Józef Banaś; Antonio Martinón

Measures of noncompactness in Banach sequence spaces

Mathematica Slovaca, Vol. 42 (1992), No. 4, 497–503

Persistent URL: http://dml.cz/dmlcz/131942

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1992

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must contain
these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped
with digital signature within the project DML-CZ: The Czech Digital Mathematics
Library http://project.dml.cz
MEASURES OF NONCOMPACTNESS IN BANACH SEQUENCE SPACES

JÓZEF BANAŚ* 1) — ANTONIO MARTINON** 2)

ABSTRACT. We construct a measure of noncompactness in the sequence space $l^p(E_i)$ which turns out to be regular but not equivalent to the Hausdorff measure of noncompactness. Apart from that a formula for the Hausdorff measure in the sequence space $c_0(E_i)$ is derived.

1. Introduction

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The theory connected with this notion was initiated by Kuratowski [12] and Darbo [7], but the main applications of measures of noncompactness were pointed out by Sadovskii [15], Ambrosetti [2], Nussbaum [14] and Danesš [6], among others. The current state of this theory and its applications are presented in the books [1, 3], for example.

In this paper we shall study measures of noncompactness in some Banach sequence spaces.

At the beginning we establish some notation.

Assume that $E$ is a Banach space with the norm $\| \cdot \|$ and the zero element $\theta$. By $K(x, r)$ we denote the closed ball centered at $x$ and radius $r$. The unit ball $K(\theta, 1)$ will be denoted by $B_E$ or shortly by $B$. Moreover, $\bar{X}$, $\text{Conv} X$ denote the closure and the convex closure of a set $X$, respectively.

Finally, denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

AMS Subject Classification (1991): Primary 47H09.
Key words: Measure of noncompactness, Banach sequence space.

1) This paper was written during the visit at the University of La Laguna
2) Supported in part by DGCYT grant PB88-0417
JÓZEF BANAS — ANTONIO MARTINON

**DEFINITION 1.** A function $\mu : \mathcal{M}_E \to \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ will be called a measure of noncompactness in $E$ if it satisfies the following conditions:

1. $\mu(X) = 0 \iff X \in \mathcal{M}_E$,
2. $X \subseteq Y \implies \mu(X) \leq \mu(Y)$,
3. $\mu(\text{Conv} \ X) = \mu(X)$,
4. $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$,
5. $\mu(X + Y) \leq \mu(X) + \mu(Y)$,
6. $\mu(cX) = |c|\mu(X)$, $c \in \mathbb{R}$.

Notice that in book [3] measures of noncompactness defined above are called regular.

**Remark 1.** Let us mention that Definition 1 implies that the measure $\mu$ has also the following property:

7. If $X_n \supseteq X_{n+1}$ and $\overline{X}_n = X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $\bigcap_{n=1}^{\infty} X_n$ is nonempty.

This fact may be proved in the same fashion as Theorem 5 in [4].

Recall that the functions $\chi, I : \mathcal{M}_E \to \mathbb{R}_+$ defined by

$$
\chi(X) = \inf\{r > 0 : \text{there exists a finite set } Y \subset E \text{ such that } X \subset Y + rB\},
$$

$$
I(X) = \sup_{(x_n) \subset X} \left\{\inf\{\|x_j - x_i\| : i \neq j, i, j = 1, 2, \ldots\}\right\}
$$

are measures of noncompactness in the sense of Definition 1 possessing also some additional properties [1]. For example, $\chi(B) = 1$.

The function $\chi$ is said to be the *Hausdorff measure* and it seems to be the most convenient in applications (cf. [1, 3]). The function $I$ is referred to as the *Istrătescu measure of noncompactness*. Recall that this function was introduced by Istrătescu in [9]. In paper [5] Danes raised the question whether $I$ is a measure of noncompactness in the above defined sense.

This question was answered in the affirmative by Erzakova [8] (cf. also [1]).

Let us observe that the measure $\chi$ and $I$ are equivalent, i.e. the following inequality holds

$$
\chi(X) \leq I(X) \leq 2\chi(X) \quad (1)
$$

for any $X \in \mathcal{M}_E$ (cf. [1]).

498
2. Auxiliary facts concerning Banach sequence spaces

Assume that \((E_i, \| \cdot \|_i)\) is a sequence of Banach spaces. Denote by \(l^p(E_i)\), or briefly by \(l^p\), \(1 \leq p \leq \infty\), the space of all sequences \(x = (x_i), \ x_i \in E_i\) for \(i = 1, 2, \ldots\) such that \(\sum_{i=1}^{\infty} \|x_i\|_i^p < \infty\). Similarly, let \(c_0 = c_0(E_i)\) denote the space of all sequences \(x = (x_i), \ x_i \in E_i\) with the property \(\|x_i\|_i \to 0\) as \(i \to \infty\). It is well known [11, 13] that both \(l^p(E_i)\) and \(c_0(E_i)\) form Banach spaces under the norms

\[\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|_i^p\right)^{1/p},\]
\[\|x\|_0 = \max\{\|x_i\|_i: \ i = 1, 2, \ldots\},\]

respectively. Similarly we can define the space \(l^\infty(E_i)\).

In the case when \(E_i = E\) for all \(i = 1, 2, \ldots\) we shall write \(l^p(E)\) and \(c_0(E)\). Such a case was discussed in [13], for example.

For further purposes denote by \(p_n\) the projection operator \(p_n: l^p(E_i) \to E_n\) (or \(p_n: c_0(E_i) \to E_n\), \(p_n(x) = p_n(x_1, x_2, \ldots) = x_n\) \((n = 1, 2, \ldots)\)).

Moreover, let us recall the following theorem [13].

**Theorem 1.** A set \(X \subset l^p(E_i)\) is relatively compact if and only if

a) \(X\) is bounded,

b) the set \(p_n(X)\) is relatively compact in \(E_n\) for any \(n = 1, 2, \ldots\), and

c) for every \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that

\[\sum_{i=n}^{\infty} \|x_i\|_i^p < \varepsilon \quad \text{for all } x = (x_i) \in X \quad \text{whenever } n \geq n_0.\]

3. Measures of noncompactness in \(l^p(E_i)\) and \(c_0(E_i)\)

In this section we introduce rather convenient formulas for some measures of noncompactness in the Banach sequence spaces \(l^p(E_i)\) and \(c_0(E_i)\). To do this assume that \(\chi_i\) is the Hausdorff measure of noncompactness in the space \(E_i, \ i = 1, 2, \ldots\) and let \(\chi_p\) denote Hausdorff measure in the space \(l^p(E_i)\). Further, for a set \(X \in M_{l^p}\) let us denote

\[a(X) = \sup\{\chi_i(p_i(X)): \ i = 1, 2, \ldots\},\]
\[b(X) = \lim_{n \to \infty}\left(\sup\left\{\left(\sum_{i=n}^{\infty} \|x_i\|_i^p\right)^{1/p}: x = (x_i) \in X\right\}\right),\]
\[\mu_p(X) = \max\{a(X), b(X)\}.

Then we have
THEOREM 2. The function $\mu_p$ is a measure of a noncompactness in the space $l^p(E_i)$ such that

$$\mu_p(X) \leq \chi_p(X)$$

for any $X \in \mathcal{M}_{l^p}$.

Proof. The proof of the first part is very simple and is therefore omitted.

To prove the second part denote $\chi_p(X) = r$. Then for an arbitrary $\varepsilon > 0$ we can find a finite set $Y \subset l^p$ such that $X \subset Y + (r + \varepsilon)B_{l^p}$. Hence, using the equality $\mu_p(B_{l^p}) = 1$ we infer that $\mu_p(X) \leq r + \varepsilon$. The arbitrariness of $\varepsilon$ completes the proof.

In what follows we show that there does not exist a constant $c > 0$ such that

$$c\chi_p(X) \leq \mu_p(X) \quad (2)$$

for any $X \in \mathcal{M}_{l^p}$, provided the spaces $E_i \; (i = 1,2,\ldots)$ are assumed to be infinite dimensional.

Consider namely the sequence of subsets of $l^p(E_i)$ defined in the following way

$$X_n = \{x = (x_1, x_2, \ldots, x_n, \theta, \theta, \ldots) : x_k \in B_{E_k} \; \text{for} \; k = 1,2,\ldots, n\}.$$ 

Obviously, we have

$$\mu_p(X_n) = 1 \quad (3)$$

for any $n = 1,2,\ldots$.

On the other hand, by the Riesz lemma (see the improved version in [10]) for any $i, 1 \leq i \leq n$, we can select a sequence $(x^i_k)_{k=1,2,\ldots}$ of points from $B_{E_i}$ such that $\|x^i_k - x^i_m\| > 1$ for $k \neq m, k,m = 1,2,\ldots$.

Now, let us fix a natural number $n$ and consider the sequence $(x^k)$ of points from $X_n$ of the form

$$x^k = (x^1_k, x^2_k, \ldots, x^n_k, \theta, \theta, \ldots),$$

where $k = 1,2,\ldots$. For $k \neq m$ we have

$$\|x^k - x^m\|_p = \left(\sum_{i=1}^{n} \|x^i_k - x^i_m\|^p \right)^{1/p} > n^{1/p},$$

what implies that $I(x_n) \geq n^{1/p}$. Consequently, in virtue of (1) we get

$$\chi_p(X_n) \geq \frac{1}{2} n^{1/p}.$$
Combining this inequality and (3) we see that the inequality (2) does not hold for any $c > 0$.

Remark 2. In 1978 K. Goebel raised the question if each regular measure of noncompactness (i.e. a measure of noncompactness in the sense of Definition 1) has to be equivalent to the Hausdorff measure $\chi$ (cf. also [3]). The example of the measure $\mu_p$ described above answers this question in the negative.

In what follows we shall deal with a measure of noncompactness in the space $c_0(E_i)$. Let us denote by $\chi_0$ the Hausdorff measure of noncompactness in $c_0$. Next, let $\mu_0: \mathcal{M}_{c_0} \to \mathbb{R}^+$ be the function defined by the formula

$$\mu_0(X) = \max\{a(X), c(X)\},$$

where $a(X)$ was defined previously and $c(X)$ is given by

$$c(X) = \lim_{n \to \infty} \left( \sup_{x=(x_i) \in X} \left( \max\{\|x_k\|_k: k \geq n\} \right) \right).$$

Then we have the following theorem.

**Theorem 3.** $\chi_0(X) = \mu_0(X)$ for any $X \in \mathcal{M}_{c_0}$.

**Proof.** First, let us observe that $\mu_0(B_{c_0}) = 1$. It is also easily seen that the function $\mu_0$ satisfies the conditions 2° - 6° of the Definition 1. Moreover, from the definition of $\mu_0$ it follows that $\mu_0(Y) = 0$ for any finite subset $Y$ of $c_0(E_i)$. Thus, similarly as in the proof of Theorem 2 we infer that

$$\mu_0(X) \leq \chi_0(X). \tag{4}$$

In order to prove the converse inequality let us denote $\mu_0(X) = r$. Then, for an arbitrary $\varepsilon > 0$ we can find a positive integer $n$ such that

$$\|x_k\|_k \leq r + \varepsilon \tag{5}$$

for any $k \geq n$ and for each $x = (x_i) \in X$.

On the other hand $\chi_i(p_i(X)) \leq r$ for $i = 1, 2, \ldots$. So, fixing $k$, $1 \leq k \leq n$, we can find a finite $(r + \varepsilon)$-net $\{y_1^k, y_2^k, \ldots, y_q^k\}$ of the set $p_k(X)$ in the space $E_k$.

Further, consider the set

$$Y = \{y = (y_{i_1}^1, y_{i_2}^2, \ldots, y_{i_n}^n, \theta, \theta, \ldots): 1 \leq i_1 \leq q_1, 1 \leq i_2 \leq q_2, \ldots, 1 \leq i_n \leq q_n \}.$$
Obviously, $Y$ is a finite set consisting of $q_1q_2\ldots q_n$ elements. Observe, that for an arbitrary $x = (x_i) \in X$ we can find $y = (y_1^1, y_2^2, \ldots, y_n^n, \theta, \theta, \ldots) \in Y$ with the property

$$\max\{\|x_k - y_{i_k}\| : 1 \leq k \leq n\} \leq r + \varepsilon.$$ 

Hence, keeping in mind (5) we deduce that for any $x \in X$ there exists $y \in Y$ such that

$$\|x - y\|_0 \leq r + \varepsilon,$$

which means that $Y$ is a finite $(r + \varepsilon)$-net of the set $X$ in the space $c_0(E_i)$. Thus

$$\chi_0(X) \leq \mu_0(X).$$

This inequality in conjunction with (4) completes the proof.

Remark 3. Let us notice that in the classical cases of $l^p(\mathbb{R})$ and $c_0(\mathbb{R})$ both $\mu_p$ and $\mu_0$ are equal to the Hausdorff measure of noncompactness (cf. [3]).

REFERENCES


MEASURES OF NONCOMPACTNESS IN BANACH SEQUENCE SPACES


Received June 3, 1991
Revised May 11, 1992

*) Department of Mathematics
Technical University of Rzeszów
35-959 Rzeszów, W. Pola 2
Poland

**) Department of Mathematical Analysis
University of La Laguna
38271 La Laguna (Tenerife)
Spain