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ALGEBRAS WITH PRINCIPAL TOLERANCESS

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An algebra A has principal congruences or A is *congruence principal* if every compact element of *Con* A is a principal congruence, in other words, if for any elements a_i, b_i of A, $i = 1, \dots, n$ there exist elements a, b of A such that

$$\Theta(a_1, b_1) \vee \dots \vee \Theta(a_n, b_n) = \Theta(a, b)$$

in the congruence lattice *Con* A. A variety $\mathcal{V}$ is congruence principal if each $A \in \mathcal{V}$ has this property. Such varieties were characterized in [3], [7], [8].

Like numerous other concepts, this one can also be transferred for tolerances. By a *tolerance* on an algebra A is meant a reflexive and symmetrical binary relation on A having the substitution property with respect to all operations of A. Clearly every congruence on A is a tolerance on A but not vice versa. As it was proven in [5], the set of all tolerances on an algebra A forms an algebraic lattice $LT(A)$ with respect to set inclusion. Hence, for every two elements a, b of A there exists the least tolerance $T(a, b)$ containing the pair $\langle a, b \rangle$, the so called *principal tolerance*. Such concepts were studied in [1], [4], [6]. Therefore, we can introduce the following concept for tolerances:

Definition 1. *An algebra A is tolerance principal if for each $a_i, b_i \in A$, $i = 1, \dots, n$ there exist $a, b \in A$ such that*

$$T(a_1, b_1) \vee \dots \vee T(a_n, b_n) = T(a, b)$$

in $LT(A)$. A variety $\mathcal{V}$ is tolerance principal if each $A \in \mathcal{V}$ has this property.

Lemma. (Lemma 2 in [2]). *Let a_i, b_i $(i = 1, \dots, n)$ be elements of an algebra A. Then*

$$\langle x, y \rangle \in \bigvee \{T(a_i, b_i);\ i = 1, \dots, n\}$$

if and only if there exists a $2n$-ary algebraic function φ over A such that

$$x = \varphi(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$$
$$y = \varphi(b_1, a_1, b_2, a_2, \dots, b_n, a_n)$$

Theorem 1. *Let $\mathcal{V}$ be a variety of algebras. The followiing conditions are equivalent:*

(1) $\mathscr{V}$ *is tolerance principal*;

(2) *there exist an* 8*-ary polynomial p and* 6*-ary polynomials t, s such that*

$x = t(p(x, y, z, v, x, y, z, v), p(y, x, v, z, x, y, z, v), x, y, z, v)$
$y = t(p(y, x, v, z, x, y, z, v), p(x, y, z, v, x, y, z, v), x, y, z, v)$
$z = s(p(x, y, z, v, x, y, z, v), p(y, x, v, z, x, y, z, v), x, y, z, v)$
$v = s(p(y, x, v, z, x, y, z, v), p(x, y, z, v, x, y, z, v), x, y, z, v)$.

Proof. (1) $\Rightarrow$ (2): Let $F_4(x, y, z, v)$ be a free algebra of $\mathscr{V}$ with free generators x, y, z, v. Then there exist elements a, b of $F_4(x. y, z, v)$ such that

$$(*) \qquad T(a, b) = T(x, y) \vee T(z, v) \qquad \text{in } LT(F_4(x, y, z, v)).$$

Hence $\langle a, b\rangle \in T(x, y) \vee T(Z, v)$, by the Lemma this gives $a = \varphi(x, y, z, v)$, $b = \varphi(y, x, v, z)$ for some 4-ary algebraic function φ over $F_4(x, y, z, v)$ i.e.

$$a = p(x, y, z, v, x, y, z, v), \quad b = p(y, x, v, z, x, y, z, v)$$

for some 8-ary polynomial p over $\mathscr{V}$. Moreover, (*) also implies

$$x = \tau(a, b), y = \tau(b, a) \quad \text{and} \quad z = \sigma(a, b), v = \sigma(b, a)$$

for some binary algebraic functions τ, σ, i.e.there exist 6-ary polynomials t, s with

$$\tau(u, w) = t(u, w, x, y, z, v)$$
$$\sigma(u, w) = s(u, w, x, y, z, v),$$

whence (2) is evident.

(2) $\Rightarrow$ (1): Let $\mathscr{V}$ satisfy (2) and $A \in \mathscr{V}$, $x, y, z, v \in A$. Then also $a = p(x, y, z, v, x, y, z, v)$, $b = p(y, x, v, z, x, y, z, v)$ are elements of A and, by the Lemma, also

$$\langle a, b\rangle \in T(x, y) \vee T(v, z).$$

However, (2) implies

$$x = t(a, b, x, y, z, v), \quad y = t(b, a, x, y, z, v)$$
$$z = s(a, b, x, y, z, v), \quad v = s(b, a, x, y, z, v),$$

thus $\langle x, y\rangle \in T(a, b)$, $\langle z, v\rangle \in T(a, b)$. We infer

$$T(a, b) = T(x, y) \vee T(v, z).$$

By induction, we obtain (1).

Example 1. The variety of groupoids satisfying the following identities:

$$(x . z) . [(x . y) . (z . v)] = x$$
$$(y . v) . [(x . y) . (z . v)] = y$$
$$[(x . y) . (z . v)] . (x . z) = z$$
$$[(x . y) . (z . v)] . (y . v) = v$$

is tolerance principal. We can put

$$p(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = x_1 . x_3$$
$$t(a, b, x, y, z, v) = a . [(x . y) . (z . v)]$$
$$s(a, b, x, y, z, v) = [(x . y) . (z . v)] . a.$$

We can continue our investigations for varieties with a nullary operations.

Definition 2. *An algebra A with a nullary operation c is c-tolerance principal if for each $a_1, \ldots, a_n$ of A there exists an element $a \in A$ such that.*

$$T(a_1, c) \vee \ldots \vee T(a_n, c) = T(a, c) \text{ in } LT(A).$$

A variety $\mathscr{V}$ with a nullary operation c is *c-tolerance principal if each $A \in \mathscr{V}$ has this property.*

Theorem 2. *Let $\mathscr{V}$ be a variety with a nullary operation c. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is c-tolerance principal;*

(2) *there exist a 6-ary polynomial q and 4-ary polynomials u, w such that*

$$c = q(c, x, c, y, x, y)$$
$$x = u(q(x, c, y, c, x, y), c, x, y)$$
$$c = u(x, q(x, c, y, c, x, y), x, y)$$
$$y = w(q(x, c, y, c, x, y), c, x, y)$$
$$c = w(c, q(x, c, y, c, x, y), x, y).$$

Proof. (1)$\Rightarrow$(2): Let $F_2(x, y)$ be a free algebra in a variety $\mathscr{V}$ with a nullary operation c. Then there exists an element a of $F_2(x, y)$ such that

$$T(a, c) = T(x, c) \vee T(y, c).$$

Hence $\langle a, c\rangle \in T(x, c) \vee T(y, c)$, which gives

$$a = q(x, c, y, c, x, y)$$
$$c = q(c, x, c, y, x, y)$$

for some 6-ary polynomial q over $\mathscr{V}$. The remaining part of the proof is analogous to that of Theorem 1 and hence omitted.

(2) $\Rightarrow$ (1): Suppose $\mathscr{V}$ is a variety with a nullary operation c and $A \in \mathscr{V}$, $x, y \in A$. Put $a = q(x, c, y, c, x, y)$. By the Lemma we can see

$$\langle a, c\rangle = \langle q(x, c, y, c, x, y), q(c, x, c, y, x, y)\rangle \in T(x, c) \vee T(y, c).$$

Conversely,

$$\langle x, c\rangle = \langle u(a, c, x, y), u(c, a, x, y)\rangle \in T(a, c)$$
$$\langle y, c\rangle = \langle w(a, c, x, y), w(c, a, x, y)\rangle \in T(a, c),$$

thus, altogether,

$$T(a, c) = T(x, c) \vee T(y, c).$$

Example 2. Each variety of lattices with the least element 0 is 0-tolerance principal. Each variety of lattices with the greatest element 1 is 1-tolerance principal.

Proof. Put
$$q(a_1, a_2, a_3, a_4, x, y) = a_1 \vee a_3$$
$$u(a, b, x, y) = a \wedge x$$
$$w(a, b, x, y) = a \wedge y.$$

Then

$$q(0, x, 0, y, x, y) = 0 \vee 0 = 0$$
$$u(q(x, 0, y, 0, x, y), 0, x, y) = q(x, 0, y, 0, x, y) \wedge x = (x \vee y) \wedge x = x$$
$$u(0, q(x, 0, y, 0, x, y), x, y) = 0 \wedge x = 0$$
$$w(q(x, 0, y, 0, x, y), 0, x, y) = q(x, 0, y, 0, x, y) \wedge y = (x \vee y) \wedge y = y$$
$$w(0, q(x, 0, y, 0, x, y), x, y) = 0 \wedge y = 0.$$

For lattices with the greatest element 1 the proof is dual.

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АЛГЕБРЫ С ГЛАВНЫМИ ТОЛЕРАНЦИЯМЫ

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Резюме

Алгебра A толерантно главная, если каждый компактный элэмент решетки толеранций алгебры A является главная толеранция. В статье даны необходимые и достаточные условия того, чтобы многообразие было многообразием толерантно главных алгебр. Этот концепт тоже обобщается для случая алгебр с нулярными операциями.