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ORIENTATIONS OF GRAPHS MINIMIZING THE RADIUS OR THE DIAMETER

ĽUBOMÍR ŠOLTÉS

The aim of this paper is to determine the exact value of a radius [a diameter] of orientations minimizing a radius [a diameter] in the case of complete bipartite graphs and to find a result in the case of product of graphs.

1. Introduction

All *graphs* considered in this paper are finite, undirected, without loops and multiple edges. A graph is a *digraph* iff every its edge is directed.

Let G be a graph or a digraph. Then the symbol $V(G)$ $[E(G)]$ denotes the set of all vertices [edges, respectively]. The symbol $r(G)$ $[d(G)]$ denotes the radius [diameter] of G. If G is a graph, then by $G'[G'']$ we mean an arbitrary orientation of graph G which has the smallest diameter [radius] of all its orientations.

If S is a set, then $|S|$ denotes the number of elements of the set S. Throughout the paper the letters n, k denote natural numbers. By $d(v, w)$ we mean the distance from the vertex v to the vertex w.

2. COMPLETE BIPARTITE GRAPHS

Let $V=\{v_1, \ldots, v_n\}$, $W=\{w_1, \ldots, w_k\}$ be the sets and $|V|=n\geq k=|W|$. Denote by $K(n, k)$ the graph with properties $V(K(n, k))=V\cup W$ and $E(K(n, k))=\{vw \mid v\in V, w\in W\}$. Let $\hat{K}(n, k)$ be an arbitrary orientation of $K(n, k)$. The matrix $\mathbf{A}_{n,k}$ is said to be Boolean iff for all $i\leq n$, $j\leq k$ we have $a_{ij}\in\{0, 1\}$. The Boolean matrix with property $a_{ij}=1$ iff $v_iw_j\in E(\hat{K}(n, k))$ will be called the matrix of $\hat{K}(n, k)$. We denote $d(\mathbf{A}_{n,k})=d(\hat{K}(n, k))$.

Boesch and Tindell [1] showed that $d(K'(n, n))=3$ for $n\geq 2$. Plesník proved that if $n\geq k\geq 2$, then $d(K'(n, k))\geq 4$. We shall determine the exact value of $d(K'(n, k))$ and $r(K''(n, k))$ for all n, k.

Lemma 1. For $n\geq k\geq 2$ we have

$$3=r(K''(n, k))\leq d(K'(n, k))\leq 4.$$

Proof. Let $v \in V$ (for instance) be a central vertex of $K''(n, k)$. Then there exists the vertex $w \in W$ with the property $vw \in E(K'(n, k))$ and we obtain $d(w, v) \geq 3$. On the other hand we can divide the set W into two parts W_1, W_2, which are not empty. Now we denote $U = V - \{v\}$. Let K_1 be the orientation of $K(n, k)$ and $E(K_1) = \{vw_1, w_1u, uw_2, w_2v \mid u \in U, \quad w_1 \in W_1, \quad w_2 \in W_2\}$ (see figure 1.). Then $r(K_1) = 3$.

The left inequality is obvious, the right one was proved by Plesník [2].

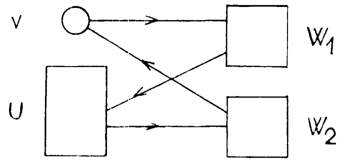

Fig 1

Definition 1. Let $\mathbf{A}_{n,k}$ be a Boolean matrix. For all $i \leq n$ we denote by $\boldsymbol{a}_i$ the k-dimensional Boolean vector $(a_{i1}, a_{i2}, \ldots, a_{ik})$, i. e. the ith row of the matrix $\mathbf{A}$. Now let

$$|\boldsymbol{a}_i| = a_{i1} + a_{i2} + \ldots + a_{ik}$$

be the length of the vector $\boldsymbol{a}_i$. For an integer $j \geq 0$ we denote the set of all k-dimensional Boolean vectors by $M(k)$ and we put $M(j, k) = \{\boldsymbol{c} \in M(k); |\boldsymbol{c}| = j\}$. By the expresion $\boldsymbol{b} \leq \boldsymbol{c}$, where $\boldsymbol{b}, \boldsymbol{c} \in M(k)$ we mean that $b_i \leq c_i$ for all $i \leq k$.

Evidently the vectors $\boldsymbol{b}$, $\boldsymbol{c}$ are incomparable iff there exist $i \leq k$, $j \leq k$ such that $b_i = 0$, $c_i = 1$, $b_j = 1$, $c_j = 0$.

Lemma 2. Let $n \geq 2$ and $\mathbf{A}_{nk}$ be the Boolean matrix of a digraph $\hat{K}(n, k)$. Then $d(\mathbf{A}) = 3$ iff every two rows and every two columns of the matrix $\mathbf{A}$ are incomparable.

Proof. Let $d(\mathbf{A}) = 3$. then $d(v_i, v_j) = 2$ in $\hat{K}(n, k)$ for every two vertices $v_i, v_j \in V$. Hence there exists a vertex $w_p \in W$ such that $a_{ip} = 1$, $a_{jp} = 0$. If we interchange i and j, we obtain that there exists a vertex $w_s \in W$ such that $a_{is} = 0$, $a_{js} = 1$. We showed that every two rows are incomparable. By interchanging V and W we can prove the same for the columns.

Let every two rows or columns be incomparable. Then $d(v_i, v_j) = d(w_p, w_r) = 2$ for every $v_i \neq v_j \in V$, $w_p \neq w_r \in W$ and there exists $m \leq k$ such that $a_{im} = 1$. Now $d(w_m, w_r)$ equals 2 or 0, hence $d(v_i, w_r) \leq 3$. Similarly we can prove that $d(w_r, v_i) \leq 3$ and using Lemma 1 we get $d(\mathbf{A}) = 3$.

Lemma 3. Let $|\boldsymbol{b}| = |\boldsymbol{c}|$ for two Boolean vectors $\boldsymbol{b}, \boldsymbol{c} \in M(k)$. Then they are incomparable iff $\boldsymbol{b} \neq \boldsymbol{c}$.

Proof is obvious.

For $k\geqslant 2$ we denote by $\mathbf{B}_{kk}$ a Boolean matrix such that for the integer $r=(j-i)(\text{mod } k)$, $r\geqslant 0$ we have

$$b_{ij}=1 \quad \text{iff} \quad r\leqslant [k/2]-1$$

(see figure 2 for $k=7$).

$$\mathrm{B}_{77}= \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Fig 2

Lemma 4. For $2\leqslant k\leqslant n\leqslant\binom{k}{[k/2]}$ we have $d(K'(n, k))=3$.

Proof. We construct the matrix $\mathbf{B}_{kn}$ by adding $n-k$ rows from the set $M([k/2], k)-\{b_1, \ldots, b_3\}$ to the matrix $\mathbf{B}_{kk}$. One can verify that $d(\mathbf{B}_{kk})=3$ (by lemmas 3, 2). Then also $d(\mathbf{B}_{kn})=3$. Hence there exists an orientation D of $K(n, k)$ such that $d(D)=3$ and from Lemma 1 $d(K(n, k))=3$ follows.

For $0\leqslant 2j<n$ we define the mapping $f(j, n)$ from $M(j, n)$ into $M(j+1, n)$ by induction. For $j=0$ and $\boldsymbol{O}_n\in M(0, n)$ we put $\boldsymbol{O}_n f(0, n)=(1, 0, \ldots, 0)\in M(1, n)$. Let $\boldsymbol{x}=(x_1, \ldots, x_n)\in M(j+1, n)$, where $n>2(j+1)$, hence $n-2>2j$ and $f(j, n-2)$ is defined. We put

$$i=i(\boldsymbol{x})=\min\{z \mid z\leqslant n,\ x_z=0,\ x_{(z+1)(\text{mod } n)}=1\}. \tag{1}$$

We denote

$$\bar{\boldsymbol{x}}=(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n) \quad \text{if } i<n \tag{2}$$

and

$$\bar{\boldsymbol{x}}=(x_2, \ldots, x_{n-1}) \qquad \text{if } i=n. \tag{3}$$

Let $\boldsymbol{r}=\bar{\boldsymbol{x}}f(j, n-2)=(r_1, \ldots, r_{n-2})$.
Let us define

$$\boldsymbol{x}f(j+1, n)=(r_1, \ldots, r_{i-1}, x_i, x_{i+1}, r_i, \ldots, r_{n-2}) \quad \text{if } i<n,$$

$$\boldsymbol{x}f(j+1, n)=(x_1, r_1, \ldots, r_{n-2}, x_n) \qquad \text{if } i=n.$$

Lemma 5. For $0\leqslant 2j<n$ $f(j, n)$ is an injection and for every vector $\boldsymbol{a}\in M(j, n)$ we have $\boldsymbol{a}\leqslant \boldsymbol{a}f(j, n)$.

Proof. We denote $f(j, n)$ by F. It is easy to verify that $\boldsymbol{a}\leqslant \boldsymbol{a}F$. If $j=0$, then F is an injection. We suppose that there exist the vectors $\boldsymbol{x}\neq \boldsymbol{y}\in M(j, n)$ such that

$xF = yF$ and $k \leqslant m$ where $k = i(x)$ and $m = i(y)$. And we can suppose that $x \leqslant xF$ for all vectors x. Now we introduce the main ideas of the proof. It can be shown that

1. $k < m$ (indirectly and by the induction hyphotesis)
2. $y_k = 0$ (from $0 = x_k = (xF)_k = (yF)_k \geqslant y_k$)
3. $y_k = y_{k+1} = \ldots = y_m = 0$ (from part 2 and from (1) for $x = y$)
4. $(yF)_{k+1} = 0 = (xF)_{k+1} = 1$.

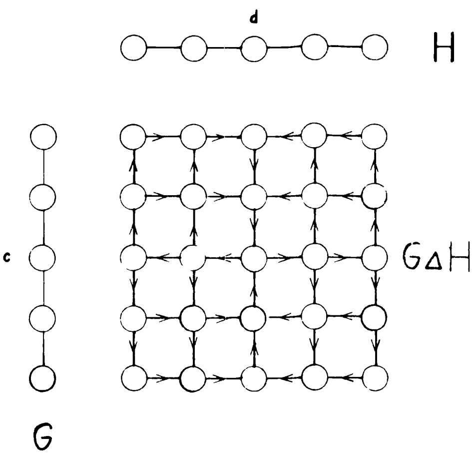

Fig. 3

To prove 4 we construct the sequence x, $\bar{x}$, $\bar{\bar{x}}$, ..., O_{n-2j}, where the next vector is obtained from the preceding one by (2) or (3), i. e. by deleting the two components. Then y_{k+1} is not the first component of O_{n-2j}, because y_k must be deleted later than y_{k+1} (it follows from 3). Hence $(yF)_{k+1} = y_{k+1} = 0$. From the definition we have $(xF)_{k+1} = x_{k+1} = 1$, which is a contradiction.

For any Boolean matrix $\mathbf{A}_{mn}$ or vector if $m = 1$ we denote $\mathbf{B} = \neg\mathbf{A}$ iff $b_{ij} = 1 - a_{ij}$ for all admissible i, j.

Lemma 6. The Boolean vectors a, b are incomparable iff $\neg a$, $\neg b$ are incomparable.

Proof is obvious.

Definition 2. Let $0 \leqslant 2j < k$, $\mathbf{A}_{nk}$ be a Boolean matrix, j be the minimal length of the rows of the matrix $\mathbf{A}_{nk}$. We will define the mappings H, h.

If $j \leqslant [k/2]$ [$j \geqslant k - [k/2]$, respectively], then $\mathbf{A}H = \mathbf{A}$[$\mathbf{A}h = \mathbf{A}$, respectively]. Otherwise for all $i \leqslant n$ we have

$$(\mathbf{A}H)_i = a_i f(j, k) \; [(\mathbf{A}h)_i = a_i f(j, k)] \quad \text{if } |a_i| = j,$$
$$(\mathbf{A}H)_i = a_i \qquad [(\mathbf{A}h)_i = a_i] \qquad \text{if } |a_i| > j.$$

Lemma 7. Let $2 \leqslant n \leqslant k$ and $\mathbf{A}_{nk}$ be a Boolean matrix. If every two rows of $\mathbf{A}$ are incomparable, then every two rows of each of the matrices $\mathbf{A}H$, $\mathbf{A}h$, $\neg\mathbf{A}$ are incomparable.

Proof. In the case of $\neg\mathbf{A}$ it follows from Lemma 6. Here we prove it in the case of $\mathbf{A}H$. For $\mathbf{A}h$ it is similar. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be two rows of $\mathbf{A}$. Now we distinguish 3 cases.
1. If $|\boldsymbol{a}| = |\boldsymbol{b}| = j$, then the proof follows from Lemma 5.
2. If $|\boldsymbol{a}| > j$, $|\boldsymbol{b}| = j$, then there exist at least two numbers $p \neq r$ such that $a_p = 1$, $b_p = b_r = 0$, $a_r = 1$ and one number t such that $a_t = 0$ and $b_t = 1$. We have $\boldsymbol{a}H = \boldsymbol{a}$ and we can make $\boldsymbol{b}H$ if we change one component of $\boldsymbol{b}$ from 0 to 1. The rest is easy.
3. If $|\boldsymbol{a}| > j$ and $|\boldsymbol{b}| > j$, then $\boldsymbol{a}H = \boldsymbol{a}$, $\boldsymbol{b}H = \boldsymbol{b}$ and the proof follows.

Theorem 1. Let $n \geqslant k$.
(a) If $k = 1$, then $d(K'(n, k)) = r(K(n, k)) = \infty$.
(b) If $k \geqslant 2$ and $n > \binom{k}{[k/2]}$, then $d(K'(n, k)) = 4$.
(c) If $k \geqslant 2$ and $n \leqslant \binom{k}{[k/2]}$, then $d(K'(n, k)) = 3$.

Proof. The part (a) is obvious.
(b): We prove it indirectly. Let $d(K'(n, k)) < 4$. From Lemma 1 we have $d(K'(n, k)) = 3$. Let $\mathbf{A}_{nk}$ be the matrix of the digraph $K'(n, k)$. Next we put $\mathbf{B}_{nk} = (\neg(\mathbf{A}h^k))H^k$. The reader can verify that every row of $\mathbf{B}$ has length $[k/2]$ and by lemmas 2,7 every two rows of $\mathbf{B}$ are incomparable. Then

$n \leqslant |M([k/2], k)| = \binom{k}{[k/2]}$, which is a contradiction.

(c): It follows from Lemma 4.

3. The product of Graphs

Definition 3. Let G, H be graphs. A graph P is said to be the product of G, H and we denote $P = G \square H$ iff $V(P) = V(G) \times V(H)$ and $(a, u)(b, v) \in E(P)$ if and only if $(a, b) \in E(G)$ and $u = v \in V(H)$ or $(u, v) \in E(H)$ and $a = b \in V(G)$.

Lemma 8. Let G, H be graphs. Then we have $r(G \square H) = r(G) + r(H)$ and $d(G \square H) = d(G) + d(H)$.
The proof is evident.

Now by $d(a, b, G)$ we denote the distance from a vertex a to the vertex b in a graph G. By a central vertex of a graph G we mean a vertex $c \in V(G)$ if we have $d(c, v, G) \leqslant r(G)$ for all $v \in V(G)$.

Definition 4. Let G, H be graphs with at least two vertices, $c(d)$ be an arbitrary central vertex of $G(H)$. Then by the symbol $G \triangle H$ we denote an arbitrary orientation of the graph $G \square H$ with following property.

Let $a, b \in V(G)$, $x, y \in V(H)$ and

$$d(a, c, G) < d(b, c, G), \ d(x, d, H) < d(y, d, H).$$

Then $(c, x)(c, y)$, $(g, y)((g, x)$, $(b, d)(a, d)$, $(a, h)(b, h) \in E(G \triangle H)$ for $g \in V(G)$, $g \neq c$ and $h \in V(H)$, $h \neq d$ (see figure 3.).

Theorem 2. Let graphs G, H contain at least two vertices and $r(G) \leqslant r(H)$. Then we have

(a) If $r(G) = 1$, then $r(G \triangle H) \leqslant r(G) + r(H) + 1$

(b) If $r(G) > 1$, then $r(G \triangle H) = r(G) + r(H) = r((G \square H)'')$.

Proof. If $r(G) = \infty$ or $r(H) = \infty$, then the theorem is true. Let us suppose that $r(G) < \infty$, $r(H) < \infty$. From lemma 8 we have $r(G \square H) = r(G) + r(H)$, hence $r(G \triangle H) \geqslant r((G \square H)'') \geqslant r(G) + r(H)$. We can verify that for $g \in V(G)$, $g \neq c$ we have

$d((g, h), (g, d), G \triangle H) \leqslant d(h, d, H) \leqslant r(H)$. And similarly in other cases.

Let $(g, h) \in V(G \triangle H)$. Now we distinguish 3 cases.

1. If $g \neq c$, then we have

$$d((g, h), (g, d), G \triangle H) \leqslant d(h, d, H) \leqslant r(H)$$
$$d((g, d), (c, d), G \triangle H) \leqslant d(g, c, G) \leqslant r(G).$$

2. If $g = c$ and $h = d$, then we have $d((g, h), (c, d), G \triangle H) = 0$.

3. If $g = c$ and $h \neq d$, then the conditions $|V(G)| > 1$ and $r(G) < \infty$ imply that there exists $g_0 \in V(G)$ such that g, g_0 are neighbours. Hence we have

$$d((g, h), (g_0, h), G \triangle H) \leqslant d(c, g_0, G) = 1$$
$$d((g_0, h), (g_0, d), G \triangle H) \leqslant d(h, d, H) \leqslant r(H)$$
$$d((g_0, d), (g, d), G \triangle H) \leqslant d(g_0, g, G) = 1.$$

We have shown that we have

$$d(a, (c, d), G \triangle H) \leqslant r(H) + \max\{2, r(G)\} \quad \text{for all } a \in V(G \triangle H). \qquad (4)$$

Now we distinguish 3 cases.

1. If $h = d$, then

$$d((c, d), (c, h), G \triangle H) \leqslant d(d, h, H) \leqslant r(H)$$
$$d((c, h), (g, h), G \triangle H) \leqslant d(c, g, G) \leqslant r(G).$$

2. If $h = d$ and $g = c$, then $d((c, d), (g, h), G \triangle H) = 0$.

3. If $h = d$ and $g \neq c$ then there exists a neighbour h_0 of d and

$$d((c, d), (c, h_0), G \triangle H) \leqslant d(d, h_0, H) = 1$$

$$d((c, h_0), (g, h_0), G \triangle H) \leq d(c, g, G) \leq r(G)$$
$$d((g, h_0), (g, h), G \triangle H) \leq d(h_0, d, H) = 1.$$

We have proved that

$$d((c, d), b, G \triangle H) \leq r(G) + \max\{2, r(H)\} \quad \text{for all } b \in V(G \triangle H). \quad (5)$$

Hence $r(G \triangle H) \leq \max\{r(H) + \max\{2, r(G)\}, r(G) + \max\{2, r(H)\}\}$. The proof follows.

Theorem 3. Let graphs G, H contain at least two vertices and $r(G) \leq r(H)$. Then we have
(a) If $r(G) = 1$, then $d(G \triangle H) \leq 2r(G) + 2r(H) + 1$
(b) If $r(G) > 1$, then $d(G \triangle H) \leq 2r(G) + 2r(H)$.

Proof.
(a): First we shall suppose that $r(H) = 1$. From the inequality $d(G \triangle H) \leq 2r(G \triangle H)$ and from the theorem 2 we get $d(G \triangle H) \leq r(H) + r(G) + 2$. Let us suppose that there exist vertices $a = (g, h)$, $b = (x, y) \in V(G \triangle H)$ such that $d(a, b, G \triangle H) = r(H) + r(G) + 2$. Then the inequalities (4), (5) change into equalities. From the proof of the theorem 2 we get the next assertion. From the equality in (4) we have $g = c$ and $h \neq d$ and from the equality in (5) we have $x \neq c$ and $y = d$. Hence $a = (c, h)$, $b = (x, d)$ and there is

$$d((c, h), (x, h), G \triangle H) \leq d(c, x, G) \leq r(G) = 1$$
$$d((x, h), (x, d), G \triangle H) \leq d(h, d, H) \leq r(H) = 1.$$

Hereby we get $d(a, b, G \triangle H) \leq 2$ and this is a contradiction.

Now let $r(H) \geq 2$, $a, b \in V(G \triangle H)$. From (4) and (5) we have

$$d(a, (c, d), G \triangle H) \leq r(H) + r(G) + 1$$
$$d((c, d), b, G \triangle H) \leq r(G) + r(H), \text{ hence}$$
$$d(a, b, G \triangle H) \leq 2r(G) + 2r(H) + 1.$$

(b): It follows from the theorem 2 and the inequality $d(G \triangle H) \leq 2r(G \triangle H)$.

Corollary. Let G, H be graphs and $1 < r(G) \leq r(H)$, $d(G) = 2r(G)$, $d(H) = 2r(H)$, then we have $d((G \square H)') = 2r(G) + 2r(H)$.

Proof. This follows from theorem 3 and lemma 8.

Remark. Theorems 2, 3 and corollary are also true if G, H are multigraphs, i. e. they can contain multiple edges.

If G, H are bipartite graphs with at least two vertices, then the inequality $d((G \square H)') \leq 1 + 2\max\{d(G), d(H)\}$ can be proved.

Cubes are a special case of the product of bipartite graphs. Plesník [2] showed that if Q_n is the graph of the n-dimensional cube, then $d(Q_n') \leq 2n - 1$ for $n \geq 2$. Now we know that $n \leq d(Q_n') \leq n + 1$ for $n \geq 4$.

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ОРИЕНТАЦИИ ГРАФОВ, МИНИМАЛИЗУЮЩИЕ РАДИУС ИЛИ ДИАМЕТР

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Резюме

В статье для всякого полного двухдольного графа найдена ориентация, которая минимализует его диаметр. Мы исследовали также ориентации продуктов двух графов.