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VOLTERRA KERNEL OPERATORS ON BANACH FUNCTION SPACES

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ABSTRACT. Let L be a Banach function space on a measurable space (X, μ). A kernel operator K on L with a kernel k is called a Volterra kernel operator if it is an operator of finite double norm and if there exists a measurable function $p\colon X \to \mathbb{R}$ such that $k(x, y) = 0$ for almost all $(x, y) \in X \times X$ with $p(x) \le p(y)$. It is shown that every Volterra kernel operator is quasinilpotent provided L and its associate space L' have order continuous norms.

Let μ be a positive σ-finite measure on a σ-algebra Σ of subsets of a set X. Let $L_0 = L_0(X, \mu)$ be the vector space of all equivalence classes of real μ-measurable functions on X. A Banach space $L \subset L_0$ is a *Banach function space* if the norm ρ on L has the additional property that if $f \in L$, $g \in L_0$, and $|g| \le |f|$, then $g \in L$ and $\rho(g) \le \rho(f)$. Here $f \le g$ with $f, g \in L_0$ means $f(x) \le g(x)$ for almost all $x \in X$. If $f \in L_0$ and $f \notin L$, then we set $\rho(f) = \infty$. Observe that $\rho(|f|) = \rho(f)$ for all $f \in L$. The norm ρ is *σ-order continuous* if $\rho(f_n) \downarrow 0$ for any decreasing sequence $f_n \downarrow 0$ in L, and it is *order continuous* if $\rho(f_\tau) \downarrow 0$ for any downwards directed system $f_\tau \downarrow 0$ in L. Since L is Dedekind σ-complete, these two notions coincide [5; Theorem 103.9]. The *carrier* of L is assumed to be the set X, i.e., if every function of L vanishes on a set $E \in \Sigma$, then $\mu(E) = 0$.

Let L' be the *associate space* of all $g \in L_0$ such that

$$\phi(f) = \int_X f(x) g(x) \, \mathrm{d}\mu(x)$$

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defines a bounded linear functional ϕ on L. The space L' is also a Banach function space with the associate norm ρ' defined by

$$\rho'(g) = \sup \left\{ \int_X |f(x)\, g(x)| \, \mathrm{d}\mu(x) : \ \rho(f) \le 1 \right\} ,$$

and it may be considered as a closed subspace of the dual Banach lattice L^*. For basic theory concerning Banach function spaces, we refer to the books of Zaanen [3] and [5].

A linear operator K on L is called a *kernel operator* if there exists a $\mu \times \mu$-measurable function $k(x,y)$ on $X \times X$ such that

$$\int_X |k(x,y) f(y)| \, \mathrm{d}\mu(y) < \infty \quad \text{a.e. for all } f \in L , \qquad \text{and}$$

$$(Kf)(x) = \int_X k(x,y) f(y) \, \mathrm{d}\mu(y) \quad \text{a.e. for every } f \in L .$$

A kernel operator K with a kernel k is called an *operator of finite double norm* (or a Hille-Tamarkin operator) if

(i) for almost every $x \in X$ the function $k_x \in L_0$ defined by $k_x(y) = k(x,y)$ is an element of L', i.e., $\rho'(k_x) < \infty$ for almost every $x \in X$,

(ii) the function $t \in L_0$ defined by $t(x) = \rho'(k_x)$ is an element of L, i.e.,

$$|\!|\!| K |\!|\!| = \rho(t) < \infty .$$

Note that $t(x)$ is a μ-measurable function on X by the result of Luxemburg (see [5; Corollary 99.3]). If L and L' have order continuous norms, then operators of finite double norm are compact (see [2; Theorem 2.3]).

Throughout the paper, the operator norm is denoted by $\| \cdot \|$.

PROPOSITION. *Let K be a kernel operator on L of finite double norm. Then K is bounded and $\|K\| \le |\!|\!| K |\!|\!|$.*

Proof. For any function $f \in L$ and for almost all $x \in X$ we have

$$|(Kf)(x)| \le \int_X |k(x,y)\, f(y)| \, \mathrm{d}\mu(y) \le \rho'(k_x)\rho(f) = t(x)\rho(f) ,$$

so that $\rho(Kf) \le |\!|\!| K |\!|\!| \rho(f)$, and finally $\|K\| \le |\!|\!| K |\!|\!|$. □

Let us now introduce Volterra kernels on $(X \times X, \mu \times \mu)$. Let K be a kernel operator on L of finite double norm with a kernel k. If there exists a measurable function $p \colon X \to \mathbb{R}$ such that $k(x,y) = 0$ for almost all $(x,y) \in X \times X$ with

$p(x) \le p(y)$, then the kernel k is called a *Volterra kernel.* Clearly, we may (and we do) assume that the function p maps X into the interval $[0,1]$ composing p by, for example, the function $q(x) = 1/2 + (1/\pi)\operatorname{arctg} x$ if necessary. The corresponding Volterra kernel operator is therefore defined by the equation

$$(Kf)(x) = \int_{D_x} k(x,y)f(y)\,\mathrm{d}\mu(y)\,,$$

where $D_x = \{y \in X : \; p(x) > p(y)\}$. The well-known result states that any Volterra kernel operator K on $L^2[0,1]$ (with $p(x) = x$, $0 \le x \le 1$) is quasinilpotent, i.e., the spectral radius $r(K)$ is equal to 0 (see Halmos [1; Problem 147]). We now extend this result.

THEOREM. *Let L be a Banach function space on a measurable space (X, Σ, μ) such that the norms of L and L' are order continuous. Let K be a Volterra kernel operator on L with a kernel k. Then K is quasinilpotent.*

As in the book of Halmos [1; Solution 147] it is convenient to begin the proof of Theorem with the following lemma, which may be of some independent interest.

LEMMA. *Under the assumptions of Theorem, let ε be a positive number. Then there exist Volterra kernel operators A and B on L, and there exists $m \in \mathbb{N}$ such that:*

(1) $K = A + B$;
(2) $\|A\| < \varepsilon$;
(3) *every product of A's and B's in which more than m factors are equal to B is equal to* 0.

Proof. Put $E(\delta) = \{(x,y) \in X \times X : \; p(x) - p(y) \le \delta\}$ when $\delta \in (0,\infty)$. The function $k_n = k \cdot \chi_{E(1/n)}$ is the kernel of the Volterra kernel operator K_n on L. Here χ_A denotes the characteristic function of a set A. For almost all $x \in X$ the decreasing sequence $\{|(k_n)_x|\}_{n\in\mathrm{N}}$, where $(k_n)_x(y) = k_n(x,y)$ for $y \in X$, is a sequence in L' and it converges to 0. By $t_n(x) = \rho'\big((k_n)_x\big)$ we define the decreasing sequence $\{t_n\}_{n\in\mathrm{N}}$ of functions of L converging to 0, since the norm ρ' is order continuous. It follows that $|\!|\!| K_n |\!|\!| = \rho(t_n)$ also converges to 0 as $n \to \infty$, because ρ is order continuous. Finally, the inequality $\|K_n\| \le |\!|\!| K_n |\!|\!|$ implies that the sequence $\|K_n\|$ converges to 0 as well. Now, fix $m \in \mathbb{N}$ such that $\|K_m\| < \varepsilon$. Setting $A = K_m$ and $B = K - K_m$, the conditions (1) and (2) are clearly satisfied. By some simple considerations, one can show that if C is a kernel operator on L whose kernel vanishes on $E(\delta)$, $\delta \in [0,1]$, then the kernels of AC and BC vanish on $E(\delta)$ and $E(\delta + 1/m)$ respectively. It follows that the kernel of a product of A's and B's vanishes as soon as more than m factors are equal to B. This proves (3) and finishes the proof. □

The assertion of Theorem now follows from Lemma. Although the proof is the same as in [1; Solution 147], we give it here for the sake of completeness.

P r o o f o f T h e o r e m . From $K^n = (A+B)^n$ it follows that for any $n > m$

$$\|K^n\| \le \sum_{i=0}^{m} \binom{n}{i} \varepsilon^{n-i} \|B\|^i .$$

Using an obvious estimate $\binom{n}{i} \le n^m$ for $0 \le i \le m$, we obtain

$$\|K^n\|^{1/n} \le \varepsilon \cdot n^{m/n} \cdot \left(\sum_{i=0}^{m} \varepsilon^{-i} \|B\|^i \right)^{1/n} ,$$

so that

$$r(K) = \lim_{n \to \infty} \|K^n\|^{1/n} \le \varepsilon .$$

This implies the desired conclusion that $r(K) = 0$. □

Remarks.

1. The assertion of Theorem also holds if it is assumed only that some power of K is a Volterra kernel operator.

2. The following example shows that Theorem does not hold if the norm of L is not order continuous. Let S be an operator on l^∞ defined by $S(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. It is easy to see that S is a Volterra kernel operator with $\| S \| = r(S) = 1$. For a "continuous" example see also [4; p. 503].

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