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ON TOTALLY SLA-SIMPLE SEMIGROUPS

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ABSTRACT. For any subsemigroup $\boldsymbol{A}$ of the semigroup $\boldsymbol{S}$ a maximum subsemigroup $\mathcal{C}(\boldsymbol{A})$ of $\boldsymbol{S}$ in the class of all subsemigroups $\boldsymbol{B}$ of $\boldsymbol{S}$ fulfilling the property "$\boldsymbol{A}$ is an SLA-ideal of $\boldsymbol{B}$" is defined and its properties are studied.

Let $\boldsymbol{S}$ be a semigroup. In the present paper, the set of all subsemigroups of $\boldsymbol{S}$ is denoted by $\mathcal{P}(\boldsymbol{S})$. A *semigroup left almost ideal* (*SLA-ideal*) of a semigroup $\boldsymbol{S}$ is $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$ such that $s\boldsymbol{A} \cap \boldsymbol{A} \neq \emptyset$ for any $s \in \boldsymbol{S}$. This notion was defined, and its basic properties are studied in [1]. We consider a following problem. Let $\boldsymbol{S}$ be a semigroup, and $\boldsymbol{A}$ be its subsemigroup. Let $\boldsymbol{B}$ be such subsemigroup of $\boldsymbol{S}$ that $\boldsymbol{A}$ is an SLA-ideal of $\boldsymbol{B}$. Does there exist a maximum subsemigroup in the class of all subsemigroups $\boldsymbol{B}$ with the foregoing property? First of all (Theorem 1), we show that such semigroup exists, and it is uniquely determined. It will be denoted by $\mathcal{C}(\boldsymbol{A})$.

The main purpose of this paper is to study the properties of $\mathcal{C}(\boldsymbol{A})$. Namely, we will answer the following questions:

1) Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}(\boldsymbol{S})$, and $\boldsymbol{A} \subseteq \boldsymbol{B}$. Does $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{B})$ hold true?
2) What semigroups have the property that $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{S}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$?
3) What semigroups have the property that $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$?

We start with Theorem 1, which provides a possibility to define $\mathcal{C}(\boldsymbol{A})$.

THEOREM 1. *Let $\boldsymbol{S}$ be a semigroup, and $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$. Then there exists a maximum subsemigroup in the class of all subsemigroups $\boldsymbol{B}$ of $\boldsymbol{S}$ with property "$\boldsymbol{A}$ is an SLA-ideal of $\boldsymbol{B}$".*

Proof. Let $\{\boldsymbol{B}_i \mid i \in I\}$ be the set of all subsemigroups of $\boldsymbol{S}$ such that $\boldsymbol{A}$ is an SLA-ideal of $\boldsymbol{B}_i$. Let us consider the set $\left(\bigcup\{\boldsymbol{B}_i \mid i \in I\}\right)^+$, where $\boldsymbol{X}^+$ means the subsemigroup of $\boldsymbol{S}$ generated by $\boldsymbol{X}$. Let $x \in \left(\bigcup\{\boldsymbol{B}_i \mid i \in I\}\right)^+$. Then $x = b_1 b_2 \dots b_k$ with $b_i \in \boldsymbol{B}_{j_i}$. Since $\boldsymbol{A}$ is an SLA-ideal in $\boldsymbol{B}_{j_k}$, to $b_k \in \boldsymbol{B}_{j_k}$

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there exists $c_k \in \boldsymbol{A}$ such that $b_k c_k = a_k \in \boldsymbol{A}$. Therefore $x c_k = b_1 \dots b_{k-1} b_k c_k = b_1 \dots b_{k-1} a_k$. The element $b_{k-1} a_k$ is from $\boldsymbol{B}_{j_{k-1}}$. Thus, to $b_{k-1} a_k$ there exists $c_{k-1} \in \boldsymbol{A}$ such that $(b_{k-1} a_k) c_{k-1} = a_{k-1} \in \boldsymbol{A}$. We can consider the element $x c_k c_{k-1} = b_1 \dots b_{k-1} a_k c_{k-1} = b_1 \dots b_{k-2} a_{k-1}$ and repeat the foregoing process. At the end we have: To $x = b_1 \dots b_k$ there exist $c_1, \dots, c_k \in \boldsymbol{A}$ such that $x c_k c_{k-1} \dots c_1 = b_1 \dots b_k c_k \dots c_1 = a_1 \in \boldsymbol{A}$. Hence, to any $x \in \left(\bigcup\{\boldsymbol{B}_i \mid i \in I\}\right)^+$ there exists $c = c_k \dots c_1 \in \boldsymbol{A}$ such that $xc \in \boldsymbol{A}$. As $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$, $\boldsymbol{A}$ is an SLA-ideal of $\left(\bigcup\{\boldsymbol{B}_i \mid i \in I\}\right)^+$.

Obviously, $\left(\bigcup\{\boldsymbol{B}_i \mid i \in I\}\right)^+$ is the maximum subsemigroup of $\boldsymbol{S}$ in which $\boldsymbol{A}$ is an SLA-ideal, and the proof is complete. □

DEFINITION 1. The above subsemigroup $\left(\bigcup\{\boldsymbol{B}_i \mid i \in I\}\right)^+$ is called an *SLA-cover* of a semigroup $\boldsymbol{A}$ and is denoted by $\mathcal{C}(\boldsymbol{A})$.

THEOREM 2. *Let $\boldsymbol{S}$ be a semigroup, and $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$. Then $\mathcal{C}(\boldsymbol{A}) = \mathcal{C}(\mathcal{C}(\boldsymbol{A}))$.*

Proof. By Definition 1, $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\mathcal{C}(\boldsymbol{A}))$. Let $c \in \mathcal{C}(\mathcal{C}(\boldsymbol{A}))$. Then there exists $b \in \mathcal{C}(\boldsymbol{A})$ such that $cb \in \mathcal{C}(\boldsymbol{A})$. Since $\boldsymbol{A}$ is an SLA-ideal of $\mathcal{C}(\boldsymbol{A})$, there exists $a \in \boldsymbol{A}$ such that $ba \in \boldsymbol{A}$ and $(cb)a \in \mathcal{C}(\boldsymbol{A})$. To $cba \in \mathcal{C}(\boldsymbol{A})$ there exists $a' \in \boldsymbol{A}$ such that $(cba)a' \in \boldsymbol{A}$. Thus, to any $c \in \mathcal{C}(\mathcal{C}(\boldsymbol{A}))$ there exists $(ba)a' \in \boldsymbol{A}$ such that $c(baa') \in \boldsymbol{A}$. Hence, $\boldsymbol{A}$ is an SLA-ideal of $\mathcal{C}(\mathcal{C}(\boldsymbol{A}))$, and $\mathcal{C}(\mathcal{C}(\boldsymbol{A})) \subseteq \mathcal{C}(\boldsymbol{A})$, which proves the theorem. □

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}(\boldsymbol{S})$. In the following example, we show that the condition $\boldsymbol{A} \subseteq \boldsymbol{B}$ does not necessarily imply $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{B})$.

EXAMPLE. Let $\boldsymbol{X} = \{a, b\}$ and $\boldsymbol{S} = \boldsymbol{X}^+$. Let $\boldsymbol{A} = \left((ab)^2 \cup (ab)^3\right)^+$ and $\boldsymbol{B} = \left((ab)^2 \cup (ab)^3 \cup a\right)^+$. Then $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{A} = \{(ab)^n \mid n \geq 2\}$. Clearly, a is a prefix of any element of $\boldsymbol{B}$, and $\boldsymbol{B}$ does not contain an element with $abaa$ as its prefix.

First we describe the structure of the semigroup $\mathcal{C}(\boldsymbol{A})$. For any $x \in \mathcal{C}(\boldsymbol{A})$ there exist integers m, n such that $n > m \geq 2$ and $x(ab)^m = (ab)^n$. It holds if and only if $x = (ab)^{n-m}$ with $n - m \geq 1$. Since $\{(ab)^n \mid n \geq 1\} \in \mathcal{P}(\boldsymbol{S})$, $\mathcal{C}(\boldsymbol{A}) = \{(ab)^n \mid n \geq 1\}$.

Suppose $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{B})$. Then $a, ab \in \mathcal{C}(\boldsymbol{B})$ and consequently, $aba \in \mathcal{C}(\boldsymbol{B})$. To $aba \in \mathcal{C}(\boldsymbol{B})$ there exists $x \in \boldsymbol{B}$ such that $(aba)x \in \boldsymbol{B}$. But a is a prefix of any $x \in \boldsymbol{B}$. Then $\boldsymbol{B}$ contains an element $abax$ with $abaa$ as a prefix. This contradicts our assumption. Thus $\mathcal{C}(\boldsymbol{A})$ is not a subsemigroup of $\mathcal{C}(\boldsymbol{B})$, and we have a negative answer to the first question.

Now we describe all semigroups $\boldsymbol{S}$ with a property: For any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$, $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{S}$. It means that any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$ is an SLA-ideal of $\boldsymbol{S}$.

DEFINITION 2. Let $\boldsymbol{S}$ be a semigroup, and $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{S}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$. Then $\boldsymbol{S}$ is called an *SLA-universal semigroup.*

The next theorem gives a characterization of SLA-universal semigroups. One of the crucial notions in that characterization is the notion of combinatorial semigroup introduced in [2]. (A semigroup $\boldsymbol{S}$ is called *combinatorial* if for any $s \in \boldsymbol{S}$ there exists a positive integer n such that $s^n = s^{n+1}$.)

THEOREM 3. *A semigroup $\boldsymbol{S}$ is SLA-universal if and only if it is combinatorial, and each idempotent of $\boldsymbol{S}$ is a right zero of $\boldsymbol{S}$.*

P r o o f . Let $\boldsymbol{S}$ be an SLA-universal semigroup, $a \in \boldsymbol{S}$ is an arbitrary element of $\boldsymbol{S}$, and $\langle a \rangle$ be a cyclic subsemigroup of $\boldsymbol{S}$ generated by a. Then $\langle a \rangle$ is an SLA-ideal of $\boldsymbol{S}$. Therefore, for any $s \in \boldsymbol{S}$, $s\langle a \rangle \cap \langle a \rangle \neq \emptyset$. Elements with this property are called *right quasi-zeros* of $\boldsymbol{S}$ (see [2]). Hence, if $\boldsymbol{S}$ is an SLA-universal semigroup, then $\boldsymbol{S}$ consists entirely of right quasi-zeros.

Conversely, let any $a \in \boldsymbol{S}$ be a right quasi-zero of $\boldsymbol{S}$. Then for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$, we can consider an arbitrary element $a \in \boldsymbol{A}$, and $\langle a \rangle \subseteq \boldsymbol{A}$. The cyclic subsemigroup $\langle a \rangle$ of $\boldsymbol{A}$ is an SLA-ideal of $\boldsymbol{S}$. Then $\boldsymbol{A}$ is also an SLA-ideal of $\boldsymbol{S}$. (Any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$ containing an SLA-ideal of $\boldsymbol{S}$ is also an SLA-ideal of $\boldsymbol{S}$.) Therefore S is an SLA-universal semigroup.

We proved: S is an SLA-universal semigroup if and only if any $a \in \boldsymbol{S}$ is a right quasi-zero of $\boldsymbol{S}$. However, with respect to [2; Corollary 3.14], any $a \in \boldsymbol{S}$ is a right quasi-zero of $\boldsymbol{S}$ if and only if $\boldsymbol{S}$ is combinatorial, and each idempotent of $\boldsymbol{S}$ is a right zero of $\boldsymbol{S}$. This completes the proof. $\square$

At the end of the paper, we consider semigroups fulfilling the property that $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$. A notion of an SLA-simple semigroup was introduced in [1]. A semigroup $\boldsymbol{S}$ is *SLA-simple* if $\boldsymbol{S}$ does not contain a proper SLA-ideal. Obviously, if $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$, then any $\boldsymbol{B} \in \mathcal{P}(\boldsymbol{S})$ is an SLA-simple semigroup.

DEFINITION 3. Let $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$. Then the semigroup $\boldsymbol{S}$ is said to be a *totally SLA-simple semigroup.*

LEMMA 1. *Let $\boldsymbol{S}$ be totally SLA-simple, and $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$. Then $\boldsymbol{A}$ is a left simple semigroup.*

P r o o f . If $\boldsymbol{A}$ is not left simple, then there exists a proper left ideal L of $\boldsymbol{A}$. Since $\boldsymbol{S}$ is totally SLA-simple, $\mathcal{C}(L) = L$. But L is also a proper SLA-ideal of $\boldsymbol{A}$. Thus $\boldsymbol{A} \subseteq \mathcal{C}(L) = L$, which is a contradiction. $\square$

LEMMA 2. *Let $\boldsymbol{S}$ be a totally SLA-simple semigroup, and $s \in \boldsymbol{S}$. Then $\langle s \rangle$ is a finite group.*

P r o o f . The semigroup $\langle s \rangle$ is a left simple commutative semigroup. Therefore $\langle s \rangle$ is a group. As an infinite cyclic semigroup $\langle s \rangle$ is not left simple, $\langle s \rangle$ is a finite group. □

It is known that a semigroup $\boldsymbol{S}$ is a left group if and only if $\boldsymbol{S}$ is left simple and contains an idempotent. In such a case, $\boldsymbol{S}$ can be written in the form $\boldsymbol{S} = \boldsymbol{EG}$, where $\boldsymbol{G}$ is a maximal group of $\boldsymbol{S}$, and $\boldsymbol{E}$ is a left zero semigroup of all idempotents of $\boldsymbol{S}$. A left group $\boldsymbol{S} = \boldsymbol{EG}$ is a union of disjoint groups $e\boldsymbol{G}$ for $e \in \boldsymbol{E}$. Any $e \in \boldsymbol{E}$ is a right identity element of $\boldsymbol{S}$.

THEOREM 4. *A semigroup $\boldsymbol{S}$ is totally SLA-simple if and only if $\boldsymbol{S}$ is a left group $\boldsymbol{EG}$ such that $\boldsymbol{G}$ is a periodic group.*

P r o o f .

Necessity: Let $\boldsymbol{S}$ be totally SLA-simple. By Lemma 1 and 2, $\boldsymbol{S}$ is a left group $\boldsymbol{EG}$, and $\boldsymbol{G}$ is a periodic group.

Sufficiency: Let $\boldsymbol{S} = \boldsymbol{EG}$ be a left group, and $\boldsymbol{G}$ be a periodic group. It is known (see [1; Corollary 1]) that any periodic group does not contain a proper SLA-ideal. Since any subsemigroup of a periodic group is a periodic group, $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{G})$. Let $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$ and $\tilde{\boldsymbol{E}} = \{e \in \boldsymbol{E} \mid \boldsymbol{A} \cap e\boldsymbol{G} \neq \emptyset\}$. Then, for any $e \in \tilde{\boldsymbol{E}}$, $\boldsymbol{A} \cap e\boldsymbol{G}$ is a subsemigroup of a periodic group $e\boldsymbol{G}$. Hence it is a group, and e is its identity element. Therefore $\tilde{\boldsymbol{E}} \subseteq \boldsymbol{A}$. Now we consider $\boldsymbol{B} \in \mathcal{P}(\boldsymbol{S})$ such that $\boldsymbol{A}$ is an SLA-ideal of $\boldsymbol{B}$. Let $e \notin \tilde{\boldsymbol{E}}$ and $\boldsymbol{B} \cap e\boldsymbol{G} \neq \emptyset$. Then for any $b \in \boldsymbol{B} \cap e\boldsymbol{G}$ and any $a \in \boldsymbol{A}$, $ba \in e\boldsymbol{G}$. But $e\boldsymbol{G} \cap \boldsymbol{A} = \emptyset$. For this reason, to $b \in \boldsymbol{B} \cap e\boldsymbol{G}$ does not exist $a \in \boldsymbol{A}$ such that $ba \in \boldsymbol{A}$. Consequently, $\boldsymbol{B} \cap e\boldsymbol{G} \neq \emptyset$ if and only if $e \in \tilde{\boldsymbol{E}}$. Thus, $\boldsymbol{B} \cap e\boldsymbol{G} \neq \emptyset$ if and only if $\boldsymbol{A} \cap e\boldsymbol{G} \neq \emptyset$.

Let $\boldsymbol{A}$ be an SLA-ideal of $\boldsymbol{B}$ and $e \in \tilde{\boldsymbol{E}}$. Then to any $b \in \boldsymbol{B} \cap e\boldsymbol{G}$ there exists $a \in \boldsymbol{A}$ such that $ba \in \boldsymbol{A}$. Since $b \in e\boldsymbol{G}$, we have $ba \in \boldsymbol{A} \cap e\boldsymbol{G}$. On account of $\tilde{\boldsymbol{E}} \subseteq \boldsymbol{A}$, $ea \in \boldsymbol{A}$ for any $a \in \boldsymbol{A}$. Therefore, to any $b \in \boldsymbol{B} \cap e\boldsymbol{G}$ there exists $ea \in \boldsymbol{A} \cap e\boldsymbol{G}$ such that $ba = bea \in \boldsymbol{A} \cap e\boldsymbol{G}$. From the above, it follows that $\boldsymbol{A} \cap e\boldsymbol{G}$ is an SLA-ideal of $\boldsymbol{B} \cap e\boldsymbol{G}$ for any $e \in \tilde{\boldsymbol{E}}$. But $\boldsymbol{A} \cap e\boldsymbol{G}$ and $\boldsymbol{B} \cap e\boldsymbol{G}$ are periodic groups. For this reason, $\boldsymbol{A} \cap e\boldsymbol{G} = \boldsymbol{B} \cap e\boldsymbol{G}$ for any $e \in \tilde{\boldsymbol{E}}$. In this way, $\boldsymbol{A} = \boldsymbol{B}$ and $\mathcal{C}(\boldsymbol{A}) = \boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{P}(\boldsymbol{S})$, and the proof is complete. □

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