z, S, P) be a probability space, E a σ-complete vector lattice with the σ-property. A countably valued random variable $f\colon Z \to E$; $f = \sum_1 x_i \chi_{E_i}$, $x_i \in E$, $E_i \in S$, $\bigcup E_i = Z$; $E_i \cap E_j = \emptyset$, $i \neq j$ is said to have the expected value Ef if $\sum_1^\infty |x_i| P(E_i) < \infty$. The expected value is defined as follows $Ef = \sum_1^\infty x_i P(E_i)$.* It is clear that this definition is justified.

Definition 2. *Let (Z, S, P) be a probability space. A sequence (f_n) of functions from Z to E converges to a function f* almost uniformly if for every $\varepsilon > 0$ *there exists a set $A \in S$ such that $P(A) < \varepsilon$ and (f_n) converges relatively uniformly on $Z - A$; i. e. there exists a sequence (a_n) of real numbers converging to 0 and an element $r \in E$ such that $|f_n(z) - f(z)| \leq a_n r$ for each $z \in Z - A$.*

Definition 3. *A non-negative function $f\colon Z \to E$ is called a random variable if there exists a non-decreasing sequence (f_n) of non-negative countably valued random variables such that (f_n) converges to f almost uniformly.*

The random variable $f\colon Z \to E$ is said to have the expected value Ef if all f_n have the expected value $E f_n$ and the sequence (Ef_n) converges relatively uniformly (ru in short). We define the expected value of f by $Ef = ru\text{-}\lim Ef_n$. The family of such random variables will be denoted by $\mathcal{U}$.

The correctness of this definition follows from lemma 1.

Lemma 1. *Let (f_n) be a decreasing sequence of countably valued random variables with expected values Ef_n such that $f_n \downarrow 0$ almost univormly. Then $Ef_n \downarrow 0$ relatively uniformly.*

Proof. Since there are only countably many values that the functions f_n take on and since E has the σ-property we can regard all f_n as random variables in a principal ideal of E (i. e. ideal generated by a single element, say u, $u \in E^+$) I_u. For the same reason we can suppose $Ef_n \in I_u$ for each n.

Since E is a σ-complete vector lattice, I_u equipped with the order-unit norm (i. e. the norm induced by u) is a Banach space. It will be denoted by $(I_u, \|\ \|_u)$. In such a space the norm-convergence and the relatively uniform convergence are equivalent (see [5], p. 102).

As $f_n \downarrow 0$ almost uniformly in I_u, we have $\|f_n\|_u \downarrow 0$ almost everywhere. From this we obtain $E\|f_n\|_u \downarrow 0$ and consequently $\|Ef_n\|_u \downarrow 0$, i. e. $Ef_n \downarrow 0$ with respect to the norm of I_u. It means that $Ef_n \downarrow 0$ relatively uniformly in I_u.

The next procedure is well known. Let (f_n) and (g_n) be nondecreasing sequences such that $f_n \uparrow f$ and $g_n \uparrow f$ almost uniformly with their expected values ru-converging. Consider the random variable f_i for a fixed i. Since $f_i \leq f$ we have $f_i - (f_i \wedge g_n) = f_i - \frac{1}{2}(f_i + g_n - |f_i - g_n|)$ for each n. From this it follows that the sequence $f_i - (f_i \wedge g_n)$ almost uniformly converges to 0 as $n \to \infty$ and consequently, by lemma 1 that $Ef_i = ru\text{-}\lim E(f_i \wedge g_n) \leq ru\text{-}\lim Eg_n$ for each i.

Definition 4. *A function f: $Z \to E$ is called a random variable if there exist random variables f_1 and f_2 from definition 3 such that $f(z) = f_1(z) - f_2(z)$ for each z. The random variable f is said to have the expected value Ef if f_1 and f_2 are in $\mathcal{U}$. The expected value is defined by setting $Ef = Ef_1 - Ef_2$.*

This definition is justified. It is clear from the above mentioned construction that for each random variable f there exists a sequence (f_n) of countably valued random variables such that (f_n) converges to f almost uniformly. For more details about vector lattice-valued random variables see [6].

In the rest of the paper I am going to investigate sequences of random variables which have the expected values.

Lemma 2. *If (f_n) is a non-decreasing sequence of random variables from $\mathcal{U}$ almost uniformly converging to a random variable f such that ru-lim Ef_n exists, then $f \in \mathcal{U}$ and $Ef = ru\text{-}\lim Ef_n$.*

Proof. Denote $ru\text{-}\lim Ef_n$ by c. For each n let (f_n^k) be a non-decreasing sequence of countably valued random variables converging almost uniformly to f_n. We have $|Ef_n^k - Ef_n| \leq a_n^k r_n$ for some $r_n \in E$ and $a_n^k \xrightarrow{k} 0$.

Since E has the σ-property, there is an $u \in E$ such that $r_n \leq K(n)\, u$ for each n, where $K(n)$ is a function from N to N, N the set of natural numbers. Denoting

$a_n^k K(n)$ by b_n^k we have $|Ef_n^k - Ef_n| \leq b_n^k u$. As the set of real numbers has the diagonal property there exists a sequence $b_n^{k^*(n)}$ converging to 0. Hence we have $|Ef_n^{k^*(n)} - Ef_n| \leq b_n^{k^*(n)} u$. From this we obtain

$$|c - Ef_n^{k^*(n)}| \leq d_n v + b_n^{k^*(n)} u, \text{ i. e. } ru\text{-lim } Ef_n^{k^*(n)} = c.$$

One can show by repeating step by step the preceding argument that there exists a sequence $(f_n^{k(n)})$ of countably valued random variables almost uniformly converging to f. Put $h_n = \sup_{i \leq n} f_i^{k(i)} \vee f_n^{k^*(n)}$. The non-decreasing sequence (h_n) of countably valued random variables almost uniformly converges to f and (Eh_n) ru-converges to c. It means that $f \in \mathscr{U}$ and $Ef = ru\text{-lim } Ef_n$.

Theorem 1. *If (f_n) is a non-decreasing sequence of random variables with expected values Ef_n almost uniformly converging to a random variable f and such that ru-lim Ef_n exists, then the function f has the expected value Ef and $Ef = ru$-lim Ef_n.*

Proof. Consider functions k_n defined as follows: $k_1(z) = f_1(z)$, $k_n(z) = f_n(z) - f_{n-1}(z)$, $n = 2, 3, \ldots$, k_n are non-negative random variables with expected values. Moreover $f_1(z) + \sum_2^\infty k_n(z) = \lim f_n(z) = f(z)$ for each z. Since ru-lim Ef_n exists it follows that the series ΣEk_n ru-converges.

By definition 4 there are functions g_n, h_n in $\mathscr{U}$ such that $k_n = g_n - h_n$, $n = 2, 3, \ldots$ Fix a real sequence (a_n) such that $a_n \downarrow 0$. Then for each n there exists a set $A_n \in S$ such that $P(A_n^C) < a_n 2^{-n}$ and $|h_n^k - h_n| \leq d_n^k r$ for each $z \in A_n$ since E has the σ-property. Also there exists a $k(n)$ such that $d_n^{k(n)} \leq 2^{-n}$. Given $\varepsilon > 0$, there exists a natural number n_0 such that $a_{n_0} \leq 2^{-1}\varepsilon$. Put $A = \bigcup_{n=n_0}^\infty A_n^C$. We have $P(A) \leq \sum_{n_0}^\infty P(A_n^C) < \varepsilon$. It means that $h_n - h_n^{k(n)} \downarrow 0$ almost uniformly. Consider the non-decreasing sequence $\left(\sum_1^n (h_i - h_i^{k(i)})\right)$ of random variables in $\mathscr{U}$. Denoting its limit (with respect to the order) by s we shall prove that this sequence converges to s almost uniformly. Given $\varepsilon > 0$, there exists (by the preceding part of the proof) the above mentioned set A^C such that $\left|\sum_{n_0}^\infty (h_i - h_i^{k(i)})\right| \leq \sum_{n_0}^\infty 2^{-n} r$. It means that the values of the function s belong to E and moreover $\left|s - \sum_1^n (h_i - h_i^{k(i)})\right| \leq \sum_{n+1}^\infty 2^{-n} r$ for each $z \in A^C$. Omitting, if necessary, the set of probability zero, we have proved that $\sum_1^\infty (h_i - h_i^{k(i)}) \in \mathscr{U}$ by lemma 2.

We have $(g_n^k - h_n^{k(n)}) \vee 0 \uparrow g_n - h_n^{k(n)}$ almost uniformly and $E(g_n - h_n^{k(n)}) - E((g_n^k - h_n^{k(n)}) \vee 0)$ goes to 0 relatively uniformly (since Eg_n^k ru-converges to Eg_n).

In other words $g_n - h_n^{k(n)} \in \mathscr{U}$. Moreover the equality $\Sigma g_n = \Sigma k_n + \Sigma h_n$ implies the almost uniform convergence of $\left(\sum_1^n g_n\right)$.

Since we have $\sum_{n=2}^{\infty} Eg_n = \sum_{n=2}^{\infty} Ek_n + \sum_{n=2}^{\infty} Eh_n$, the consideration of functions $g_n - h_n^{k(n)}$ and $h_n - h_n^{k(n)}$ instead of g_n and h_n, respectively, shows that the series $\sum_{n=2}^{\infty} Eg_n$ ru-converges.

It follows from lemma 2 that Σg_n belongs to $\mathscr{U}$ and consequently that the series $\Sigma k_n = \Sigma g_n - \Sigma h_n$ has the expected value. Moreover we have

$$E \sum_2^{\infty} k_n = E \sum_2^{\infty} g_n - E \sum_2^{\infty} h_n = ru\text{-lim} \sum_2^n Eg_i - ru\text{-lim} \sum_2^n Eh_i =$$

$$= ru\text{-lim} \sum_2^n Ek_i = \sum_2^{\infty} Ek_n.$$

$$Ef = E\left(f_1 + \sum_2^{\infty} k_n\right) = Ef_1 + E \sum_2^{\infty} k_n = Ef_1 + ru\text{-lim} \sum_2^n Ek_i =$$

$$= ru\text{-lim}\left(Ef_1 + \sum_2^n Ek_i\right) = ru\text{-lim}\, Ef_n.$$

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Katedra teórie pravdepodobnosti
a matematickej štatistiky
Matematicko-fyzikálnej fakulty UK
Mlynská dolina
842 15 Bratislava

МАТЕМАТИЧЕСКОЕ ОЖИДАНИЕ ДЛЯ СЛУЧАЙНЫХ ВЕЛИЧИН СО ЗНАЧЕНИЯМИ В ВЕКТОРНОЙ РЕШЕТКЕ

Rastislav Potocký

Резюме

В работе определяется математическое ожидание для случайных величин со значениями в векторной решетке и доказывается одна предельная теорема.