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THE LINEAR ARBORICITY OF 10-REGULAR GRAPHS

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The concept of linear arboricity was introduced by Harary [8], [2] as one of the covering invariants of graphs.

A linear forest is a graph in which each component is a path. The linear arboricity $\Xi(G)$ of a graph $G$ is the minimum number of linear forests whose union is $G$.

The value of linear arboricity so far has been determined only for a few special classes of graphs, e.g. for trees, complete graphs and complete bipartite graphs. (See [1] and [2].) The following conjecture expressed in [2] has a fundamental importance in the research on linear arboricity.

**Conjecture.** The linear arboricity of an $r$-regular graph is $\left\lfloor \frac{r+1}{2} \right\rfloor$.

The general proof of this conjecture does not seem to be simple and so we are meanwhile satisfied with partial results. The conjecture was proved for $r = 3$ and $r = 4$ by Akiyama, Exoo and Harary in [2] and [3], the cases of $r = 5, 6$ were solved independently by Enomoto [4], Peroche [10] and Tomasta [12] (only for $r = 6$) and the case of $r = 8$ was proved by Enomoto and Peroche [5].

The aim of this paper is to prove the validity of the conjecture for the case of $r = 10$.

Let us at first introduce some necessary notions and notations. In the paper by a graph we mean an undirected finite simple graph, by a spanning linear forest we mean a linear forest which is a factor with minimum degree one. Let us denote by $V(G)$ the set of vertices of a graph $G$ and by $E(G)$ the set of edges of $G$. Further let us denote by $V_r(G)$ the set of vertices of degree $r$ of $G$, let $N_G(v)$ denote the set of vertices adjacent to a vertex $v$ in $G$, let $\langle M \rangle$ denote the subgraph induced by a subset $M$ of vertices, let $(u, v)$ denote the undirected edge joining vertices $u$ and $v$ and let $\Delta(G)$ denote the maximum degree of $G$. Any terminology not defined in the paper can be found in [9].

The basic fact which implies our new results about the conjecture is the following theorem.

**Theorem 1.** Let $G$ be a graph with the degree sequence $(6, 5; 5, \ldots, 5)$. Then $\Xi(G) = 3$. 

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As the complete proof of this result is too long for publication in a journal, we shall only outline the method of the proof. The detailed proof of Theorem 1 and of Lemmas 1—3 can be found in [6].

The proof of Theorem 1 is based on the strong Theorem 2 proved by Enomoto [4] and on Lemmas 1—3.

**Theorem 2.** Let $G$ be a graph with $\Delta(G)=4$. Let $\Delta((V_4(G)))\leq 1$. Then $\Xi(G)=2$.

**Lemma 1.** Let $G$ be a graph with one vertex $v$ of degree 6 and all other vertices of degree 5. Let $v_1, v_2, \ldots, v_6 \in V(G)$ be adjacent to $v$. Let there exist a spanning linear forest $P$ in $G$ such that

(i) $(v_1, v), (v_2, v) \in E(P)$, i.e. $v$ is not an endvertex of $P$,
(ii) $v_3, v_4, v_5$ are not endvertices of $P$,
(iii) either $v_6$ is not an endvertex of $P$ or no vertex $y \in (N_G(v_6) - \{v_6\})$ is an endvertex of $P$, where $(v_6, v_{a1}) \in E(P)$.

Then $\Xi(G)=3$.

**Lemma 2.** Let $G$ be a graph with one vertex $v$ of degree 6 and all other vertices of degree 5. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be adjacent to $v$. Let there exist a spanning linear forest $P'$ in $G$ such that

(i) $(v, v_1) \in E(P')$ and $(v, v_2), (v, v_3), \ldots, (v, v_6) \notin E(P')$, i.e. $v$ is an endvertex of $P'$,
(ii) $v_2, v_3, \ldots, v_6$ are not endvertices of $P'$.

Then $\Xi(G)=3$.

**Lemma 3.** Let $G$ be a graph, let $V(G)=M_1 \cup M_2$, $M_1 \cap M_2 = \emptyset$. Let $N(M_1) = \{y \in M_2; \exists x \in M_1, (y, x) \in E(G)\}$, let $N(M_1) \neq \emptyset$. Let there exist a spanning linear forest $P$ in $\langle M_2 \rangle$ such that $\deg_P(x) = 2$ for all $x \in N(M_1)$. Let there exist an integer $\delta > 1$ such that $\delta \leq \deg_G(u) \leq 2\delta$ for all $u \in M_2$. Then there exists a spanning linear forest $P$, in $\langle M_2 \rangle$ with the property that there exists a vertex $v_0 \in N(M_1)$ such that $\deg_P(v_0) = 1$ and for all $y \in N(M_1), y \neq v_0$ we have $\deg_P(y) = 2$.

The main idea of the proof of Theorem 1 is the following: Let $v$ be the vertex of degree 6 and let $v_1, v_2, v_3, v_4, v_5, v_6$ be adjacent to $v$. Consider the graph $G_1 = G - (v, v_1)$. From [4] or [10] it follows that $G_1$ can be decomposed into three linear forests $F_1, F_2, F_3$. Then after adding the edge $(v, v_1)$ back to $G_1$ we can either find a decomposition of $G$ into three linear forests $F_1, F_2, F_3$ by modifying $F_1, F_2, F_3$ or determine a linear forest $P$ (resp. $P'$) in $G$ which fulfils the conditions of Lemma 1 (resp. Lemma 2). There are considered 4 main cases in this proof, according to the values of the numbers

$$p_i = \sum_{j=1}^{\delta} \deg_{F_1 - u}(v_j) \quad \text{for} \quad i = 1, 2, 3.$$
The main cases are then analysed and divided into more detailed subcases, the whole proof consists of verifying 19 subcases altogether.

**Theorem 3.** Let $r$ be an odd integer, $r \geq 5$. Let the linear arboricity of every $r$-regular graph be $\left\{ \frac{r+1}{2} \right\}$, i.e. let the Akiyama—Exoo—Harary conjecture hold for $r$. Then the linear arboricity of every $(r+5)$-regular graph is $\left\{ \frac{(r+5)+1}{2} \right\}$, i.e. the conjecture holds also for the case of $r+5$.

**Proof.** Let $G$ be an arbitrary $(r+5)$-regular graph. The inequality $\mathcal{E}(G) \geq \frac{(r+5)+1}{2}$ is obvious. By Petersen [11] we can decompose $G$ into a 10-regular factor $H$ and an $(r-5)$-regular factor $(G-H)$.

I. Let $|V(G)|$ be even. Consider a Eulerian trail in $H$. Colour the edges of this trail alternately with two colours. We obtain two 5-regular factors $H_1$ and $H_2$. By [4] or [10] $H_1$ can be decomposed into three linear forests and $\mathcal{E}(G-H_1)=\left\{ \frac{r+1}{2} \right\}$ follows from the assumption. Hence $\mathcal{E}(G)=\left\{ \frac{(r+5)+1}{2} \right\}$.

II. Let $|V(G)|$ be odd. Once again consider a Eulerian trail in $H$. Colour the edges of this trail alternately with two colours. We obtain a decomposition of $H$ into two factors $H_1$ and $H_2$. The factor $H_1$ has the degree sequence $(6, 5, 5, \ldots, 5)$ and $H_2$ has $(5, 5, \ldots, 5, 4)$. By Theorem 2 $H_1$ can be decomposed into 3 linear forests and $\mathcal{E}(G-H_1)=\left\{ \frac{r+1}{2} \right\}$ follows from the assumption. Hence $\mathcal{E}(G)=\left\{ \frac{(r+5)+1}{2} \right\}$.

As the conjecture has already been proved for the case of $r=5$, Theorem 3 implies the following important corollary.

**Corollary.** The linear arboricity of every 10-regular graph is 6.

A similar implication as in Theorem 3 from an odd $r$ to $r+3$ can be easily proved by a generalization of Tomast'a's method of the proof of conjecture for $r=6$. The implication from an odd $r$ to $r+1$ follows easily from the fact that every $(r+1)$-regular graph (for an odd $r$) contains a spanning linear forest (because every 2-factor contains a spanning linear forest) and so we can formulate the following theorem in the conclusion.

**Theorem 4.** If the Akiyama—Exoo—Harary conjecture holds for some odd $r$, then it holds for $r+1$, $r+3$, $r+5$, too.
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ЛИНЕЙНАЯ ДРЕВЕСНОСТЬ 10-ПРАВИЛЬНЫХ ГРАФОВ

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Резюме

Линейная древесность $E(G)$ графа $G$ — это минимальное число линейных лесов, объединение которых равно $G$. В работе [2] была высказана гипотеза, что линейная древесность $r$-правильного графа равна

$$\left\{ \frac{r+1}{2} \right\}.$$

До сих пор была доказана для $r = 2, 3, 4, 5, 6, 8$. В этой статье показывается, что гипотеза правильна тоже для $r = 10$. 

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