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ABSTRACT. The relation between the adjoint operator of degree k, $k = 0, 1, \dots, n$ of the differential operator

$$Lu = r_n(r_{n-1} \dots (r_1(r_0 u)')' \dots)' + \delta u, \qquad \delta \in \mathbf{R} \setminus \{0\}$$

and the two-point boundary value problem is given.

Let $\langle \alpha, \beta \rangle \subset \mathbf{R}$ be a compact interval, where $\mathbf{R}$ is the set of all real numbers. In [3] O h r i s k a introduced the adjoint operator of degree k ($k = 0, 1, \dots, n$) for the linear differential operator of the form

$$Lu = r_n(r_{n-1} \dots (r_1(r_0 u)')' \dots)' + \delta u,$$

where $r_i \colon \langle \alpha, \beta \rangle \to \mathbf{R} \setminus \{0\}$, $i = 0, 1, \dots, n$ are continuous functions and δ is a nonzero constant. The purpose of the present paper is to give the relation between this new notion and the boundary value problems.

We introduce the notation:

$$\begin{aligned} L_0 u &= r_0 u, \\ L_i u &= r_i (L_{i-1} u)', \quad i = 1, 2, \dots, n, \\ \boldsymbol{L} u &= \operatorname{col}(L_0 u, L_1 u, \dots, L_{n-1} u) \quad (\boldsymbol{L} u \text{ is a column-vector}). \end{aligned}$$

For $k \in \{0, 1, \dots, n\}$ denote

$$\begin{aligned} L_0^k v &= v, \\ L_i^k v &= r_{k-i}(L_{i-1}^k v)', \quad i = 1, 2, \dots, k, \quad (\text{if } k > 0) \\ L_{k+1}^k v &= r_{n-1}(r_n L_k^k v)', \quad (\text{if } k < n) \\ L_i^k v &= r_{n+k-i}(L_{i-1}^k v)', \quad i = k+2, k+3, \dots, n, \quad (\text{if } k < n-1), \\ \boldsymbol{L}^k v &= \operatorname{col}(L_0^k v, L_1^k v, \dots, L_{n-1}^k v). \end{aligned}$$

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The differential operator L can be rewritten as

$$Lu = L_n u + \delta u.$$

The corresponding adjoint operator of degree k has the form

$$\bar{L}^k v = (-1)^n L_n^k v + \delta v \qquad \text{for} \quad k = 0, 1, \dots, n-1,$$
$$\bar{L}^n v = (-1)^n r_n \cdot L_n^n v + \delta v.$$

We mention that $\bar{L}^0$ is the usual adjoint operator of L. The domain $\mathcal{D}(L)$ ($\mathcal{D}(\bar{L}^k)$) of L ($\bar{L}^k$) is defined to be the set of all functions u such that $L_i u$ ($L_i^k u$), $i = 0, 1, \dots, n$ exist and are continuous.

Let us now define the *two-point boundary value problem.* Let $f\colon \langle \alpha, \beta \rangle \to \mathbb{R}$ be a continuous function, $A_{ij}, B_{ij}, c_i \in \mathbb{R}$, $i = 1, 2, \dots, l$, $j = 1, 2, \dots, n$. The problem is to find the solution u of

$$Lu = f \tag{1}$$

on the interval $\langle \alpha, \beta \rangle$, which satisfies

$$\mathbf{A}\boldsymbol{L}u(\alpha) + \mathbf{B}\boldsymbol{L}u(\beta) = \mathbf{c}. \tag{2}$$

($\mathbf{A} = (A_{ij})$, $\mathbf{B} = (B_{ij})$ are $l \times n$ matrices, $\mathbf{c} = \operatorname{col}(c_1, c_2, \dots, c_l)$).
By a solution of the equation (1) we mean a function $u \in \mathcal{D}(L)$ which satisfies (1).

For $k \in \{0, 1, \dots, n\}$ let $\Phi^k(t)$, $t \in \langle \alpha, \beta \rangle$ be the following matrix-function of type $n \times n$:

$$\begin{aligned}
\Phi^k_{i,j}(t) &= (-1)^{k-i} && \text{for} \quad i + j = k + 1, \quad (\text{if } k > 0) \\
\Phi^k_{k+1,n}(t) &= r_n(t), && (\text{if } k < n) \\
\Phi^k_{i,j}(t) &= (-1)^{i-k-1} && \text{for} \quad i + j = n + k + 1,\ i \neq k + 1, \quad (\text{if } k < n) \\
\Phi^k_{i,j}(t) &= 0 && \text{for} \quad i + j \notin \{k + 1, n + k + 1\}\ .
\end{aligned}$$

For a given matrix $\mathbf{D}$ let $\mathbf{D}^{\mathrm{T}}$ denote the transpose of the matrix $\mathbf{D}$.

Let $k \in \{0, 1, \dots, n\}$ and consider the following problem. Find $v \in \mathcal{D}(\bar{L}^k)$ and $\boldsymbol{w} \in \mathbb{R}^l$ such that

$$\bar{L}^k v = 0, \tag{3}$$
$$\mathbf{A}^{\mathrm{T}}\boldsymbol{w} - [\Phi^k]^{\mathrm{T}}(\alpha)\boldsymbol{L}^k v(\alpha) = \boldsymbol{O}, \quad \mathbf{B}^{\mathrm{T}}\boldsymbol{w} + [\Phi^k]^{\mathrm{T}}(\beta)\boldsymbol{L}^k v(\beta) = \boldsymbol{O}. \tag{4}$$

The problem (3), (4) is called the *adjoint parametrical boundary value problem of degree* k; $\boldsymbol{w}$ is the parameter. The aim of this paper is to prove the following

Theorem 1. *Let* $k \in \{0, 1, \dots, n\}$. *The problem* (1), (2) *has a solution if and only if*

$$\int_{\alpha}^{\beta} f(t) L_k^k v(t)\,\mathrm{d}t + \mathbf{c}^{\mathrm{T}} \mathbf{w} = 0 \tag{5}$$

for every solution $(v, \mathbf{w})$ *of the problem* (3), (4).

For $k = 0$ Theorem 1 gives the well-known relation between the boundary value problem (1), (2) and its adjoint parametrical boundary value problem ([2], p. 172).

Proof of Theorem 1. Let $AC_n[\alpha, \beta]$ and $L_n^1[\alpha, \beta]$ denote the space of functions $\mathbf{y}\colon \langle \alpha, \beta \rangle \to \mathbb{R}^n$ having absolutely continuous, resp. L^1-integrable components, where $\mathbb{R}^n$ is the real n-dimensional space (elements in $\mathbb{R}^n$ are regarded as column vectors). $L_n^{\infty}[\alpha, \beta]$ is the space of functions $\mathbf{y}\colon \langle \alpha, \beta \rangle \to \mathbb{R}^n$ essentially bounded and $W_n^{1,\infty}[\alpha, \beta] = \{\mathbf{y} \in AC_n[\alpha, \beta] : \mathbf{y}' \in L_n^{\infty}[\alpha, \beta]\}$.

The problem (1), (2) can be written in the following vector form:

$$\mathbf{P}_0 \boldsymbol{\xi}' + \mathbf{P} \boldsymbol{\xi} = \mathbf{g}, \tag{6}$$

$$\mathbf{A} \boldsymbol{\xi}(\alpha) + \mathbf{B} \boldsymbol{\xi}(\beta) = \mathbf{c}, \tag{7}$$

where

$$\mathbf{P}_0 = \begin{pmatrix} r_1, & 0, & 0, & \dots, & 0 \\ 0, & r_2, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & r_n \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0, & -1, & 0, & 0, & \dots, & 0 \\ 0, & 0, & -1, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0, & \dots, & -1 \\ \delta/r_0, & 0, & 0, & 0, & \dots, & 0 \end{pmatrix},$$

$$\boldsymbol{\xi} = \mathbf{L} u \qquad \text{and} \qquad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

We shall consider the problem (6), (7) as an operator equation

$$\mathcal{L}(\boldsymbol{\xi}) = \varphi, \qquad \varphi = \begin{pmatrix} \mathbf{g} \\ \mathbf{c} \end{pmatrix}), \tag{8}$$

where $\mathcal{L}\colon AC_n[\alpha, \beta] \to L_n^1[\alpha, \beta] \times \mathbb{R}^l$ is a linear bounded operator defined by

$$\mathcal{L}(\boldsymbol{\xi}) = \begin{pmatrix} \mathbf{P}_0 \boldsymbol{\xi}' + \mathbf{P} \boldsymbol{\xi} \\ \mathbf{A} \boldsymbol{\xi}(\alpha) + \mathbf{B} \boldsymbol{\xi}(\beta) \end{pmatrix}.$$

We obtain the analytic form of the adjoint operator $\mathcal{L}^*: L_n^\infty[\alpha,\beta] \times \mathbb{R}^l \to W_n^{1,\infty}[\alpha,\beta]$ of the operator $\mathcal{L}$ from the equation

$$\int_\alpha^\beta \boldsymbol{\eta}^{\mathrm{T}}(\mathbf{P}_0\boldsymbol{\xi}' + \mathbf{P}\boldsymbol{\xi})\,\mathrm{d}t + \boldsymbol{w}^{\mathrm{T}}(\mathbf{A}\boldsymbol{\xi}(\alpha) + \mathbf{B}\boldsymbol{\xi}(\beta))$$

$$= \int_\alpha^\beta \Big[\boldsymbol{\eta}^{\mathrm{T}}(t)\mathbf{P}_0(t) + \int_t^\beta \boldsymbol{\eta}^{\mathrm{T}}(s)\mathbf{P}(s)\,\mathrm{d}s + \boldsymbol{w}^{\mathrm{T}}\mathbf{B}\Big]\boldsymbol{\xi}'(t)\,\mathrm{d}t + \Big[\boldsymbol{w}^{\mathrm{T}}(\mathbf{A}+\mathbf{B}) + \int_\alpha^\beta \boldsymbol{\eta}^{\mathrm{T}}\mathbf{P}\,\mathrm{d}t\Big]\boldsymbol{\xi}(\alpha),$$

which holds for every $\boldsymbol{\eta} \in L_n^\infty[\alpha,\beta]$, $\boldsymbol{w} \in \mathbb{R}^l$ and $\boldsymbol{\xi} \in AC_n[\alpha,\beta]$ (see [1], p. 14–16 for details). The adjoint equation

$$\mathcal{L}^*(\boldsymbol{\eta}, \boldsymbol{w}) = \boldsymbol{O} \tag{9}$$

of equation (8) is equivalent to the following system of equations for $\boldsymbol{\eta} \in L_n^\infty[\alpha,\beta]$ and $\boldsymbol{w} \in \mathbb{R}^l$ such that $\boldsymbol{\eta}^{\mathrm{T}}\mathbf{P}_0 \in AC_n[\alpha,\beta]$:

$$(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{P}_0)'(t) - (\boldsymbol{\eta}^{\mathrm{T}}\mathbf{P})(t) = 0 \qquad \text{for almost every} \quad t \in \langle\alpha,\beta\rangle, \tag{10}$$

$$(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{P}_0)(\alpha) = \boldsymbol{w}^{\mathrm{T}}\mathbf{A}, \qquad -(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{P}_0)(\beta) = \boldsymbol{w}^{\mathrm{T}}\mathbf{B}. \tag{11}$$

(In virtue of the continuity of $\mathbf{P}_0$ and $\mathbf{P}$ (10) holds for every $t \in \langle\alpha,\beta\rangle$.) Because $\det(\mathbf{P}_0(t)) \neq 0$ on $\langle\alpha,\beta\rangle$, by Theorem 3.12 from [1] $\mathcal{L}$ has a closed range in $L_n^1[\alpha,\beta] \times \mathbb{R}^l$. So the solvability of (8) can be stated in the form of Fredholm Alternatives (see [1], Theorem 3.14), i.e. the problem (6), (7) has a solution if and only if the right-hand side $(\boldsymbol{g}, \boldsymbol{c})$ is orthogonal to each solution $(\boldsymbol{\eta}, \boldsymbol{w})$ of the system (10), (11). The equation (10) can be rewritten as follows

$$r_0(r_1\eta_1)' - \eta_n\delta = 0 \qquad \text{on } \langle\alpha,\beta\rangle, \tag{12}$$

$$(r_i\eta_i)' + \eta_{i-1} = 0 \qquad \text{on } \langle\alpha,\beta\rangle, \quad i = 2,3,\ldots,n. \tag{13}$$

Let $k \in \{0,1,\ldots,n\}$ be fixed. Denote $r_k\eta_k = (-1)^{k-1}\delta v$ and $\eta_n = v$ for $k \in \{1,2,\ldots,n\}$ and $k = 0$, respectively, and express the remaining components of $\boldsymbol{\eta}$ from (12), (13) by v. A simple calculation shows that:

(i) v is a solution of (3),

(ii) $\eta_n = L_k^k v$,

(iii) $\boldsymbol{\eta}^{\mathrm{T}}\mathbf{P}_0 = \big([\boldsymbol{\Phi}^k]^{\mathrm{T}}\boldsymbol{L}^k v\big)^{\mathrm{T}}$.

The assertion of Theorem 1 is just a modified formulation of the above mentioned Fredholm Alternative. The proof of Theorem 1 is complete.

Remark 1: It follows from the preceding proof that all adjoint problems of degree k $(k = 0, 1, \dots, n)$ are equivalent to the "functional-analytic" adjoint problem (9).

Remark 2. To determine whether the problem (1), (2) has a solution we have to find all solutions $(v, \boldsymbol{w})$ of the problem (3), (4) for some $k \in \{0, 1, \dots, n\}$. Let $\mathcal{S}_k((3),(4))$ denote the vector space of all solutions of (3), (4).

Let $v^{[1]}, v^{[2]}, \dots, v^{[n]}$ be a fundamental system of (3), $\mathbf{V}^k(t) = (\boldsymbol{L}^k v^{[1]}(t), \boldsymbol{L}^k v^{[2]}(t), \dots, \boldsymbol{L}^k v^{[n]}(t))$. Put $\boldsymbol{v}(t) = \mathbf{V}^k(t) \cdot \boldsymbol{b}$, $\boldsymbol{b} \in \mathbb{R}^n$. Then $(v_1, \boldsymbol{w}) \in \mathcal{S}_k((3),(4))$ if and only if

$$\begin{pmatrix} [-\Phi^k]^{\mathrm{T}}(\alpha)\mathbf{V}^k(\alpha), & \mathbf{A}^{\mathrm{T}} \\ [\Phi^k]^{\mathrm{T}}(\beta)\mathbf{V}^k(\beta), & \mathbf{B}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{w} \end{pmatrix} = \boldsymbol{O}. \tag{14}$$

If $\begin{pmatrix} \boldsymbol{b}^{[i]} \\ \boldsymbol{w}^{[i]} \end{pmatrix}$, $i = 1, 2, \dots, m$ is a basis of the solution space of the equation (14), then $\left(\sum_{j=1}^{n} v^{[j]} b_j^{[i]}, \boldsymbol{w}^{[i]}\right)$, $i = 1, 2, \dots, m$ is a basis of $\mathcal{S}_k((3),(4))$. By Theorem 1 the problem (1), (2) has a solution if and only if

$$\int_{\alpha}^{\beta} f(t) \sum_{j=1}^{n} L_k^k v^{[j]}(t) b_j^{[i]} \,\mathrm{d}t + \boldsymbol{c}^{\mathrm{T}} \boldsymbol{w}^{[i]} = 0 \qquad \text{for} \quad i = 1, 2, \dots, m.$$

Example (i). Consider the problem

$$t^2 u'' - 3tu' + 3u = f(t), \tag{15}$$

$$-4u(1) + 3u'(1) + u'(2) = c, \tag{16}$$

where $f\colon \langle 1, 2\rangle \to \mathbb{R}$ is a continuous function and $c \in \mathbb{R}$.
Put $Lu = t^2 u'' - 3tu' + 3u$. The operator L can be written as $Lu = t^3\left(t\left(\frac{1}{t^2}u\right)'\right)' - u$; $r_0(t) = \frac{1}{t^2}$, $r_1(t) = t$, $r_2(t) = t^3$ and the problem (15), (16) can be reformulated as

$$Lu = f \tag{17}$$

$$\mathbf{A}\boldsymbol{L}u(1) + \mathbf{B}\boldsymbol{L}u(2) = c \tag{18}$$

where $\mathbf{A} = (2, 3)$, $\mathbf{B} = (4, 2)$.

We shall investigate the solvability of the problem (17), (18) using the operator $\bar{L}^2$. We calculate $\bar{L}^2 v = t(tv')' - v$,

$$\mathbf{\Phi}^2(t) = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \qquad \mathbf{V}^2(t) = \begin{pmatrix} t, & 1/t \\ t, & -1/t \end{pmatrix},$$

$$[\mathbf{\Phi}^2]^{\mathrm{T}}(t)\mathbf{V}^2(t) = \begin{pmatrix} -t, & 1/t \\ t, & 1/t \end{pmatrix},$$

$$\begin{pmatrix} [-\mathbf{\Phi}^2]^{\mathrm{T}}(1)\mathbf{V}^2(1), & \mathbf{A}^{\mathrm{T}} \\ [\mathbf{\Phi}^2]^{\mathrm{T}}(2)\mathbf{V}^2(2), & \mathbf{B}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 1, & -1, & 2 \\ -1, & -1, & 3 \\ -2, & \frac{1}{2}, & 4 \\ 2, & \frac{1}{2}, & 2 \end{pmatrix}. \tag{19}$$

Since the rank of the matrix (19) is 3, the equation (14) has only a trivial solution and the problem (17), (18) has a solution for all continuous functions f and all $c \in \mathbb{R}$.

E x a m p l e (i i). Consider the problem (15),

$$-7u(1) + 5u'(1) + 2u(2) - 2u'(2) = c. \tag{20}$$

A simple calculation shows that this problem is equivalent to the problem (17), (18) with $\mathbf{A} = (3, 5)$, $\mathbf{B} = (0, -4)$.
In this case

$$\begin{pmatrix} [-\mathbf{\Phi}^2]^{\mathrm{T}}(1)\mathbf{V}^2(1), & \mathbf{A}^{\mathrm{T}} \\ [\mathbf{\Phi}^2]^{\mathrm{T}}(2)\mathbf{V}^2(2), & \mathbf{B}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 1, & -1, & 3 \\ -1, & -1, & 5 \\ -2, & \frac{1}{2}, & 0 \\ 2, & \frac{1}{2}, & -4 \end{pmatrix}. \tag{21}$$

The rank of the matrix (21) is 2, the vector $\operatorname{col}(1, 4, 1)$ is a solution of (14), $v(t) = t + \frac{4}{t}$, $\mathbf{w} = 1$ is a solution of the problem (3), (4) $(k - 2)$. Hence the problem (15), (20) has a solution if and only if

$$\int_1^2 f(t)\Big(\frac{1}{t^2} + \frac{1}{t^4}\Big)\,\mathrm{d}t + c = 0.$$

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REFERENCES

[1] BROWN, R. C.—TVRDÝ, M.: Generalized boundary value problems with abstract side conditions and their adjoints I. Czech. Math. J. 30(105) (1980), 7–27.

[2] KURZWEIL, J.: Ordinary Differential Equations. (Czech), SNTL, Praha, 1978.

[3] OHRISKA, J.: Adjoint differential equations. Math. Slovaca (To appear).

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