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HASHIMOTO TOPOLOGIES
AND QUASI-CONTINUOUS MAPS

JANINA EWERT

ABSTRACT. In a topological space \((X, T)\) any ideal \(\mathcal{P}\) of subsets of \(X\) induces a new topology \(T(\mathcal{P})\). If \(Y\) is a regular space, then under some assumptions on \(\mathcal{P}\), for any upper \(T(\mathcal{P})\)-quasi-continuous multivalued map \(F: X \to Y\) the sets of all points at which \(F\) is lower quasi-continuous (lower semicontinuous) with respect to \(T(\mathcal{P})\) coincide. If \(F\) is compact-valued lower \(T(\mathcal{P})\)-quasi-continuous, then the symmetrical result holds.

For a subset \(A\) of a topological space \((X, T)\) the symbols \(\text{cl}(A)\) and \(\text{int}(A)\) denote the closure and the interior of \(A\) respectively. A set \(A\) is said to be:

- semi-open, if \(A \subseteq \text{cl}(\text{int}(A))\), [9]
- semi-closed, if \(X \setminus A\) is semi-open, [2, 3].

The union of all semi-open sets contained in \(A\) is called the semi-interior of \(A\) and it is denoted as \(\text{sint}(A)\). The intersection of all semiclosed sets containing \(A\) is called the semi-closure of \(A\) and we denote it by \(\text{scl}(A)\), [2, 3].

Now, let \(\mathcal{P}\) be an ideal of subsets of \(X\) and let

\[ B(\mathcal{P}) = \{U \setminus H: U \in T, H \in \mathcal{P}\}. \]

Then \(B(\mathcal{P})\) is a base of some topology \(T(\mathcal{P})\) in \(X\) and \(T \subseteq T(\mathcal{P})\). For any set \(A \subset X\) by \(\text{cl}_\mathcal{P}(A)\), \(\text{int}_\mathcal{P}(A)\), \(\text{scl}_\mathcal{P}(A)\) and \(\text{sint}_\mathcal{P}(A)\) are denoted the closure, interior, semi-closure and semi-interior of \(A\) in \((X, T(\mathcal{P}))\). Let us put

\[ D_\mathcal{P}(A) = \{x \in X: U \cap A \notin \mathcal{P}\text{ for each }T\text{-neighbourhood }U\text{ of }x\}. \]

Then we have \(A \cup D_\mathcal{P}(A) = \text{cl}_\mathcal{P}(A)\) for each set \(A \subset X\), [6].

Let us consider the following two properties:

\[
\begin{align*}
(\ast) & \quad A \in \mathcal{P} \iff D_\mathcal{P}(A) = \emptyset \iff A \cap D_\mathcal{P}(A) = \emptyset \\
(\ast\ast) & \quad \text{If } \{A_j: j \in J\} \text{ is a family of sets belonging to } \mathcal{P} \text{ and each } A_j \text{ is an open set in the subspace } \bigcup\{A_j: j \in J\}, \text{ then } \bigcup\{A_j: j \in J\} \in \mathcal{P}.
\end{align*}
\]

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1. Proposition. The conditions (*) and (**) are equivalent.

Proof. Let (*) be satisfied and let \( \{A_j : j \in J\} \) be a family of sets belonging to \( \mathcal{P} \) such that every \( A_j \) is open in \( \bigcup \{A_j : j \in J\} \). For every point \( x \in \bigcup \{A_j : j \in J\} \) there exist \( j_x \in J \) and an open set \( U \) in \( (X, T) \) such that \( x \in U \cap \bigcup \{A_j : j \in J\} = A_{j_x} \in \mathcal{P} \). Thus

\[
\left( \bigcup \{A_j : j \in J\} \right) \cap D_\mathcal{P} \left( \bigcup \{A_j : j \in J\} \right) = \emptyset,
\]

which implies

\[
\bigcup \{A_j : j \in J\} \in \mathcal{P}.
\]

Conversely, let (**) hold. Since the implications \( A \in \mathcal{P} \implies D_\mathcal{P}(A) = \emptyset \) and \( D_\mathcal{P}(A) = \emptyset \implies A \cap D_\mathcal{P}(A) = \emptyset \) are true, it suffices to prove \( A \cap D_\mathcal{P}(A) = \emptyset \implies A \in \mathcal{P} \). If \( A \cap D_\mathcal{P}(A) = \emptyset \) holds, then each point \( x \in A \) has a \( T \)-neighbourhood \( U_x \) such that \( U_x \cap A \in \mathcal{P} \). So \( \{U_x \cap A : x \in A\} \) is a family of sets belonging to \( \mathcal{P} \) and every \( U_x \cap A \) is open in the subspace \( \bigcup \{U_x \cap A : x \in A\} \). From the assumption \( A = \bigcup \{U_x \cap A : x \in A\} \in \mathcal{P} \) and the proof is completed.

If an ideal \( \mathcal{P} \) satisfies (**), then \( \mathcal{B}(\mathcal{P}) = T(\mathcal{P}) \), [6]; in the other case this equality need not be satisfied. For instance, if \((\mathbb{R}, T)\) is the set of real numbers with the natural topology, \( \mathbb{Q} \) is the set of all rational numbers and \( \mathcal{P} \) is the ideal of all bounded subsets of \( \mathbb{Q} \), then

\[
\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (-n, n) \setminus [-n, n] \cap \mathbb{Q} \in T(\mathcal{P})\text{ and } \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{B}(\mathcal{P}).
\]

Let us denote

\[
D_\mathcal{P}^*(A) = \{x \in X : U \cap A \notin \mathcal{P} \text{ for each } T\text{-semi-open set } U \text{ containing } x\}.
\]

If an ideal \( \mathcal{P} \) has the property (**), then \( A \cup D_\mathcal{P}^*(A) = \text{scl}_\mathcal{P}(A) \) for each set \( A \subset X \), [4].

2. Lemma. If an ideal \( \mathcal{P} \) in a space \((X, T)\) satisfies (**), then for each set \( A \subset X \) the sets \( \text{cl}_\mathcal{P}(A) \setminus \text{scl}_\mathcal{P}(A) \) and \( \text{sint}_\mathcal{P}(A) \setminus \text{int}_\mathcal{P}(A) \) are nowhere dense in \((X, T)\).

Proof. Let \( x \in \text{int}(D_\mathcal{P}(A)) \) and let \( W \) be a semi-open set containing \( x \). Then the set \( U = \text{int}(D_\mathcal{P}(A)) \cap \text{int}(W) \) is open non-empty, so \( U \cap A \notin \mathcal{P} \), which implies \( x \in D_\mathcal{P}^*(A) \). Thus we have \( \text{int}(D_\mathcal{P}(A)) \subset D_\mathcal{P}^*(A) \). Since \( D_\mathcal{P}(A) \) is a closed set [6] and \( D_\mathcal{P}(A) \setminus D_\mathcal{P}^*(A) \subset D_\mathcal{P}(A) \setminus \text{int}(D_\mathcal{P}(A)) \) it follows that \( D_\mathcal{P}(A) \setminus D_\mathcal{P}^*(A) \) is nowhere dense.

Now we have \( \text{cl}_\mathcal{P}(A) \setminus \text{scl}_\mathcal{P}(A) = (A \cup D_\mathcal{P}(A)) \setminus (A \cup D_\mathcal{P}^*(A)) \subset D_\mathcal{P}(A) \setminus D_\mathcal{P}^*(A) \);
so $\text{cl}_p(A) \setminus \text{scl}_p(A)$ is nowhere dense.

For each subset $A$ of $(X,T)$ the following formula is true:

$$\text{sint}(A) = X \setminus \text{scl}(X \setminus A); \ [2].$$

Thus we obtain $\text{sint}_p(A) \setminus \text{int}_p(A) = (X \setminus \text{cl}_p(X \setminus A)) \setminus (X \setminus \text{cl}_p(X \setminus A)) = \text{cl}_p(X \setminus A) \setminus \text{scl}_p(X \setminus A)$, so $\text{sint}_p(A) \setminus \text{int}_p(A)$ is a nowhere dense set in $(X,T)$.

Let $(X,T), (Y,\tau)$ be topological spaces and $F: X \to Y$ a multivalued map which assigns non-empty subsets of $Y$. Following [1], for any set $V \subseteq Y$ we will denote $F^+(V) = \{x \in X: F(x) \cap V \neq \emptyset\}$ and $F^-(V) = \{x \in X: F(x) \cap V = \emptyset\}$. A map $F: X \to Y$ is said to be upper (lower) quasi-continuous at a point $x_0 \in X$ if for each open set $V \subseteq Y$ with $F(x_0) \subseteq V$ (resp. $F(x_0) \cap V = \emptyset$) and each neighbourhood $U$ of $x_0$ there exists an open non-empty set $U_1 \subseteq U$ such that $F(x) \subseteq V$ (resp. $F(x) \cap V = \emptyset$) for $x \in U_1$, [10].

A map $F$ is upper (lower) quasi-continuous at $x_0$ if and only if for each open set $V \subseteq Y$ with $F(x_0) \subseteq V$ (resp. $F(x_0) \cap V = \emptyset$) we have $x_0 \in \text{sint}(F^+(V))$, (resp. $x_0 \in \text{sint}(F^-(V))$).

Any single-valued map $f: X \to Y$ can be considered as the multivalued map with values $\{f(x)\}$ for $x \in X$. In this case both upper and lower quasi-continuity mean quasi-continuity in the sense of Kempisty [8].

For a multivalued map $F: (X,T) \to (Y,\tau)$ we denote by $E^+(F,T,\tau)$ and $E^-(F,T,\tau)$ the sets of all points at which $F$ is upper or lower quasi-continuous. Similarly $C^+(F,T,\tau)$ and $C^-(F,T,\tau)$ denote the sets of points of upper or lower semicontinuity, respectively. When there is no possibility of confusion, then the letter $\tau$ above will be omitted.

A multivalued map is called upper (lower) quasi-continuous if it is upper (lower) quasi-continuous at each point.

3. Theorem. Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,T)$ satisfying $(\ast)$. Assume that $(Y,\tau)$ is a second countable space and $F: X \to Y$ a multivalued map. Then:

(a) If $F$ has compact values, then $E^+(F,T(\mathcal{P}),\tau) \setminus C^+(F,T(\mathcal{P}),\tau)$ is of the first category in $(X,T)$.

(b) The set $E^-(F,T(\mathcal{P}),\tau) \setminus C^-(F,T(\mathcal{P}),\tau)$ is of the first category in $(X,T)$.

Proof. Let $\{V_n: n \geq 1\}$ be an open base of the topology $\tau$ in $Y$ and let $A$ be the set of all finite one-to-one sequences of natural numbers. Then $A = \{\alpha_k: k \geq 1\}$, where $\alpha_k = (n_{k,1}, n_{k,2}, \ldots, n_{k,j(k)})$. Assume $W_k = \bigcup\{V_{n_{k,i}}: i = 1, 2, \ldots, j(k)\}$. Since $F$ has compact values we have

$$E^+(F,T(\mathcal{P}),\tau) \setminus C^+(F,T(\mathcal{P}),\tau) \subseteq \bigcup_{k=1}^{\infty} \text{sint}_\mathcal{P}(F^+(W_k)) \setminus \text{int}_\mathcal{P}(F^+(W_k)).$$

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It follows from Lemma 2 that $E^+(F, T(\mathcal{P}), \tau) \setminus C^+(F, T(\mathcal{P}), \tau)$ is of the first category in $(X, T)$.

Similarly the second part follows from Lemma 2 and the inclusion

$$E^-(F, T(\mathcal{P}), \tau) \setminus C^-(F, T(\mathcal{P}), \tau) \subset \bigcup_{n=1}^{\infty} \text{ sint}_\mathcal{P}(F^-(V_n)) \setminus \text{ int}_\mathcal{P}((F^-(V_n))$$

A set $Y$ with two topologies is called a bitopological space [7]. In a bitopological space $(Y, \tau_1, \tau_2)$ the topology $\tau_1$ is said to be regular with respect to $\tau_2$ if for each $\tau_1$-open set $U$ and each point $x \in U$ there exists a set $U_1 \in \tau_1$ such that $x \in U_1 \subset \text{ cl}(U_1) \subset U$, where $\text{ cl}(A)$ denotes the $\tau_i$-closure of $A$.

$(Y, \tau_1, \tau_2)$ is said to be pairwise regular if $\tau_i$ is regular with respect to $\tau_j$ for $i, j \in \{1, 2\}, i \neq j$.

A bitopological space $(Y, \tau_1, \tau_2)$ is called pairwise normal if for each $\tau_1$-closed set $A$ and $\tau_2$-closed set $B$ with $A \cap B = \emptyset$ there exist disjoint sets $U \in \tau_2, V \in \tau_1$ such that $A \subseteq U$ and $B \subseteq V$, [7].

4. THEOREM. Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X, T)$ satisfying $T \cap \mathcal{P} = \emptyset$ and let $(Y, \tau_1, \tau_2)$ be a bitopological space in which $\tau_2$ is regular with respect to $\tau_1$. If $F: (X, T(\mathcal{P})) \to (Y, \tau_1)$ is an upper quasi-continuous map, then

$$E^-(F, T, \tau_2) = E^-(F, T(\mathcal{P}), \tau_2),$$

$$C^-(F, T, \tau_2) = C^-(F, T(\mathcal{P}), \tau_2).$$

Proof. The inclusion $E^-(F, T, \tau_2) \subset E^-(F, T(\mathcal{P}), \tau_2)$, is evident. So let us assume that $x_0 \in E^-(F, T(\mathcal{P}), \tau_2) \setminus E^-(F, T, \tau_2)$. Then there exists a $\tau_2$-open set $V_0$ with $F(x_0) \cap V_0 \neq \emptyset$ and a $T$-neighbourhood $U$ of $x_0$ such that every non-empty $T$-open set $U' \subset U$ contains a point $x'$ for which $F(x') \cap V_0 = \emptyset$ holds. Let $y_0 \in F(x_0) \cap V_0$. By the regularity of $\tau_2$ with respect to $\tau_1$ we can choose a set $V \in \tau_2$ satisfying

$$y_0 \in V \subset \text{ cl}(V) \subset V_0.$$ 

Since $x_0 \in E^-(F, T(\mathcal{P}), \tau_2)$, there exist $U_1 \in T, H_1 \in \mathcal{P}$ such that $U_1 \setminus H_1 \subset U$ and

$$F(x) \cap V \neq \emptyset \quad \text{for} \quad x \in U_1 \setminus H_1. \quad (1)$$

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On the other hand there exists a point \( x_1 \in U_1 \cap U \) such that \( F(x_1) \subseteq Y \setminus \text{cl}_1(V) \). The map \( F: (X, T(P)) \to (Y, \tau_1) \) is upper quasi-continuous at \( x_1 \), so for some \( U_2 \in T, \ H_2 \in \mathcal{P} \) we have \( U_2 \setminus H_2 \subseteq U_1 \cap U \) and

\[
F(x) \subseteq Y \setminus \text{cl}_1(V) \quad \text{for} \quad x \in U_2 \setminus H_2.
\]

But \( (U_1 \setminus H_1) \cap (U_2 \setminus H_2) \neq \emptyset \), hence it contradicts (1) and the proof of the first equality is completed.

The second part of the proof is analogous, so it is omitted.

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5. **Theorem.** Let \( \mathcal{P} \) be an ideal of subsets of a topological space \( (X, T) \) which satisfies \( T \cap \mathcal{P} = \{\emptyset\} \) and let \( (Y, \tau_1, \tau_2) \) be a bitopological space. Assume that \( F: (X, T(P)) \to (Y, \tau_1) \) is a lower quasi-continuous multivalued map. If one of the following conditions holds:

(a) \( \tau_2 \) is regular with respect to \( \tau_1 \) and the map \( F \) has \( \tau_2 \)-compact values,

(b) \( (Y, \tau_1, \tau_2) \) is pairwise normal and \( F \) has \( \tau_1 \)-closed values,

then

\[
E^+(F, T, \tau_2) = E^+(F, T(P), \tau_2),
\]

\[
C^+(F, T, \tau_2) = C^+(F, T(P), \tau_2).
\]

6. **Corollary.** Let \( \mathcal{P} \) be an ideal of subsets of a topological space \( (X, T) \) satisfying \( T \cap \mathcal{P} = \{\emptyset\} \) and let \( Y \) be a regular topological space. For a multivalued map \( F: X \to Y \) with compact values the following properties are equivalent:

(a) \( F: (X, T) \to Y \) is upper and lower quasi-continuous;

(b) \( F: (X, T(P)) \to Y \) is upper and lower quasi-continuous.

7. **Corollary.** If \( \mathcal{P} \) is an ideal of subsets of a topological space \( (X, T) \) such that \( T \cap \mathcal{P} = \{\emptyset\} \) and \( (Y, \tau_1, \tau_2) \) is pairwise regular, then for any map \( f: X \to Y \) the following are equivalent:

(a) \( f: (X, T) \to (Y, \tau_i) \) is quasi-continuous for \( i \in \{1, 2\} \);

(b) \( f: (X, T(P)) \to (Y, \tau_i) \) is quasi-continuous for \( i \in \{1, 2\} \).

Denoting by \( C(f, T) \) the set of all points at which \( f: (X, T) \to Y \) is continuous, from Theorem 4 we obtain
8. **Corollary.** Let \((X,T)\) be a topological space, \(\mathcal{P}\) an ideal of subsets of \(X\) such that \(T \cap \mathcal{P} = \{\emptyset\}\) and let \(Y\) be a regular space. If \(f: X \to Y\) is a quasi-continuous map, then \(C(f,T) = C(f,T(\mathcal{P}))\).

Let us remark that in Corollary 8 regularity of a space \(Y\) is not necessary.

9. **Example.** In the space \((R, \tau)\) of real numbers with the natural topology we denote by \(\mathcal{P}_1\) the ideal of sets of the first category. The space \((R, \tau(\mathcal{P}_1))\) is not regular. For instance, let \(x_0 = 0\) and let \(W = (-1,1) \setminus \left\{ \frac{1}{n} : n \geq 1 \right\}\). Evidently \(x_0 \in W \in \tau(\mathcal{P}_1)\). Every \(\tau(\mathcal{P}_1)\)-neighbourhood \(W_1\) of \(x_0\) is of the form \(W_1 = U \setminus H\), where \(U \in \tau\), \(H \in \mathcal{P}_1\) and \(cl_{\mathcal{P}_1}(W_1) = cl(W_1) = cl(U) = \emptyset\), so \(cl_{\mathcal{P}_1}(W_1) \notin W\).

Now let \(T = \{(a,\infty): a \in R\} \cup \{\emptyset, R\}\) and let \(\mathcal{P}\) be an ideal satisfying \(T \cap \mathcal{P} = \{\emptyset\}\). A map \(f: (R, T(\mathcal{P})) \to (R, \tau(\mathcal{P}_1))\) is quasi-continuous if and only if it is constant. Thus we have \(C(f, T(\mathcal{P})) = C(f, T)\) for each \(T(\mathcal{P})\)-quasi-continuous map \(f\).

Under some assumptions we can characterize regular spaces in terms of quasi-continuous maps.

10. **Theorem.** Let \(Y\) be a first countable \(T_1\) Baire space. Then the following conditions are equivalent:

(a) \(Y\) is regular;
(b) for any topological space \((X,T)\), an ideal \(\mathcal{P}\) satisfying \(T \cap \mathcal{P} = \{\emptyset\}\) and for each upper quasi-continuous multivalued map \(F: (X, T(\mathcal{P})) \to Y\) we have \(C^-(F,T) = C^-(F,T(\mathcal{P}))\);
(c) for any topological space \((X,T)\), an ideal \(\mathcal{P}\) satisfying \(T \cap \mathcal{P} = \{\emptyset\}\) and for each lower quasi-continuous multivalued map \(F: (X, T(\mathcal{P})) \to Y\) with compact values we have \(C^+(F,T) = C^+(F,T(\mathcal{P}))\);
(d) for any topological space \((X,T)\), an ideal \(\mathcal{P}\) satisfying \(T \cap \mathcal{P} = \{\emptyset\}\) and for each quasi-continuous map \(f: (X, T(\mathcal{P})) \to Y\) we have \(C(f,T) = C(f,T(\mathcal{P}))\).

Proof. The implications (a) \(\implies\) (b) and (a) \(\implies\) (c) are consequences of Theorems 4 and 5; (b) \(\implies\) (d) and (c) \(\implies\) (d) are evident. Thus it suffices to prove (d) \(\implies\) (a). Assume that \(Y\) is not regular. Then there exists an open set \(W_0 \subset Y\) and a point \(y_0 \in W_0\) such that
\[
cl(V) \notin W_0 \quad \text{for every neighbourhood } V \text{ of } y_0.
\]
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Let \( \{W_n: n \geq 1\} \) be an open neighbourhoods base at \( y_0 \) such that \( W_{n+1} \subseteq W_n \subseteq W_0 \) for \( n \geq 1 \). Let us put

\[
B' = \left\{ A \subseteq Y: A \subseteq Y \setminus \left( \{y_0\} \cup \bigcup_{n=0}^{\infty} \text{Fr}(W_n) \right) \right\} \cup \{ \text{cl}(W_n): n \geq 0 \},
\]

where \( \text{Fr}(W_n) \) denotes the boundary of the set \( W_n \), and

\[
P = \left\{ A \subseteq Y: A \subseteq \bigcup_{n=0}^{\infty} \text{Fr}(W_n) \right\}.
\]

Then \( B' \) is a base of some topology \( T \) on \( Y \), \( P \) is an ideal and \( T \cap P = \{\emptyset\} \). Let us consider the map \( f: (Y, T) \to Y \) given by \( f(x) = x \) for \( x \in Y \). Immediately we have

\[
Y \setminus \left( \{y_0\} \cup \bigcup_{n=0}^{\infty} \text{Fr}(W_n) \right) \subseteq C(f, T).
\]

Let \( V \) be a neighbourhood of \( f(y_0) \). Then for some \( m \geq 1 \) we have \( W_m \subseteq V \). Hence we obtain \( y_0 \in U = \text{cl}(W_m) \setminus \bigcup_{n=0}^{\infty} \text{Fr}(W_n) \in T(P) \) and \( f(U) \subseteq V \). Thus, taking into account (1), we have shown

\[
y_0 \in C(f, T(P)) \setminus C(f, T).
\]

Now let \( x \in \bigcup_{n=0}^{\infty} \text{Fr}(W_n) \) and let \( V \) be a neighbourhood of \( f(x) \). Then \( \text{cl}(W_m) \) is a \( T \)-neighbourhood of \( x \), \( W_m \cap V \neq \emptyset \). Since \( Y \) is a \( T_1 \)-space, the condition (1) implies that \( y_0 \) is not an isolated point. Thus \( W_m \cap V \setminus \{y_0\} \) is an open non-empty set. From the assumption that \( Y \) is a Baire space it follows that \( U = W_m \cap V \setminus \left( \{y_0\} \cup \bigcup_{n=0}^{\infty} \text{Fr}(W_n) \right) \neq \emptyset \). Moreover we have \( U \in T(P) \), \( U \subseteq \text{cl}(W_m) \) and \( f(U) \subseteq V \). It means that \( f \) is \( T(P) \)-quasi-continuous at \( x \) and in consequence

\[
\bigcup_{n=0}^{\infty} \text{Fr}(W_n) \subseteq E(f, T(P)),
\]

where \( E(f, T(P)) \) is the set of all points at which \( f: (Y, T(P)) \to Y \) is quasi-continuous. From (2), (3) and (4) we have that \( f: (Y, T(P)) \to Y \) is quasi-continuous but \( C(f, T) \neq C(f, T(P)) \).
In a topological space \((Y, \tau)\) let \(\tau\) be the Vietoris topology on the set \(\mathcal{Z}(Y)\) of all non-empty compact subsets of \(Y\). For open sets \(W_1, \ldots, W_m \subset Y\) we will denote
\[
\mathcal{V}(W_1, \ldots, W_m) = \left\{ B \in \mathcal{Z}(Y) : B \subset \bigcup_{i=1}^{m} W_i \text{ and } B \cap W_i \neq \emptyset \text{ for } i \leq m \right\}.
\]
If \(F\) is a multivalued map defined on a topological space \((X, \mathcal{T})\) with non-empty compact values in \(Y\), then it can be considered also as the single valued map \(F : (X, \mathcal{T}) \to (\mathcal{Z}(Y), \mathcal{T})\). For the set \(E(F, \mathcal{T}, \tau)\) of points at which this single valued map is quasi-continuous we have \(E(F, \mathcal{T}, \tau) \subset E^+(F, \mathcal{T}, \tau) \cap E^-(F, \mathcal{T}, \tau)\) and the inclusion cannot be replaced by the equality.

**11. Theorem.** Let \(\mathcal{P}\) be an ideal of subsets of a topological space \((X, \mathcal{T})\) such that \(\mathcal{T} \cap \mathcal{P} = \{\emptyset\}\) and let \((Y, \tau)\) be a regular space. If \(F : X \to Y\) is a multivalued map with compact values which is upper and lower \(\mathcal{T}(\mathcal{P})\)-quasi-continuous, then \(E(F, \mathcal{T}, \tau) = E(F, \mathcal{T}(\mathcal{P}), \tau)\).

**Proof.** Assume that \(x_0 \in E(F, \mathcal{T}(\mathcal{P}), \tau) \setminus E(F, \mathcal{T}, \tau)\). Then there exists a \(\mathcal{T}\)-neighbourhood \(U\) of \(x_0\) and \(\tau\)-neighbourhood \(\mathcal{V}(V_1, \ldots, V_n)\) of \(F(x_0)\) such that each non-empty \(\mathcal{T}\)-open set \(U' \subset U\) contains a point \(x'\) for which \(F(x') \notin \mathcal{V}(V_1, \ldots, V_n)\). Since \(F(x_0)\) is compact and \(F(x_0) \subset \bigcup_{i=1}^{n} V_i\) we can choose open sets \(W_1, \ldots, W_n\) such that \(\text{cl}(W_i) \subset V_i\) for \(i \leq n\) and \(F(x_0) \in \mathcal{V}(W_1, \ldots, W_n)\).

Let us put \(W = \bigcup_{i=1}^{n} W_i\). The condition \(x_0 \in E(F, \mathcal{T}(\mathcal{P}), \tau)\) implies the existence of sets \(U_1 \in \mathcal{T}, H_1 \in \mathcal{P}\) such that \(\emptyset \neq U_1 \subset U\) and
\[
F(x) \in \mathcal{V}(W_1, \ldots, W_n) \quad \text{for } x \in U_1 \setminus H_1.
\]
On the other hand for some point \(x_1 \in U_1\) there holds \(F(x_1) \notin \mathcal{V}(\text{cl}(W_1), \ldots, \text{cl}(W_n))\). Then
\[
F(x_1) \cap \text{cl}(W_i) = \emptyset \quad \text{for some } i \leq n,
\]
or
\[
F(x_1) \notin \text{cl}(W).
\]
If (2) holds, then using the upper \(\mathcal{T}(\mathcal{P})\)-quasi-continuity of \(F\) at \(x\) we can choose a non-empty set \(U_2 \in \mathcal{T}\) and \(H_2 \in \mathcal{P}\) such that
\[
F(x) \subset Y \setminus \text{cl}(W_i) \quad \text{for } x \in U_2 \setminus H_2.
\]
But \( U_1 \cap U_2 \neq \emptyset \), so \((U_2 \setminus H_2) \cap (U_1 \setminus H_1) \neq \emptyset \). Thus (4) is the contradiction to (1).

Now we assume that (3) is satisfied. Since \( F \) is lower \( T(P) \)-quasi-continuous at \( x_1 \) there exist sets \( U_3 \in T \), \( H_3 \in P \) with \( \emptyset \neq U_3 \subset U_1 \) such that

\[
F(x) \cap (Y \setminus \text{cl}(W)) \neq \emptyset \quad \text{for} \quad x \in U_3 \setminus H_3.
\]

(5)

Because \((U_1 \setminus H_1) \cap (U_3 \setminus H_3) \neq \emptyset \) the condition (5) is in contradiction to (1), which finishes the proof.

The next results are consequences of Theorem 4 and 5.

A multivalued map \( F: (X, T) \to (Y, \tau) \) is called upper (lower) \( c \)-quasi-continuous at \( x_0 \in X \) if for each open set \( V \subset Y \) with \( F(x_0) \subset V \) (resp. \( F(x_0) \cap V \neq \emptyset \)) and \( Y \setminus V \) compact, and for each neighbourhood \( U \) of \( x_0 \) there exists a non-empty open set \( U_1 \subset U \) such that \( F(x) \subset V \) (resp. \( F(x) \cap V \neq \emptyset \)) for \( x \in U_1 \). A map \( F \) is called upper (lower) \( c \)-quasi-continuous if it is upper (lower) \( c \)-quasi-continuous at each point.

In a topological space \((Y, \tau)\) the family \( \tau_c = \{ V \in \tau : Y \setminus V \text{ is compact} \} \cup \{ \emptyset \} \) is a topology and the upper (lower) \( c \)-quasi-continuity of a map \( F: (X, T) \to (Y, \tau_c) \) coincides with the upper (lower) quasi-continuity of \( F: (X, T) \to (Y, \tau) \). Moreover we have

12. "\textbf{Lemma.} If \((Y, \tau)\) is a locally compact \( T_2 \)-space, then the bitopological space \((Y, \tau_c, \tau)\) is pairwise regular."

\textbf{Proof.} One can readily see that for any relatively compact set \( V \) there holds \( \text{cl}(V) = \text{cl}_c(V) \), where \( \text{cl}_c \) denotes the \( \tau_c \)-closure.

Let \( U \) be a \( \tau \)-open set and \( x \in U \). Then there exists a set \( V \in \tau \) with \( \text{cl}(V) \) compact such that \( x \in V \subset \text{cl}(V) \subset U \). Since \( \text{cl}(V) = \text{cl}_c(V) \), we have \( \text{cl}_c(V) \subset U \), so \( \tau \) is regular with respect to \( \tau_c \).

Conversely, let us take \( U \in \tau_c \) and a point \( x \in U \). We can choose a \( \tau \)-open set \( V \) such that \( Y \setminus U \subset V \) and \( x \notin \text{cl}(V) \). Furthermore we can choose a \( \tau \)-open relatively compact set \( W \) satisfying \( Y \setminus U \subset W \subset \text{cl}(W) \subset V \) and \( x \in Y \setminus \text{cl}(W) \subset U \). The set \( Y \setminus \text{cl}(W) \) is \( \tau_c \)-open and \( \text{cl}(Y \setminus \text{cl}(W)) \subset U \), thus \( \tau_c \) is regular with respect to \( \tau \).

13. "\textbf{Theorem.} Let \( \mathcal{P} \) be an ideal of subsets of a topological space \((X, T)\) with \( \mathcal{P} \cap T = \{ \emptyset \} \) and let \((Y, \tau)\) be a locally compact \( T_2 \)-space. If \( F: X \to Y \) is an upper (lower) \( c \)-quasi-continuous multivalued map with compact values, then \( E^-(F, T, \tau) = E^-(F, T(\mathcal{P}), \tau) \) and \( C^-(F, T, \tau) = C^-(F, T(\mathcal{P}), \tau) \), (resp. \( E^+(F, T, \tau) = E^+(F, T(\mathcal{P}), \tau) \) and \( C^+(F, T, \tau) = C^+(F, T(\mathcal{P}), \tau) \))."

\textbf{Proof.} It is direct consequence of Lemma 12 and Theorems 4 and 5.
Finally we will consider real functions. A function \( f: X \rightarrow \mathbb{R} \) is said to be upper (lower) quasi-continuous at \( x_0 \in X \) if for each \( \varepsilon > 0 \) and each neighbourhood \( U \) of \( x_0 \) there exists a non-empty open set \( U_1 \subset U \) such that \( f(x) < f(x_0) + \varepsilon \) (resp. \( f(x_0) - \varepsilon < f(x) \)) for \( x \in U_1 \), [5]. By \( E_u(f, T) \) and \( E_l(f, T) \) will be denoted the sets of all points at which \( f \) is upper or lower quasi-continuous respectively; moreover \( E(f, T) \) is the set of quasi-continuity points. Similarly \( C_u(f, T) \) and \( C_l(f, T) \) denote the sets of points at which \( f \) is upper or lower semicontinuous.

A function \( f \) is called upper (lower) quasi-continuous if \( E_u(f, T) = X \) (\( E_l(f, T) = X \)).

14. **Theorem.** Let \( \mathcal{P} \) be an ideal of subsets of a topological space \( (X, T) \) such that \((*)\) is satisfied. For any function \( f: X \rightarrow \mathbb{R} \) the sets \( E_u(f, T(\mathcal{P})) \setminus C_u(f, T(\mathcal{P})) \), \( E_l(f, T(\mathcal{P})) \setminus C_l(f, T(\mathcal{P})) \) and \( E(f, T(\mathcal{P})) \setminus C(f, T(\mathcal{P})) \) are of the first category in \( (X, T) \).

**Proof.** Let us put \( \tau_1 = \{(-\infty, a): a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \), \( \tau_2 = \{(a, \infty): a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \) and let \( \tau \) denotes the natural topology on \( \mathbb{R} \). Then the conclusion is the consequence of Theorem 3 applied to the function \( f: (X, T) \rightarrow (\mathbb{R}, \tau_1) \) or \( f: (X, T) \rightarrow (\mathbb{R}, \tau_2) \) or \( f: (X, T) \rightarrow (\mathbb{R}, \tau) \), respectively.

15. **Theorem.** Let \( \mathcal{P} \) be an ideal of subsets of a topological space \( (X, T) \) with \( \mathcal{P} \cap T = \{\emptyset\} \) and \( f: X \rightarrow \mathbb{R} \) any function.

(a) If \( f \) is lower \( T(\mathcal{P}) \)-quasi-continuous, then \( E_u(f, T) = E_u(f, T(\mathcal{P})) \) and \( C_u(f, T) = C_u(f, T(\mathcal{P})) \).

(b) If \( f \) is upper \( T(\mathcal{P}) \)-quasi-continuous, then \( E_l(f, T) = E_l(f, T(\mathcal{P})) \) and \( C_l(f, T) = C_l(f, T(\mathcal{P})) \).

**Proof.** Assume \( \tau_1 = \{(-\infty, a): a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \) and \( \tau_2 = \{(a, \infty): a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \). The upper (lower) quasi-continuity means the quasi-continuity with respect to \( \tau_1 \) (or \( \tau_2 \) resp.). Since the bitopological space \( (\mathbb{R}, \tau_1, \tau_2) \) is pairwise regular it suffices to use Theorem 4 to the single valued map \( f \).

**References**


HASHIMOTO TOPOLOGIES AND QUASI-CONTINUOUS MAPS


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