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ON THE KURZWEIL INTEGRAL FOR FUNCTIONS WITH VALUES IN ORDERED SPACES II

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ABSTRACT. A limit theorem is formulated and proved for uniform convergent sequences of Kurzweil-Henstock integrable functions from a compact interval to a Riesz space.

The paper is a continuation of the article [5]. In the article there were presented the definitions and some elementary properties of the Kurzweil integral. This paper contains a limit theorem.

We recall that a function $f: I \rightarrow X$ ($I = (a, b) \subset \mathbb{R}$, $X$ being a boundedly $\sigma$-complete, $\sigma$-distributive linear lattice, i.e. for every bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \downarrow 0$ ($j \rightarrow \infty$, $i = 1, 2, \ldots$) it is $\bigwedge_{\varphi \in \mathbb{N}} \bigvee_{i} a_{i\varphi(i)} = 0$), is called integrable (in the Kurzweil sense) if there exist $x \in X$ and a bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \downarrow 0$ ($j \rightarrow \infty$, $i = 1, 2, \ldots$) and for every $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ there exists $\sigma: I \rightarrow (0, \infty)$ such that for every $D \in A(\sigma)$

$$|x - S(f, D)| < \bigvee_{i} a_{i\varphi(i)}.$$  

Here $A(\sigma)$ consists of all decompositions $D$ of $I$ such that $D = \{(J_1, t_1), (J_2, t_2), \ldots, (J_n, t_n)\}$, where $J_i \subset (t_i - \sigma(t_i), t_i + \sigma(t_i))$, and $S(f, D) = \sum_{i=1}^{n} f(t_i)m(J_i)$, where $m(J_i)$ is the measure of the interval $J_i$, is the integral sum.

If $x_n, x \in X$, then $x_n \rightarrow x$ (with respect to the ordering) if and only if there exist $a_n \in X$, $a_n \downarrow 0$ and $|x_n - x| \leq a_n$ for all $n$.

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It is possible to prove that a sequence \((x_n)_n \subset X\) converges to \(x \in X\) if and only if \((x_n)_n\) is bounded and

\[
x = \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} x_i = \bigvee_{n=1}^{\infty} \bigwedge_{i=n}^{\infty} x_i.
\]

We say that \(f_n \to f\) uniformly \((f_n, f : I \to X)\) if and only if there exist \(a_n \in X, a_n \searrow 0\) such that

\[
|f_n(t) - f(t)| \leq a_n
\]

for every \(t \in I\) and every \(n\).

**Lemma 1.** If \(f_n : I \to X\) is integrable for \(n = 1, 2, \ldots\), \(f_n \to f\) uniformly and \(f\) is bounded, then \(\lim_{n \to \infty} \int f_n \, dm\) exists.

**Proof.** It is sufficient to show that the sequence \((\int f_n \, dm)_n\) is bounded and

\[
\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \int f_i \, dm \leq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} \int f_j \, dm.
\]

The function \(f\) is bounded, then there exists \(h \in X, h > 0\) such that \(|f(t)| \leq h\) for all \(t \in I\).

If \(f_n \to f\) uniformly, then there exists a sequence \((a_n)_n \subset X, a_n \to 0\) \((n \to \infty)\) and for any \(t \in I\)

\[
|f_n(t) - f(t)| \leq a_n
\]

for all \(n\). Hence

\[-h - a_1 \leq f_n(t) \leq h + a_1 \quad \text{and} \quad -2a_n \leq f_i(t) - f_j(t) \leq 2a_n\]

for any \(t \in I\) and \(i, j \geq n\). It is evident that if for \(f : I \to X, f(t) = a\) for all \(t \in I\), then \(\int f \, dm = am(I)\). By Theorems 5 and 6 in [5] for any \(n\) we have

\[
(-h - a_1)m(I) \leq \int f_n \, dm \leq (h + a_1)m(I)
\]

and

\[-2a_nm(I) \leq \int (f_i - f_j) \, dm = \int f_i \, dm - \int f_j \, dm \leq 2a_nm(I)\]
for $i,j \geq n$.

Then the sequence $\left( \int f_n \, dm \right)_n$ is bounded and

$$\bigvee_{i=n}^{\infty} \int f_i \, dm \leq \bigwedge_{j=n}^{\infty} \int f_j \, dm + 2a_n m(I)$$

for all $n$ and hence

$$\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} f_i \, dm \leq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} f_j \, dm.$$

**Theorem 2.** Let $f_n : I \to X$ be integrable for $n = 1, 2, \ldots$, $f_n \to f$ uniformly and $f$ be bounded. Then $f$ is integrable and $\int f \, dm = \lim_{n \to \infty} \int f_n \, dm$.

**Proof.** By Lemma 1 $\lim_{n \to \infty} \int f_n \, dm = c$ exists and hence there exists a sequence $(c_n)_n \subset X$, $c_n \searrow 0 \ (n \to \infty)$ and

$$\left| \int f_n \, dm - c \right| \leq c_n.$$

for any $n$.

The function $f_n$ is integrable and then there exists a bounded double sequence $(a_{nij})_{i,j} \subset X$ such that $a_{nij} \searrow 0 \ (j \to \infty, \ i,n = 1,2,\ldots)$ and for every $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $\sigma_n : I \to (0, \infty)$ such that for every $D \in A(\sigma_n)$

$$\left| \int f_n \, dm - S(f_n, D) \right| < \bigvee_i a_{nij}(i+n+1).$$

When $f_n \to f$ uniformly, then there exists a sequence $(b_n)_n \subset X$, $b_n \searrow 0$ and $|f_n(t) - f(t)| \leq b_n$ for any $t \in I$ and all $n$.

Let $\varphi \in \mathbb{N}^N$. Put $k = \min_j \varphi(j+1)$ and take $D \in A(\sigma_k)$, $D = \{(J_1, t_1), (J_2, t_2), \ldots, (J_r, t_r)\}$.

Then

$$|S(f, D) - c| \leq |S(f, D) - S(f_k, D)| + |S(f_k, D) - \int f_k \, dm| + |\int f_k \, dm - c|$$

$$< \sum_{i=1}^{r} |f(t_i) - f_k(t_i)| m(J_i) + \bigvee_i a_{k\chi \varphi(i+k+1)} + c_k$$

$$\leq b_k \sum_{i=1}^{r} m(J_i) + \bigvee_i a_{k\chi \varphi(i+k+1)} + c_k$$

$$= b_k m(I) + c_k + \bigvee_i a_{k\chi \varphi(i+k+1)} = d_k + \bigvee_i a_{k\chi \varphi(i+k+1)},$$

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where \( d_j = b_j m(I) + c_j \) for \( j = 1, 2, \ldots \), \( d_j \downarrow 0 \ (j \to \infty) \), \( d_k = d_{\min \varphi(j+1)} \) \( = \bigvee_i d_{\varphi(i+1)} \). Put \( b_{1ij} = d_j \) for \( i, j = 1, 2, \ldots \) and \( b_{n+1ij} = a_{nij} \) for \( n, i, j = 1, 2, \ldots \).

Now

\[
|S(f, D) - c| < \bigvee_i d_{\varphi(i+1)} + \bigvee_i a_{k+i\varphi(i+k+1)}
\]

\[
= \bigvee_i b_{1i\varphi(i+1)} + \bigvee_i b_{k+i1\varphi(i+k+1)}
\]

\[
\leq \sum_{n=1}^{\infty} \bigvee_i b_{n+i\varphi(i+n)}.
\]

There exists \( h \in X, \ h > 0 \) such that \( |f(t)| \leq h \) for any \( t \in I \), since \( f \) is bounded. Then

\[
|S(f, D) - c| \leq h \cdot m(I) + |c| = a,
\]

when \( a \in X, \ a > 0 \) and

\[
|S(f, D) - c| \leq a \wedge \left( \sum_{n=1}^{\infty} \bigvee_i b_{n+i\varphi(i+n)} \right).
\]

By Lemma 2 in [7] there exists a bounded double sequence \((a_{ij})_{i,j} \subset X\), \( a_{ij} \downarrow 0 \ (j \to \infty, \ i = 1, 2, \ldots) \) and

\[
a \wedge \left( \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} b_{n+i\varphi(i+n)} \right) \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.
\]

Therefore there exists \( c \in X, \ c = \lim_{n \to \infty} \int f_n \ dm \) and the sequence \((a_{ij})_{i,j} \subset X\), \( a_{ij} \downarrow 0 \ (j \to \infty, \ i = 1, 2, \ldots) \) and for every \( \varphi \in \mathbb{N}^\mathbb{N} \) there exists \( \sigma: I \to (0, \infty) \ (\sigma = \sigma_{\min \varphi(j+1)}) \) such that

\[
|S(f, D) - c| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}
\]

for any \( D \in A(\sigma) \). Hence \( f \) is integrable and

\[
\int f \ dm = \lim_{n \to \infty} \int f_n \ dm.
\]
ON THE KURZWEIL INTEGRAL ...

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