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THIN SETS IN TRIGONOMETRICAL SERIES AND QUASINORMAL CONVERGENCE

ZUZANA BUKOVSKÁ

1. Introduction

J. Arbault [1] introduced the following notion of a thin set: a set $E \subseteq \langle 0, 1\rangle$ is called an N_O-set if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the series $\sum_{k=0}^{\infty} |\sin n_k \pi x|$ is pointwise convergent on E. He showed that every countable subset of $\langle 0, 1\rangle$ is an N_0-set. N. K. Bari [2, p. 737—738] presented another proof of this fact based on an idea by V. V. Niemyckij [10]. Later, N. N. Cholščevnikova [5] extended this result showing that every set of power less than $\mathfrak{m}$ (for definition of the Martin number $\mathfrak{m}$ see, e.g., D. Fremlin [8]) is an N_O-set. Both N. K. Bari and N. N. Cholščevnikova proved more. Actually they showed that the series $\sum_{k=0}^{\infty} |\sin n_k \pi x|$ is convergent in stronger way than pointwise. This type of convergence was investigated under the name of "equal convergence" by Á. Császár and M. Laczkovich [6, 7]. In [3], we call it quasinormal convergence.

Definition 1. Let f_n, f be real valued functions, $n = 0, 1, 2, \ldots$, defined on a set X. We say that the sequence $\{f_n\}_{n=0}^{\infty}$ quasinormally converges to f on X if there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of nonnegative reals, $\lim_{n\to\infty} \varepsilon_n = 0$ such that for every $x \in X$ there is an index n_x with $|f_n(x) - f(x)| \leq \varepsilon_n$ for $n \geq n_x$.

This convergence lies between pointwise and uniform convergences. We shall need the following simple facts proved in [3, 6, 7].

Theorem 1. a) *If $\{f_n\}_{n=0}^{\infty}$ quasinormally converges to f on X, then there are sets X_k, $k = 0, 1, 2, \ldots$ such that $X = \bigcup_{k=0}^{\infty} X_k$ and $\{f_n\}_{n=0}^{\infty}$ converges uniformly to f on each X_k, $k = 0, 1, 2, \ldots$. Moreover, if X is a topological space and f_n is continuous on X for every n, then we can suppose that X_k are closed sets.*

b) *If* $\{f_n\}_{n=0}^{\infty}$ *quasinormally converges to* f *on* X_k, $k = 0, 1, \ldots$, *then it does so on the union* $\bigcup_{k=0}^{\infty} X_k$.

For every real number x we denote by $\|x\|$ the distance of x to the nearest integer, i.e. if z is the integer for which $z \leq x < z + 1$, then $\|x\| = \min\{x - z, z + 1 - x\}$.

Remark 1. One can easily show that for any real x the following inequalities hold true

$$\frac{\|x\|}{2} \leq \frac{|\sin \pi x|}{\pi} \leq \|x\|.$$

Therefore the sequences $\{\|n_k x\|\}_{k=0}^{\infty}$, $\{\sin \pi n_k x\}_{k=0}^{\infty}$ behave equally as concerns convergence to zero. Similarly the series $\sum_{k=0}^{\infty} \|n_k x\|$ converges if and only if $\sum_{k=0}^{\infty} |\sin \pi n_k x|$ converges.

We conclude with a classical result which we shall need (for proof see, e.g., J. W. S. Cassels [4] or N. K. Bari [2]).

Theorem 2 (Dirichlet—Minkowski). *Let* $x_1, \ldots, x_m \in \mathbf{R}$, $\varepsilon > 0$, $k \in \mathbf{N}$. *Then there exists a natural number* $n > k$ *such that* $\|nx_i\| < \varepsilon$ for $i = 1, 2, \ldots, m$.

The main purpose of this paper is to classify some thin sets in the theory of trigonometrical series using the quasinormal convergence and its properties. Moreover, we shall strengthen the main theorem of N. N. Cholščevnikova [5].

2. Some thin sets in trigonometrical series

Let us recall some notions of thin sets (see N. K. Bari [2] or T. W. Körner [9]). A set $E \subseteq \langle 0, 1 \rangle$ is called an R-set if there exists a trigonometrical series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi nx + b_n \sin 2\pi nx \tag{*}$$

convergent for every $x \in E$ but $\lim_{n \to \infty} (a_n^2 + b_n^2)$ is not zero. A set $E \subseteq \langle 0, 1 \rangle$ is called an N-set if there exists a series (*) absolutely convergent on E and such that $\sum_{n=0}^{\infty} |a_n| + |b_n| = +\infty$. It is well known (see e.g. N. K. Bari [2]) that every N_0-set is both an N-set and an R-set. A set $E \subseteq \langle 0, 1 \rangle$ is called Dirichlet if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\{e^{2\pi i n_k x}\}_{k=0}^{\infty}$ converges uniformly to 1 on E. One can easily show that a set E is Dirichlet if and only if for every $\varepsilon > 0$, every k, there exists a natural number $n > k$ such that $|\sin 2\pi n x| < \varepsilon$ for every $x \in E$. Finally, a set $E \subseteq \langle 0, 1 \rangle$ is called almost Dirichlet if every $F \subseteq E$, $F \neq E$, F closed in E is Dirichlet.

The relationship between those sets is discussed, e.g., by T. W. Körner [9]. However, T. W. Körner requires that all those sets be closed. We introduce some new types of thin sets and we shall use them for the investigation of the above mentioned thin sets.

Definition 2. A set $E \subseteq \langle 0, 1 \rangle$ is called a D-set (a strong D-set, a weak D-set) if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges to 0 on E quasinormally (uniformly, pointwise, respectively). Similarly, a set $E \subseteq \langle 0, 1 \rangle$ is called a DS-set (a strong DS-set, a weak DS-set) if for a suitable increasing sequence $\{n_k\}_{k=0}^{\infty}$ the series $\sum_{k=0}^{\infty} \|n_k x\|$ converges on E quasinormally (uniformly, pointwise).

Using Remark 1 one can easily show that the notions of a strong D-set, a strong DS-set and a Dirichlet set coincide. Similarly, the D-set and the DS-set coincide. A weak DS-set is exactly the N_0-set. A weak D-set was introduced also by J. Arbault [1] as a set admitting "une suite de limite nulle".

Theorem 3. *Let $E \subseteq \langle 0, 1 \rangle$. The following are equaivalent*

a) *E is a D-set,*

b) *E is a union of an increasing sequence of Dirichlet sets,*

c) *E is a union of an increasing sequence of Dirichlet sets closed in E.*

Proof. Evidently c) implies b). We show that b) implies a) and a) implies c).

Let $E = \bigcup_{n=0}^{\infty} E_n$, $E_0 \subseteq E_1 \subseteq \ldots$ being Dirichlet sets. By induction we can choose n_k such that $n_k > \max\{n_i; i < k\}$ and $\|n_k x\| < \frac{1}{k+1}$ for every $x \in E_k$ (E_k is a dirichlet set !).

If $x \in E$, then $x \in E_m$ for some m. For any $k \geq m$ we have $x \in E_k$ and therefore $\|n_k x\| \leq \frac{1}{k+1}$. Thus, E is a D-set.

Now assume that E is a D-set, i.e. there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\{\|n_k x\|\}_{k=0}^{\infty}$ quasinormally converges to zero on E. By Theorem 1, a) the set E is a union of an increasing sequence of closed (in E) sets E_n, $n = 0, 1,$

... such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges uniformly to zero on every E_n. Thus every E_n is Dirichlet and closed in E.

q.e.d.

A simple sequence of this theorem is a strenghtening of Lemma 4.3 of T. W. Körner [9].

Corollary. *Every almost Dirichlet set is a D-set.*

Proof. Let E be an almost Dirichlet set. If E is finite, then E is a D-set (actually a Dirichlet set) by Theorem 2.

Assume E to be infinite. Then there exists an accumulation point $a \in \langle 0, 1\rangle$ of the set E. Denote

$$E_n = \left(\bar{E} - \left(a - \frac{1}{n+1}, a + \frac{1}{n+1}\right)\right) \cup \{a\}, \quad n = 0, 1, 2, \ldots.$$

Every E_n is a closed set, $E_n \subseteq E_{n+1}$ and $\bigcup_{n=0}^{\infty} (E_n \cap E) = E$. Since a is an accumulation point of E, $E \cap E_n \neq E$ for every n. Thus, by the definition of an almost Dirichlet set the set $E \cap E_n$ is Dirichlet and the set E satisfies the condition c) of Theorem 3.

q.e.d.

Contrary to T. W. Körner [9], defining the thin sets we have omitted the condition that a thin set be closed. This omission is important since, e.g., $\mathbf{Q} \cap \langle 0, 1\rangle$ is a D-set and it is not closed, neither is its closure a thin set. However, sometimes the thin sets ought to be nice, say Borel. We show that, in a certain sense, this can be achieved.

Theorem 4. *For every set $E \subseteq \langle 0, 1\rangle$ there exists a set $F \supseteq E$ such that*

a) *if E is Dirichlet, then F is also Dirichlet and F is closed;*

b) *if E is a D-set, then F is also a D-set and F is an F_σ-set;*

c) *if E is an N_0-set, then F is also an N_0-set and F is an $F_{\sigma\delta}$-set;*

d) *if E is a weak D-set, then F is also a weak D-set and F is an $F_{\sigma\delta}$-set.*

Proof. Assume, e.g., that E is a D-set and is not Dirichlet. Then there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges quasinormally to zero on E. By Theorem 1 there are sets E_n, $n = 0, 1, \ldots$, such that $E = \bigcup_{n=0}^{\infty} E_n$ and $\{\|n_k x\|\}_{k=0}^{\infty}$ converges uniformly to zero on each E_n. Thus it converges uniformly on every finite union $\bigcup_{i=0}^{n} E_i$.

Since every function $\|n_k x\|$ is continuous one can easily find a closed set $F_n \supseteq \bigcup_{i=0}^{n} E_i$ such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges uniformly to zero on F_n.

By Theorem 3 the set $F = \bigcup_{n=0}^{\infty} F_n$ is the desired D-set which is an F_σ-set.

The other cases can be shown in a similar way.

q.e.d.

In literature (see e.g. T. W. Körner [9, p. 222]) the notion of a weak Dirichlet set is investigated: a set $E \subseteq \langle 0, 1\rangle$ is called weak Dirichlet if for every positive finite Borel measure μ on $\langle 0, 1\rangle$, for any $\varepsilon > 0$, $\eta > 0$ and given n_0, we can find $n \geq n_0$ such that $\mu^*(\{x \in E;\ |e^{2\pi i n x} - 1| \geq \varepsilon\}) \leq \eta$ (μ^* is the exterior measure associated with μ). Using Lebesque's Dominated Convergence Theorem (see, e.g., W. Rudin [11]), from part d) of Theorem 4 we have

Corollary. *Every weak D-set is a weak Dirichlet set.*

3. Thin sets and groups

The set $\langle 0, 1\rangle$ can be considered as an Abelian group with the addition modulo 1. If $E \subseteq \langle 0, 1\rangle$, we denote by $G(E)$ the subgroup of $\langle 0, 1\rangle$ generated by E. Group-theoretical properties of thin sets were investigated by several authors (see N. K. Bari [2]). We shall use them for distinguishing D-sets from almost Dirichlet sets. We start with a rather elementary result.

Theorem 5. *Let $E \subseteq \langle 0, 1\rangle$. If E is a D-set (an N_O-set, a weak D-set), then $G(E)$ is also a D-set (an N_O-set, a weak D-set).*

Proof. Let E be a D-set. Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of integers such that $\{\|n_k x\|\}_{k=0}^{\infty}$ quasinormally converges to zero on E. Let $\{\varepsilon_k\}_{k=0}^{\infty}$ witness this convergence, i.e. if $x \in E$, then $\|n_k x\| \leq \varepsilon_k$ for sufficiently big k and $\lim_{k \to \infty} \varepsilon_k = 0$. We denote

$$F_0 = \{x \in \langle 0, 1\rangle;\ x \in E \text{ or } 1 - x \in E\},$$

$$F_{n+1} = \{x \in \langle 0, 1\rangle;\ (\exists y, z \in F_n)\, x = y + z \bmod 1\}.$$

Then $G(E) = \bigcup_{n=0}^{\infty} F_n$. If $x \in F_n$, then $\|n_k x\| \leq 2^n \varepsilon_k$ for sufficiently big k. Therefore, $\{\|n_k x\|\}_{k=0}^{\infty}$ quasinormally converges to zero on F_n. By Theorem 1, b) the set $G(E)$ is a D-set.

For N_O-sets and weak D-sets the proof is straightforward.

q.e.d.

The aim of this part is

Theorem 6. *If $E \subseteq \langle 0, 1\rangle$ is an almost Dirichlet set which is also an additive group, then E is finite.*

Proof. Assume E is infinite. Then there is an accumulation point a of E, i.e. $a = \lim_{n\to\infty} x_n$ for some $x_n \in E$, $x_n \neq x_{n+1}$. Setting $y_n = |x_{n+1} - x_n|$ we have $\lim_{n\to\infty} y_n = 0$, $y_n \in E$ and $y_n \neq 0$. Thus 0 is an accumulation point of E.

Since E is a group we have $E \cap \left\langle 0, \frac{1}{4} \right\rangle \neq E$. We show that $\left\langle 0, \frac{1}{4} \right\rangle \cap E$ is not a Dirichlet set. It suffices, for given $\varepsilon > 0$, $n \in \mathbf{N}$, to find an $x \in \left\langle 0, \frac{1}{4} \right\rangle \cap E$ such that $\|nx\| > \varepsilon$.

So, let $\varepsilon > 0$, $\varepsilon < \frac{1}{4}$, n arbitrary. Since 0 is an accumulation point of E there exists $y \in E$ such that $0 < y < \frac{\varepsilon}{n}$. Let k be the smallest natural number for which $ky > \frac{\varepsilon}{n}$. Set $x = ky$. Then

$$0 < nx = nky = n(k-1)y + ny < \varepsilon + \varepsilon < \frac{1}{2}$$

and therefore

$$\|nx\| = nx = nky > \varepsilon.$$

q.e.d.

Corollary 1. *If $E \subseteq \langle 0, 1 \rangle$ is a Dirichlet set and an additive group, then* **E** *is finite.*

Corollary 2. *There exist D-sets which are not almost Dirichlet.*

Proof. Take any infinite countable $E \subseteq \langle 0, 1 \rangle$. Then E is a D-set (see, e.g., Corollary 2 to Theorem 10) and $G(E)$ is also a D-set. $G(E)$ is not almost Dirichlet.

q.e.d.

4. A set of cardinality smaller than $\mathfrak{p}$ is a D-set

The cardinal numbers $\mathfrak{m}$ and $\mathfrak{p}$ are defined, e.g., in D. Fremlin [8, p. 3]. Always $\aleph_1 \leq \mathfrak{m} \leq \mathfrak{p}$ and the consistency of the inequality $\mathfrak{m} < \mathfrak{p}$ is known (for detailed discussion see D. Fremlin [8, p. 290]).

As we have already noted N. N. Cholščevnikova [5] has shown that every $E \subseteq \langle 0, 1 \rangle$, $|E| < \mathfrak{m}$ is a D-set. We shall improve this result replacing $\mathfrak{m}$ by $\mathfrak{p}$. For to make this paper self contained we repeat the definition of the cardinal number $\mathfrak{p}$.

A family $\mathscr{F}$ of subsets of $\mathbf{N}$ (the set of all natural numbers) has the finite intersection property if $A_0 \cap \ldots \cap A_n$ is infinite whenever $A_0, \ldots, A_n \in \mathscr{F}$, $n \in \mathbf{N}$. The cardinal $\mathfrak{p}$ is the least cardinal such that there exists a family $\mathscr{F}$ of subsets of $\mathbf{N}$ with the finite intersection property, $|\mathscr{F}| = \mathfrak{p}$ and such that for any $B \subseteq \mathbf{N}$ infinite, there exists $A \in \mathscr{F}$ with $B - A$ infinite.

One can easily show that $\aleph_0 < \mathfrak{p} \leq 2^{\aleph_0}$. The set $\mathbf{N}$ in this definition can be replaced by any countable infinite set.

Theorem 10. *Let $E_s \subseteq \langle 0, 1\rangle$ be a Dirichlet set, $s \in S$, $|S| < \mathfrak{p}$. If for every $T \subseteq S$, T finite the union $\bigcup_{s \in T} E_s$ is Dirichlet, then $E = \bigcup_{s \in S} E_s$ is a D-set.*

Proof. For any $T \subseteq S$ we denote

$$B(T,m) = \left\{[k,n] \in \mathbf{N} \times \mathbf{N};\ k,n \geq m \,\&\left(\forall x \in \bigcup_{s \in T} E_s\right) \|nx\| < \frac{1}{k+1}\right\}.$$

First we show that $B(T,m)$ is infinite whenever T is finite. So, suppose T is finite.

By the assumptions the set $\bigcup_{s \in T} E_s$ is Dirichlet. Therefore there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\|n_k x\| < \dfrac{1}{k+1}$ for every $k \in \mathbf{N}$, every $x \in \bigcup_{s \in T} E_s$. Since $n_k \geq k$, we have $[k, n_k] \in B(T,m)$ for every $k \geq m$.

From the definition of $B(T,m)$ we obtain directly

$$B(T,m) \subseteq B(T_0, m_0) \cap \ldots \cap (T_k, m_k),$$

where $T = T_0 \cup \ldots \cup T_k$, $m = \max\{m_0, \ldots, m_k\}$. therefore the family

$$\mathscr{F} = \{B(T,m);\ T \subseteq S \text{ finite, } m \in \mathbf{N}\}$$

has the finite intersection property. Since $|\mathscr{F}| \leq \aleph_0 \cdot |S| < \mathfrak{p}$ there exists an infinite set $C \subseteq \mathbf{N} \times \mathbf{N}$ such that $C - B(T,m)$ is finite for every $T \subseteq S$ finite, every $m \in \mathbf{N}$.

If m is an arbitrary integer, then $C \cap B(T,m) \neq \emptyset$ for any $T \subseteq S$ finite. Hence, there are $k, n \geq m$ with $[k,n] \in C$.

Now, we shall construct two increasing sequences $\{k_i\}_{i=0}^{\infty}$, $\{n_i\}_{i=0}^{\infty}$ of natural numbers. Let k_0 be the smallest natural number for which there is an n_0 such that $[k_0, n_0] \in C$. By induction, let k_{i+1} be the least natural number greater than k_i for which there is $n_{i+1} > n_i$ such that $[k_{i+1}, n_{i+1}] \in C$.

We show that for every $x \in E = \bigcup_{s \in S} E_s$ there exists an i_0 such that

$\|n_i x\| < \frac{1}{k_i+1}$ for $i \geq i_0$. Let $x \in E$. then $x \in E_s$ for some $s \in S$ and the set $C - B(\{s\}, 0)$ is finite. Since $[k_i, n_i] \in C$ for every $i \in \mathbf{N}$, there exists an i_0 such that $[k_i, n_i] \in B(\{s\}, 0)$ for $i \geq i_0$. By the definition of $B(\{s\}, 0)$ we have $\|n_i x\| < \frac{1}{k_i+1}$.

q.e.d.

Theorem 3 can be strengthened as

Corollary 1. *Let E_ξ, $\xi < a$ be an increasing sequence of Dirichlet sets, a being an ordinal. If $a < \mathfrak{p}$, then $\bigcup_{\xi \in a} E_\xi$ is a D-set.*

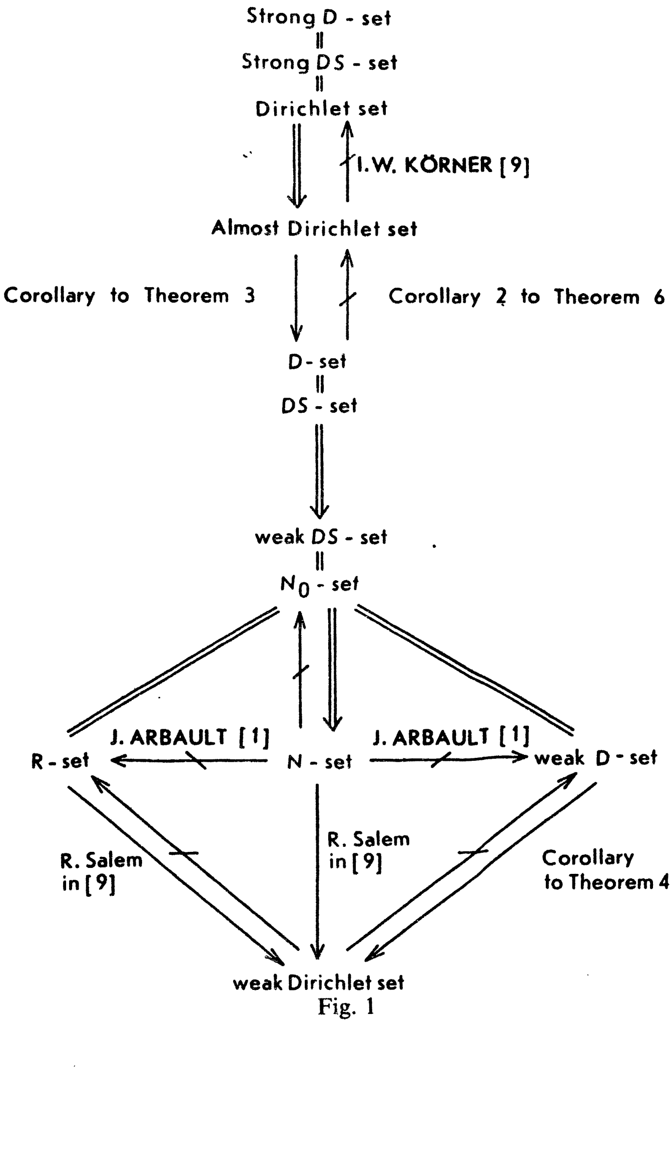

Fig. 1

By the Dirichlet—Minkowski theorem every finite set $E \subseteq \langle 0, 1 \rangle$ is Dirichlet. therefore

Corollary 2. *If $E \subseteq \langle 0, 1 \rangle$ is of cardinality smaller than $\mathfrak{p}$, then E is a D-set.*

5. Concluding remarks

The main purpose of introducing new types of thin sets was to make the relationship between some classical notions of thin sets clear. The obtained result is presented in Figure 1. A double arrow indicates the immediate consequence of definitions. A simple arrow indicates a nontrivial result with a corresponding reference. A simple arrow without reference means a consequence of related results indicated in the figure. Similarly for crossed arrows.

Of course, the absence of an arrow indicates an open problem. The most important are the following:

1) *Is there an R-set which is not an N-set* (see N. K. Bari [2])?
2) *Is there an N_O-set which is not an D-set?*
3) *Is there a weak D-set which is not an N_O-set?*

The first question is open for more than thirty years. We show a relation of this question to the third one.

Theorem 11. *If there exists an R-set which is not an N-set, then there exists a weak D-set which is not an N-set.*

Proof. Let $E \subseteq \langle 0, 1 \rangle$ be an R-set which is not an N-set. Take any $x_0 \in E$. Then

$$F = \{x - x_0;\ x \in E\}$$

is an R-set (see [2, p. 731]) and $0 \in F$. Therefore, F is a weak D-set (see [1] or [2, p. 732]).

Suppose that F is an N-set. Then by [2, p. 752] the set $E = F + x_0$ is also an N-set — a contradiction.

q.e.d.

REFERENCES

[1] ARBAULT, J.: Sur l'ensemble de convergence absolue d'une série trigonométrique. Bull. Soc. Math. France, **80**, 1952, 253—317.

[2] БАРИ, Н. К.: Тригонометрические ряды, Москва 1961,
English translation: BARY, N. K.: A Treatise on Trigonometrical Series, Vol. I and II, Oxford 1964.

[3] BUKOVSKÁ, Z.: Quasinormal convergence, to appear.

[4] CASSELS, J. W. S.: An Introduction to Diophantine Approximation, Cambridge 1965.

[5] ХОЛЩЕВНИКОВА, Н. Н.: О несчетных R- и N-множествах. Мат. заметки **38**, 1985, 270—277.
[6] CSÁSZÁR, Á. and LACZKOVICH, M.: Discrete and equal convergence. Studia Sci. Math. Hungar., **10**, 1975, 463—472.
[7] CSÁSZÁR, Á. and LACZKOVICH, M.: Some remarks on discrete Baire classes. Acta Math. Acad. Sci. Hungr., **33**, 1979, 51—70.
[8] FREMLIN, D.: Consequences of Martin's axiom. Cambridge 1984.
[9] KÖRNER, T. W.: Some results on Kronecker, Dirichlet and Helson sets. Ann. Inst. Fourier (Grenoble) **20**, Fasc. 2, 1970, 219—324.
[10] НЕМЫЦКИЙ, В. В.: О некоторых классах линейных множеств в связи с абсолютной сходимостью тригонометрических рядов. Мат. сборник **33**, 1926, 5—32.
[11] RUDIN, W.: Real and Complex Analysis. New York 1966.

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ТОНКИЕ МНОЖЕСТВА В ТРИГОНОМЕТРИЧЕСКИХ РЯДАХ И КВАЗИНОРМАЛЬНАЯ СХОДИМОСТЬ

Zuzana Bukovská

Резюме

Используя квазинормальную сходимость, вводим новое понятие тонкого множества в теории тригонометрических рядов — множество типа D. Всякое почти Дирихле множество [9] является множеством типа D, а множество типа D является множеством типа N_0 [2]. Доказано, что всякое множество мощности меньше $\mathfrak{p}$ [8] есть D-множество, расширяя так основную теорему [5]. Показано, что почти Дирихле множества и D-множества не совпадают.