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ADJOINT OPERATORS
AND BOUNDARY VALUE PROBLEMS
FOR LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The relation between the adjoint operator of degree $k$, $k = 0, 1, \ldots, n$ of the differential operator

$$Lu = r_n(r_{n-1} \ldots (r_1(r_0u)')') \ldots)' + \delta u, \quad \delta \in \mathbb{R} \setminus \{0\}$$

and the two-point boundary value problem is given.

Let $(\alpha, \beta) \subset \mathbb{R}$ be a compact interval, where $\mathbb{R}$ is the set of all real numbers. In [3] Ohriska introduced the adjoint operator of degree $k$ ($k = 0, 1, \ldots, n$) for the linear differential operator of the form

$$Lu = r_n(r_{n-1} \ldots (r_1(r_0u)')') \ldots)' + \delta u,$$

where $r_i : (\alpha, \beta) \to \mathbb{R} \setminus \{0\}, \ i = 0, 1, \ldots, n$ are continuous functions and $\delta$ is a nonzero constant. The purpose of the present paper is to give the relation between this new notion and the boundary value problems.

We introduce the notation:

$$L_0u = r_0u,$$

$$L_iu = r_i(L_{i-1}u)', \quad i = 1, 2, \ldots, n,$$

$$Lu = \text{col}(L_0u, L_1u, \ldots, L_{n-1}u) \quad (Lu \text{ is a column-vector}).$$

For $k \in \{0, 1, \ldots, n\}$ denote

$$L_0^kv = v,$$

$$L_i^kv = r_{k-i}(L_{i-1}^kv)', \quad i = 1, 2, \ldots, k, \quad (\text{if } k > 0)$$

$$L_{k+1}^kv = r_{n-1}(r_nL_k^kv)', \quad (\text{if } k < n)$$

$$L_i^kv = r_{n+k-i}(L_{i-1}^kv)', \quad i = k + 2, k + 3, \ldots, n, \quad (\text{if } k < n - 1),$$

$$L_n^kv = \text{col}(L_0^kv, L_1^kv, \ldots, L_{n-1}^kv).$$
The differential operator $L$ can be rewritten as

$$Lu = L_nu + \delta u.$$ 

The corresponding adjoint operator of degree $k$ has the form

$$\tilde{L}^k v = (-1)^n L^n v + \delta v \quad \text{for} \quad k = 0, 1, \ldots, n - 1,$$

$$\tilde{L}^n v = (-1)^n r_n \cdot L^n v + \delta v.$$ 

We mention that $\tilde{L}^0$ is the usual adjoint operator of $L$. The domain $\mathcal{D}(L)$ ($\mathcal{D}(\tilde{L}^k)$) of $L$ ($\tilde{L}^k$) is defined to be the set of all functions $u$ such that $L_i u (L^k_i u)$, $i = 0, 1, \ldots, n$ exist and are continuous.

Let us now define the two-point boundary value problem. Let $f: (\alpha, \beta) \rightarrow \mathbb{R}$ be a continuous function, $A_{ij}, B_{ij}, c_i \in \mathbb{R}$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, n$. The problem is to find the solution $u$ of

$$Lu = f$$

on the interval $(\alpha, \beta)$, which satisfies

$$ALu(\alpha) + BLu(\beta) = c.$$  

($A = (A_{ij}), \ B = (B_{ij})$ are $l \times n$ matrices, $c = \text{col}(c_1, c_2, \ldots, c_l)$).

By a solution of the equation (1) we mean a function $u \in \mathcal{D}(L)$ which satisfies (1).

For $k \in \{0, 1, \ldots, n\}$ let $\Phi^k(t), t \in (\alpha, \beta)$ be the following matrix-function of type $n \times n$:

$$\Phi^k_{i,j}(t) = (-1)^{k-i} \quad \text{for} \quad i + j = k + 1, \quad (\text{if} \ k > 0)$$

$$\Phi^k_{k+1,n}(t) = r_n(t), \quad (\text{if} \ k < n)$$

$$\Phi^k_{i,j}(t) = (-1)^{i-k-1} \quad \text{for} \quad i + j = n + k + 1, \quad i \neq k + 1, \quad (\text{if} \ k < n)$$

$$\Phi^k_{i,j}(t) = 0 \quad \text{for} \quad i + j \notin \{k + 1, n + k + 1\}.$$ 

For a given matrix $D$ let $D^T$ denote the transpose of the matrix $D$.

Let $k \in \{0, 1, \ldots, n\}$ and consider the following problem. Find $v \in \mathcal{D}(\tilde{L}^k)$ and $w \in \mathbb{R}^l$ such that

$$\tilde{L}^k v = 0,$$ 

$$A^T w - [\Phi^k]^T(\alpha)L^k v(\alpha) = O, \quad B^T w + [\Phi^k]^T(\beta)L^k v(\beta) = O.$$ 

The problem (3), (4) is called the adjoint parametrical boundary value problem of degree $k$; $w$ is the parameter. The aim of this paper is to prove the following
Theorem 1. Let \( k \in \{0,1,\ldots,n\} \). The problem (1), (2) has a solution if and only if
\[
\int_{\alpha}^{\beta} f(t)L_k^kv(t)\,dt + c^T w = 0
\]  
(5)
for every solution \((v, w)\) of the problem (3), (4).

For \( k = 0 \) Theorem 1 gives the well-known relation between the boundary value problem (1), (2) and its adjoint parametrical boundary value problem ([2], p. 172).

Proof of Theorem 1. Let \( AC_n[\alpha, \beta] \) and \( L^1_n[\alpha, \beta] \) denote the space of functions \( y: (\alpha, \beta) \to \mathbb{R}^n \) having absolutely continuous, resp. \( L^1 \)-integrable components, where \( \mathbb{R}^n \) is the real \( n \)-dimensional space (elements in \( \mathbb{R}^n \) are regarded as column vectors). \( L_\infty^n[\alpha, \beta] \) is the space of functions \( y: (\alpha, \beta) \to \mathbb{R}^n \) essentially bounded and \( W^{1,\infty}_n[\alpha, \beta] = \{ y \in AC_n[\alpha, \beta] : y' \in L_\infty^n[\alpha, \beta] \} \).

The problem (1), (2) can be written in the following vector form:
\[
P_0 \xi' + P \xi = g, 
\]  
(6)
\[
A \xi(\alpha) + B \xi(\beta) = c, 
\]  
(7)
where
\[
P_0 = \begin{pmatrix} r_1, & 0, & 0, & \ldots, & 0 \\ 0, & r_2, & 0, & \ldots, & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & \ldots, & r_n \end{pmatrix}, 
\]
\[
P = \begin{pmatrix} 0, & -1, & 0, & 0, & \ldots, & 0 \\ 0, & 0, & -1, & 0, & \ldots, & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta/r_0, & 0, & 0, & 0, & \ldots, & -1 \end{pmatrix}, 
\]
\[
\xi = Lu \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix} .
\]

We shall consider the problem (6), (7) as an operator equation
\[
\mathcal{L}(\xi) - \varphi, \quad \varphi = \begin{pmatrix} g \\ c \end{pmatrix} , 
\]  
(8)
where \( \mathcal{L}: AC_n[\alpha, \beta] \to L^1_n[\alpha, \beta] \times \mathbb{R}^1 \) is a linear bounded operator defined by
\[
\mathcal{L}(\xi) = \left( P_0 \xi' + P \xi \right) (A \xi(\alpha) + B \xi(\beta)) .
\]
We obtain the analytic form of the adjoint operator \( \mathcal{L}^* : L_\infty^n[\alpha, \beta] \times \mathbb{R}^l \rightarrow W_1^n, \infty[\alpha, \beta] \) of the operator \( \mathcal{L} \) from the equation

\[
\int_\alpha^\beta \eta^T(P_0 \xi' + P \xi) \, dt + w^T(A \xi(\alpha) + B \xi(\beta)) = 0.
\]

which holds for every \( \eta \in L_\infty^n[\alpha, \beta], \ w \in \mathbb{R}^l \) and \( \xi \in AC_n[\alpha, \beta] \) (see [1], p. 14–16 for details). The adjoint equation

\[
\mathcal{L}^*(\eta, w) = 0
\]

of equation (8) is equivalent to the following system of equations for \( \eta \in L_\infty^n[\alpha, \beta] \) and \( w \in \mathbb{R}^l \) such that \( \eta^T P_0 \in AC_n[\alpha, \beta] \):

\[
(\eta^T P_0)'(t) - (\eta^T P)(t) = 0 \quad \text{for almost every} \quad t \in (\alpha, \beta),
\]

\[
(\eta^T P_0)(\alpha) = w^T A, \quad -(\eta^T P_0)(\beta) = w^T B.
\]

(In virtue of the continuity of \( P_0 \) and \( P \) (10) holds for every \( t \in (\alpha, \beta) \).) Because \( \det(P_0(t)) \neq 0 \) on \( (\alpha, \beta) \), by Theorem 3.12 from [1] \( \mathcal{L} \) has a closed range in \( L_1^n[\alpha, \beta] \times \mathbb{R}^l \). So the solvability of (8) can be stated in the form of Fredholm Alternatives (see [1], Theorem 3.14), i.e., the problem (6), (7) has a solution if and only if the right-hand side \( (g, c) \) is orthogonal to each solution \( (\eta, w) \) of the system (10), (11). The equation (10) can be rewritten as follows

\[
\begin{align*}
r_0(r_1 \eta_1)' - \eta_n \delta &= 0 \quad \text{on} \ (\alpha, \beta), \\
(r_i \eta_i)' + \eta_{i-1} &= 0 \quad \text{on} \ (\alpha, \beta), \quad i = 2, 3, \ldots, n.
\end{align*}
\]

Let \( k \in \{0, 1, \ldots, n\} \) be fixed. Denote \( r_k \eta_k = (-1)^{k-1} \delta v \) and \( \eta_n = v \) for \( k \in \{1, 2, \ldots, n\} \) and \( k = 0 \), respectively, and express the remaining components of \( \eta \) from (12), (13) by \( v \). A simple calculation shows that:

(i) \( v \) is a solution of (3),

(ii) \( \eta_n = L_k v \),

(iii) \( \eta^T P_0 = ([\Phi^k]^T L_k v)^T \).

The assertion of Theorem 1 is just a modified formulation of the above mentioned Fredholm Alternative. The proof of Theorem 1 is complete.
Remark 1. It follows from the preceding proof that all adjoint problems of degree \( k \) \((k = 0, 1, \ldots, n)\) are equivalent to the "functional-analytic" adjoint problem (9).

Remark 2. To determine whether the problem (1), (2) has a solution we have to find all solutions \((v, w)\) of the problem (3), (4) for some \( k \in \{0, 1, \ldots, n\}\). Let \( S_k((3), (4)) \) denote the vector space of all solutions of (3), (4).

Let \( v^{[1]}, v^{[2]}, \ldots, v^{[n]} \) be a fundamental system of (3), \( V^k(t) = (L^kv^{[1]}(t), L^kv^{[2]}(t), \ldots, L^kv^{[n]}(t)) \). Put \( v(t) = V^k(t) \cdot b, b \in \mathbb{R}^n \). Then \((v_1, w) \in S_k((3), (4))\) if and only if

\[
\begin{pmatrix}
[-\Phi^k]^T(\alpha)V^k(\alpha), & A^T \\
[\Phi^k]^T(\beta)V^k(\beta), & B^T \\
\end{pmatrix}
\begin{pmatrix}
b \\
w \\
\end{pmatrix} = O. \tag{14}
\]

If \( \begin{pmatrix} b^{[i]} \\ w^{[i]} \end{pmatrix}, \ i = 1, 2, \ldots, m \) is a basis of the solution space of the equation (14), then \( \sum_{j=1}^n v^{[j]} b^{[j]}_j, w^{[i]} \) is a basis of \( S_k((3), (4)) \). By Theorem 1 the problem (1), (2) has a solution if and only if

\[
\int_{\alpha}^{\beta} f(t) \sum_{j=1}^n L^k_j v^{[j]}(t) b^{[j]}_j dt + c^T w^{[i]} = 0 \quad \text{for} \quad i = 1, 2, \ldots, m.
\]

Example (i). Consider the problem

\[
t^2u'' - 3tu' + 3u = f(t), \tag{15}
\]
\[
-4u(1) + 3u'(1) + u'(2) = c, \tag{16}
\]

where \( f: (1, 2) \rightarrow \mathbb{R} \) is a continuous function and \( c \in \mathbb{R} \).

Put \( Lu = t^2u'' - 3tu' + 3u \). The operator \( L \) can be written as

\( Lu = t^3 (\frac{1}{t^2} u)' = u; \ r_0(t) = \frac{1}{t^2}, \ r_1(t) = t, \ r_2(t) = t^3 \)

and the problem (15), (16) can be reformulated as

\[
Lu = f \tag{17}
\]
\[
ALu(1) + BLu(2) = c \tag{18}
\]

where \( A = (2, 3), \ B = (4, 2) \).
We shall investigate the solvability of the problem (17), (18) using the operator $L^2$. We calculate $L^2v = t(tv')' - v$,

$$
\Phi^2(t) = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \\
V^2(t) = \begin{pmatrix} t, & 1/t \\ t, & -1/t \end{pmatrix},
$$

$$
\begin{pmatrix} \Phi^2 \end{pmatrix}^T(t)V^2(t) = \begin{pmatrix} -t, & 1/t \\ t, & -1/t \end{pmatrix},
$$

$$
\begin{pmatrix} [-\Phi^2]^T(1)V^2(1), & A^T \\ [-\Phi^2]^T(2)V^2(2), & B^T \end{pmatrix} = \begin{pmatrix} 1, & -1, & 2 \\ -1, & -1, & 3 \\ -2, & \frac{1}{2}, & 4 \\ 2, & \frac{1}{2}, & 2 \end{pmatrix}.
$$

Since the rank of the matrix (19) is 3, the equation (14) has only a trivial solution and the problem (17), (18) has a solution for all continuous functions $f$ and all $c \in \mathbb{R}$.

Example (i i). Consider the problem (15),

$$-7u(1) + 5u'(1) + 2u(2) - 2u'(2) = c. \quad (20)$$

A simple calculation shows that this problem is equivalent to the problem (17), (18) with $A = (3, 5), \ B = (0, -4)$.

In this case

$$\begin{pmatrix} [-\Phi^2]^T(1)V^2(1), & A^T \\ [-\Phi^2]^T(2)V^2(2), & B^T \end{pmatrix} = \begin{pmatrix} 1, & -1, & 3 \\ -1, & -1, & 5 \\ -2, & \frac{1}{2}, & 0 \\ 2, & \frac{1}{2}, & 4 \end{pmatrix}.
$$

The rank of the matrix (21) is 2, the vector $\text{col}(1, 4, 1)$ is a solution of (14), $v(t) = t + \frac{4}{t}, \ w = 1$ is a solution of the problem (3), (4) ($k - 2$). Hence the problem (15), (20) has a solution if and only if

$$\int_1^2 f(t) \left(\frac{1}{t^2} + \frac{1}{t^4}\right) dt + c = 0.$$

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