LINES IN DIRECTED DISTRIBUTIVE MULTILATTICES

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Introduction

M. Kolibiar [8] has studied properties of lines in a lattice and he has shown that the Jordan-Hölder theorem for lines is true in modular lattices. E. Gedeonová [4] has investigated lines in nonmodular lattices and she has proved that the Jordan-Hölder theorem for lines is valid in a p-modular and semimodular lattice. The aim of this paper is to investigate properties of lines in directed distributive multilattices. The main result of the paper is Jordan-Hölder theorem for lines in a directed distributive multilattice. The method of this paper is a modification of the methods used in [8].

A multilattice [2] is a poset $M$ in which the condition (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leq h, b \leq h$, then there exists $v \in M$ such that (a) $v \leq h, v \geq a, v \geq b$, and (b) $z \in M, z \leq v, z \geq a, z \geq b$ implies $z = v$.

Analogously as in [2] denote by $(a \vee b)_h$ the set of all elements $v \in M$ of (i). Let $(a \land b)_d$ have the dual meaning. Set

$$a \vee b = \bigcup_{a \leq h, h \leq b} (a \vee b)_h, \quad a \land b = \bigcup_{d \leq a, d \leq b} (a \land b)_d.$$ 

Let $A$ and $B$ be nonempty subsets of $M$, then we define

$$A \vee B = \bigcup \{a \vee b \mid a \in A, b \in B\}, \quad A \land B = \bigcup \{a \land b \mid a \in A, b \in B\}.$$ 

In the whole paper we write $[(a \vee x) \land (b \vee x)]_x = x$ ([(a \land x) \lor (b \land x)]_x = x) instead of $[(a \vee x) \land (b \lor x)]_x = \{x\}$ ([(a \land x) \lor (b \land x)]_x = \{x\}).

A poset $A$ is called directed if for each pair of elements $a, b \in A$ there exist elements $d, h \in A$ such that $d \leq a, d \leq b$ and $a \leq h, b \leq h$.

A multilattice $M$ is distributive [2] iff for every $a, b, b', d, h \in M$ satisfying the conditions $d \leq a \leq h, d \leq b \leq h, d \leq b' \leq h, (a \lor b)_h = (a \lor b')_h = h, (a \land b)_d = (a \land b')_d = d$ we have $b = b'$.

Let $M$ be a multilattice and $N$ a nonempty subset of $M$. $N$ is called a submultilattice [2] of $M$ iff $N \cap (a \lor b)_h \neq \emptyset$ and $N \cap (a \land b)_d \neq \emptyset$ for every $a, b, d, h \in N$ satisfying $a \leq h, b \leq h, d \leq a, d \leq b$. 

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Recall the following definition and results from [7]:
Multilattices $M$ and $M'$ are said to be isomorphic (denoted as $M \sim M'$) if there exists a bijection $f$ of $M$ onto $M'$ satisfying: $x \preceq y$ iff $f(x) \preceq f(y)$ ($x, y \in M$).

Let $M$ be the Cartesian product of two posets $M_1$, $M_2$. $M$ is directed iff $M_1$ and $M_2$ are. $M$ is a multilattice iff $M_1$ and $M_2$ are. For each element $x \in M$ denote by $x_1$, $x_2$ ($x \in M_1$) its Cartesian coordinates. Then for all $a$, $b$, $h$, $v \in M$, $v \in (a \vee b)_h$ ($v \in (a \wedge b)_h$) iff $v_i \in (a_i \vee b_i)_h$ ($v_i \in (a_i \wedge b_i)_h$) for $i = 1, 2$.

Properties of lines in directed distributive multilattices

Throughout the paper $M$ and $M'$ denote directed distributive multilattices. Let $a$, $b$, $x \in M$. We say that $x$ is between $a$ and $b$ and write $axb$ if

$$(r) \quad [(a \wedge x) \vee (b \wedge x)] = x = [(a \vee x) \wedge (b \vee x)].$$

**Theorem A** ([6, Theorem 1]). Let $M$ be a directed distributive multilattice, $a$, $b$, $x \in M$. Then $(r)$ is equivalent with

$$(s) \quad (a \wedge x) \vee (b \wedge x) \subseteq a \wedge b, \quad (a \vee x) \wedge (b \vee x) \subseteq a \vee b.$$  

Analogously as in [6] denote by $B(a, b)$ the set of all elements $x \in M$ for which $axb$ holds.

**Theorem B** ([6, Lemma 12]). Let $M$ be a directed distributive multilattice. $a$, $b \in M$. Then

$$B(a, b) = \bigcup_{u \in a \wedge b} \langle u, v \rangle.$$  

**Lemma 1.** The relation $(r)$ in $M$ has the properties:

1. $xyz$ implies $zyx$.
2. $xyz$ and $xzy$ iff $y = z$.
3. $xyz$ and $xzu$ imply $yzu$.
4. $xyx$ implies $x = y$.
5. $xyz$ and $xzu$ imply $xyu$.

**Proof.** The assertions (1), (2), (3) follow from [6, Lemma 6, Lemma 14, Lemma 15]. We prove (4). By Theorem A, from $xyx$ we get $(x \wedge y) \wedge (x \wedge y) \subseteq x \wedge x = x$, hence $x \in (x \wedge y) \wedge (x \wedge y)$ and $x \preceq y$. By duality we get $y \preceq x$, hence $x = y$. Proof of (5): By Theorem A, from $xyz$ we get $(x \wedge y) \wedge (y \wedge z) \subseteq x \wedge z$ and $(x \vee y) \vee (y \vee z) \subseteq x \vee z$. Let us choose $u_i \in (x \wedge y) \wedge (y \wedge z)$ and $v_i \in (x \vee y) \vee (y \vee z)$. Obviously $u_i \preceq y \preceq v_i$. Using Theorem A, from $xzu$ we get $u_i \wedge (z \wedge u) \subseteq (x \wedge z) \wedge (z \wedge u) \subseteq x \wedge u,$
\[ v_1 \lor (z \lor u) \subseteq (x \lor z) \lor (z \lor u) \subseteq x \lor u. \]

Choose \( u_2 \in u_1 \land (z \land u) \), \( v_2 \in v_1 \lor (z \lor u) \). Then

\[ u_2 \leq u_1 \leq y \leq v_1 \leq v_2, \]

hence \( y \in \langle u_2, v_2 \rangle \subset B(x, u) \) (Theorem B).

Four different elements \( a, b, c, d \in M \) form a pseudolinear quadruple [8] when they satisfy \( abc, bcd, cda, dab \).

If \( A, B \) are subsets of some multilattices and a bijection \( \varphi \) from \( A \) onto \( B \) is given so that \( abc \) iff \( \varphi(a)\varphi(b)\varphi(c) \), we say that \( A, B \) are \( b \)-equivalent. A subset \( A \) of \( M \) is called a line if there exists a chain that is \( b \)-equivalent to \( A \). An element \( a \) is an end element of a line \( A \), if \( a \in A \) and for each two elements \( x, y \) of the line \( A \), \( axy \) or \( ayx \). Evidently, a chain in \( M \) is a line in \( M \).

**Lemma 2** (see [1]). Let \( C \) be a chain in \( M \). The relation \((r)\) in \( C \) has the property:

\[ (6) \quad xyz, yzu, y \neq z \text{ imply } xuy. \]

Remark. If \( A \) is a line in \( M \), then \( A \) is \( b \)-equivalent with a chain, hence \((6)\) holds in \( A \).

Let \( A \) be a line in \( M \). Two elements \( a, b \in A, a \neq b \), are called neighbouring if \( \{x \mid x \in A, axb\} = \{a, b\} \).

A length of a finite line is defined to be \( n - 1 \), if \( n \) is a number of the elements of \( A \).

A line \( A \subset M \) is called connected when it has the following property: If \( x \in M \) and if there exist elements \( a, b \in A \) such that \( axb \) and \( A \cup \{x\} \) is a line in \( M \), then \( x \in A \).

An interval \( \langle a, b \rangle (a < b) \) in \( M \) is called a prime interval, if \( \langle a, b \rangle = \{a, b\} \).

**Lemma 3.** Let \( \varphi : M \to M' \) be a \( b \)-equivalence of \( M \) onto \( M' \). Then the image of each prime interval in \( M \) is a prime interval in \( M' \).

Proof. Let \( \langle a, b \rangle \) be a prime interval in \( M \). Denote \( \varphi(a) = a', \varphi(b) = b' \). Since \( B(a, b) = \{a, b\} \), we get \( B(a', b') = \{a', b'\} \). If \( a', b' \) are incomparable, then there exists \( v' \in a' \lor b' \), \( v' \neq a' \), \( v' \neq b' \) and \( v' \in B(a', b') \), which is impossible. Hence the elements \( a', b' \) form a prime interval.

We shall use the following result from [1]:

**Theorem C.** Let \( A \) be a set with a ternary relation \( axb \) satisfying the conditions (1), (3), (4) of Lemma 1, (6) of Lemma 2 and

\[ (7) \quad \text{for each three elements } x, y, z \in A \text{ at least one of the relations } xyz, yzx, zxy \text{ holds.} \]
Then there exists a partial order \( \preceq \) on \( A \) in which \( xyz \) if \( x \preceq y \preceq z \) or \( z \preceq y \preceq x \).

**Theorem 1.** A subset \( A \) of a directed distributive multilattice is a line iff it satisfies the following conditions:

(7) from Theorem C and

(8) \( A \) does not contain any pseudolinear quadruple.

**Remark.** The proof of this Theorem is a modification of the proof of [8, Theorem 2.1].

**Proof.** Let \( A \) be a line. Then \( A \) is \( b \)-equivalent to a chain. In the chain the condition (7) is valid, hence it is valid in \( A \), too. Assume that \( A \) contains a pseudolinear quadruple \( a, b, c, d \). From \( abc, bcd, b \neq c \) we get \( a=db \) (Lemma 2). Using (2), from \( dba, dab \) we get \( a=b \), which is contradictory. Hence, if \( A \) is a line, then the conditions (7) and (8) are valid.

Conversely, let \( A \) satisfy (7) and (8). We have to verify that the relation (r) in \( A \) satisfies the conditions (1), (3), (4), (6), (7) of Theorem C. Obviously, (1), (3), (4), (7) are valid in \( A \). We shall prove the validity of (6). Let \( xyz, yzu, y+z \). From (7) it follows that we get the following cases: 1. \( yux \), 2. \( uxy \), 3. \( xuy \). In the first case from \( xyz \) and \( xuy \) we get \( uy \) (use (3)). This and \( yzu \) implies \( y=z \), which is contradictory. In the second case for \( x, z, u \) at least one of the relations \( zux, uxz, xzu \) holds by (7). If \( zux \) holds, then the elements \( x, y, u, z \) are not pairwise different by (8). Let \( x = y \). Then from \( yzu \) and \( zuy \) we get \( z = u \), and from this \( xzu \), hence (6) is valid. Let \( x = u \). Then from \( yzx \) and \( xyz \) we get \( y = z \), which is contradictory. Let \( x = z \). Then from \( zux \) we get \( xux \) and this implies \( x = u \), which is contradictory. Let \( z = u \), then from \( xyz \) we have \( xyu \), hence (6) is valid. Let \( y = u \), then from \( uxu \) we get \( uu \) and this implies \( x = u \), which is contradictory. If \( uzx \) holds, then from \( uzx, uzx \) we get \( xuy \) by (3) from Lemma 1. From \( xzu, xyz \) we have \( y = z \), which is contradictory. If \( xzu \) holds, then from \( xzu \) and \( xyz \) we get \( xuy \) by (5) from Lemma 1. Hence (6) is valid. This completes the proof.

**Corollary.** Let \( A \) be a line in \( M \), \( a, b \in A \), and let \( a, b \) be incomparable. Then there exists at most one element \( v \in a \lor b \) and at most one element \( u \in a \land b \), such that \( u, v \in A \).

**Proof.** Let \( v_1, v_2 \in A \), \( v_1, v_2 \in a \lor b \), \( v_1 \neq v_2 \). Then it holds \( av_1b, av_2b \). From (7) it follows that for the elements \( a, v_1, v_2 \), we get the following cases: 1. \( av_1v_2 \), 2. \( av_2v_1 \), 3. \( v_1av_2 \). Analogously for \( b, v_1, v_2 \) we have one of the possibilities: 1'. \( bv_1v_2 \), 2'. \( bv_2v_1 \), 3'. \( v_1bv_2 \). In the first case from \( a \preceq v_2 \) we get \( v_1 \preceq v_2 \), which is impossible. The second case is analogous. Similarly we verify that neither \( bv_1v_2 \) nor \( bv_2v_1 \) can hold. Hence the elements \( a, b \), \( v_1, v_2 \) satisfy \( av_1b, v_1bv_2, bv_2a, v_2av_1 \). We get a pseudolinear quadruple, which contradicts (8). The second part of the assertion is dual.

**Theorem 2.** Each two neighbouring elements of a connected line \( A \) in a directed distributive multilattice \( M \) form a prime interval in \( M \).
Remark. The proof of this Theorem is a modification of the proof [8, Theorem 2.3].

Proof. Let $a, b$ be neighbouring elements of a connected line $A$ in $M$. Let $t \in M$ and let $atb$ hold. We shall prove that the set $A \cup \{t\}$ is a line in $M$. Hence we have to verify:

(a) for each $x, y \in A$ at least one of the relations $txy, xty, xyt$ is valid;
(b) if $x, y, z \in A$, then the elements $x, y, z, t$ do not form a pseudolinear quadruple.

In view of the symmetry, for the elements $a, b, x, y \in A$ it suffices to consider the following cases: 1. $xab, yab$; 2. $xab, aby$. In the first case it holds $xya$ or $yxa$. If $xya$ holds, then we get $xyb$ by (6) of Lemma 2. From $bay$ and $bta$ we have $bty$ by (5) of Lemma 1. The relations $bty$ and $byx$ imply $txy$ by (3) of Lemma 1. If $yxa$ holds, then the proof is analogous. In the second case we get $xay$ by (6) of Lemma 2. From $aby$ and $atb$ we have $aty$ by (5) of Lemma 1. The relations $yax$ and $yta$ imply $ytb$ by (5) of Lemma 1. Consequently (a) is proved.

Now we shall prove (b). Let the elements $x, y, z, t \in A$ form a pseudolinear quadruple. In view of the symmetry it suffices to consider the following cases for the elements $a, b, x, y, z \in A$: 1. $xab, yab, zab$; 2. $xab, yab, zba$; 3. $xab, yba, zab$. In the first case we have either $xza$ or $zxa$. Assume that $xza$ holds. From $xza$ and $xyz$ we get $xya$ by (5) of Lemma 1. The relations $xya$ and $xab$ imply $xyb$. From $bay$ and $bta$ we have $bty$. This and $byx$ imply $txy$ from $txy$ and $txy$ it follows that $x = y$, which is contradictory. The case $zxa$ is analogous.

In the second case we get $bty$ from $bta$ and $bay$. The relations $yhb$ and $yzt$ imply $ztb$. This and $zba$ imply $tba$. From $tba$ and $bta$ we get $t = b$, which is contradictory.

We have proved that $A \cup \{t\}$ is a line in $M$. Since $A$ is connected, then $t \in A$. Because $a, b$ are neighbouring in $A$, we get $a = t$ or $b = t$. By Theorem B we get that the elements $a, b$ are comparable, hence they form a prime interval. Consequently Theorem 2 is proved.

Theorem 3. Each line $A$ in $M$ is a submultilattice of $M$.

Proof. Let $A$ be a line in $M$, $a, b \in A$ and assume that there exists $h \in A$ such that $a \leq h, b \leq h$. Obviously $(a \lor b) \cap A \neq \emptyset$, if $a, b$ are comparable. Let $a, b$ be incomparable. At least one of the relations $abh, abh, bah$ holds. From $abh$ it follows that $h \in \langle u, v \rangle$, where $u \in a \land b, v \in a \lor b$ (Theorem B), hence $h \leq v$, and $h = v$. Consequently $(a \lor b) \cap A \neq \emptyset$. In the case $abh$, from $a \leq h$ we get $a \leq b \leq h$, which is impossible. Analogously $bah$ is impossible. Hence $(a \lor b) \cap A \neq \emptyset$, if $a, b, h \in A, a \leq h, b \leq h$. The dual assertion can be proved analogously.

Lemma 4. Let $A$ be a line in $M$ with end elements $a, b$. If $a < b$, then $A$ is a chain in $M$. 59
Proof. Let \( x, y \in A \). From \( axb \) and \( ayb \) we get \( a \leq x \leq b, a \leq y \leq b \). For the elements \( a, x, y \) one of the relations \( axy, ayx \) holds. From \( axy \) and \( ayb \) we get \( xyb \), hence \( x \leq y \leq b \). Analogously from \( ayx \) we get \( y \leq x \leq b \). Hence \( x \) and \( y \) are comparable.

Lemma 5. Let \( A \) be a finite connected line in \( M \) with end elements \( a, b \). Then there exists an interval \( \langle u, v \rangle \), \( u \in a \land b, v \in a \lor b \) such that \( A \subset \langle u, v \rangle \).

Proof. The assertion is evident, if \( a \) and \( b \) are comparable. Let \( a, b \) be incomparable. We prove the assertion by induction with respect to the length of \( A \). Since \( A \) is connected and its end elements are incomparable, then \( A \) has at least three elements. If \( A \) has three elements \( a, x, b \), then we get the following cases: 1. \( a \leq x, x \leq b \); 2. \( a \geq x, x \leq b \). In the first case obviously \( x \in a \lor b \) and \( A \subset \langle u, x \rangle \) for an element \( u \in a \land b \). The second case is dual. Now we assume that the assertion is true for lines having length \( n-1 \) \((n \geq 3)\) and prove it for \( n \). Let \( A \) have the length \( n \) \((n \geq 3)\) and denote its elements \( a = a_0, a_1, \ldots, a_n = b \), where \( a_0, a_1, \ldots \) are neighbouring elements \((i = 1, \ldots, n-1)\). The elements \( a_0, \ldots, a_n \) form a line with length \( n-1 \), hence there exist \( u_i \in a_i \land a_{i-1} \) and \( v_i \in a_i \lor a_{i+1} \) such that \( A \subset \langle u_i, v_i \rangle \). Let \( u_2 \in a_{n-1} \land a_n, v_2 \in a_{n-1} \lor a_n \). From \( a_0 a_{i-1} a_n \) by Theorem A it follows that there exist \( u \in u_1 \land u_2, u \in a_0 \land a_n \) and \( v \in v_1 \lor v_2, v \in a_0 \lor a_n \) such that \( A \subset \langle u, v \rangle \).

Jordan-Hölder Theorem for Lines

A subset \( \{a, b, u, v\} \) of \( M \) is called an elementary quadruple if \( u \in a \land b, v \in a \lor b \) and the intervals \( \langle u, a \rangle, \langle u, b \rangle, \langle a, v \rangle, \langle b, v \rangle \) are prime intervals.

Lemma 6. Let \( \varphi : M \to M' \) be a \( b \)-equivalence. Then the image of an elementary quadruple in \( M \) is an elementary quadruple in \( M' \).

Proof. Let \( a, b, u, v \in M, u \in a \land b, v \in a \lor b \) and \( \{a, b, u, v\} \) be an elementary quadruple. Denote \( x' = \varphi(x) \) for each \( x \in M \). By Lemma 3 the images of \( \langle u, a \rangle, \langle u, b \rangle, \langle a, v \rangle, \langle b, v \rangle \) are prime intervals in \( M' \). Now the assertion of the lemma follows immediately from [5, Lemma 5 and Lemma 6].

Let \( \langle u, a \rangle \) and \( \langle b, v \rangle \) be intervals of a directed distributive multilattice. The intervals \( \langle u, a \rangle \) and \( \langle b, v \rangle \) are called transposes if \( u \in a \land b \) and \( v \in a \lor b \). The intervals \( \langle u, a \rangle \) and \( \langle b, v \rangle \) are called projective if there exists a finite sequence of intervals \( \langle a, y_n \rangle = \langle x_0, y_n \rangle, \langle x_1, y_1 \rangle, \ldots, \langle x_n, y_n \rangle = \langle b, v \rangle \) such that \( \langle x_i, y_{i-1} \rangle \) and \( \langle x_i, y_i \rangle \) are transposes for \( i = 1, 2, \ldots, n \). The intervals \( \langle x_i, y_i \rangle \) are called middle for \( i = 1, 2, \ldots, n-1 \).

From the paper [2] it follows that the following theorem is true.

Theorem D. Let \( A, B \) be finite connected chains with end elements \( a, b \) in a directed distributive multilattice. Then the chains \( A, B \) have the same length and
there exists a one-to-one mapping of the set of all prime intervals of the chain $A$ onto the set of all prime intervals of the chain $B$ such that the corresponding prime intervals are projective and the middle intervals $\langle x_i, y_i \rangle$ satisfy $\langle x_i, y_i \rangle \subset \langle a, b \rangle$.

**Lemma 7.** Let $a_0, a_1, ..., a_{n+k} \in M$ and $a_0 < a_1 < ... < a_n, a_n > a_{n+1} > ... a_{n+k}$. The elements $a_0, a_1, ..., a_{n+k}$ form a finite line with end elements $a_0, a_{n+k}$ in $M$ if and only if $a_n \in a_0 \lor a_{n+k}$.

**Proof.** Let $a_n \in a_0 \lor a_{n+k}$. We prove that the elements $a_0, a_1, ..., a_{n+k}$ form a finite line in $M$. According to Theorem 1 we have to verify that the conditions (7) and (8) hold. First we prove the condition (7). Since the elements $a_0, a_1, ..., a_n$ form a chain, we get $a_i a_m$ for $i < j < m$, $i = 0, 1, ..., n - 2$, $m = 2, 3, ..., n$. Analogously we get $a_i a_r$ for $r < s < t$, $r = n, n + 1, ..., n + k - 2$, $t = n + 2, n + 3, ..., n + k$. Let $i = 0, 1, ..., n, j = n, n + 1, ..., n + k$. Since $a_n \in a_0 \lor a_{n+k}$, then $a_n \in a_i \lor a_r$ and we have $a_i a_r$. From this and $a_i a_r$ ($i < p < n$) we get $a_i a_r$. The relations $a_i a_r, a_r a_s$ ($n < q < j$) imply $a_i a_r$. Hence we have proved that for each three elements $a_i, a_r, a_s \in A$, $x < y < z$, $x = 0, 1, ..., n + k - 2$, $z = 2, 3, ..., n + k$ it holds $a_i a_r$. Consequently the condition (7) is true. Now we prove the condition (8). Let the elements $a_i, a_r, a_s, a_w \in A$ form a pseudolinear quadruple $(x, y, z, w = 0, 1, ..., n + k)$, hence we have $a_i a_r, a_i a_w, a_r a_w, a_i a_w$. In view of the symmetry it suffices to consider the following case: $x < y < z < w$. Then $a_i a_r$. This and $a_i a_r$ imply $a_i = a_w$, which is contradictory. We have proved that the elements $a_0, a_1, ..., a_{n+k}$ form a line in $M$. Evidently the elements $a_0, a_{n+k}$ are end elements of this line. Conversely, we prove that $a_n \in a_0 \lor a_{n+k}$. Since $a_n > a_0, a_n > a_{n+k}$, $a_0 a_n a_{n+k}$ we get $a_n \in a_0 \lor a_{n+k}$ by [5, Lemma 2].

**Lemma 8.** Let $a_0, a_1, ..., a_{n+k} \in M$, $a_0 < a_1 < ... < a_n, a_n > a_{n+1} > ... a_{b+k}$ and let these elements form a finite connected line $A$ with end elements $a_0, a_{n+k}$ in $M$. Let $b_i \in M, b_k \in a_0 \lor a_{n+k}$. Then there exists a finite connected line $B$ with end elements $a_0, a_{n+k}$, which has elements $b_i, b_k$, $b_i < b_k$ such that $b_0, b_1, ..., b_{n+k}$ is a transposes and $\langle a_0, a_{n+k} \rangle, \langle b_0, b_{n+k} \rangle$ are transposes for $i = 0, 1, ..., n - 1, j = 0, 1, ..., k - 1$.

**Proof.** Denote $a_0 = b_0, a_{n+k} = b_{n+k}$. By Lemma 7 $a_0 \lor b_{n+k}$. By [5, Lemma 10] the intervals $\langle a_i, a_w \rangle, \langle b_i, b_{n+k} \rangle$ are isomorphic and they exist elements $b_{i+k} \in \langle b_i, b_{n+k} \rangle$ such that $b_0, b_1, ..., b_{n+k}$ is a transposes and $\langle a_0, a_{n+k} \rangle, \langle b_0, b_{n+k} \rangle$ are transposes for $i = 0, 1, ..., n - 1$. Next it holds $b_{i+k} \in (a_i \lor b_{k+i+1})a_n$ and $a_{i+k} \in (a_i \lor b_{k+i+1})a_n$. Consequently the intervals $\langle a_i, a_{i+k} \rangle, \langle b_0, b_{n+k} \rangle$ are transposes for $i = 0, 1, ..., n - 1$. Analogously we get the elements $b_i \in \langle b_i, b_{n+k} \rangle$ and the validity of the assertion “the intervals $\langle a_i, a_{i+k} \rangle, \langle b_i, b_{j+i+1} \rangle$ are transposes”, for $j = 0, 1, ..., k - 1$. From this and from Lemma 7 it follows that the elements $b_0, b_1, ..., b_{n+k}$ form the finite connected line $B$.

**Remark.** Evidently, the dual of Lemma 7 and the dual of Lemma 8 are valid too.
Lemma 9. Let \( a, b \) be incomparable elements in \( M \). Let \( A \) be a finite connected line with end elements \( a, b \) in \( M \). Further we assume that \( u, v \in M, u \in a \land b, v \in a \lor b \) such that \( A \subset \langle u, v \rangle \). Then there exists a finite connected line \( B \) with end elements \( a, b \), which has the elements \( a = b_0 > b_1 > \ldots > b_k = u, u < b_{k+1} < b_{k+2} = b \) \((a = b_0 < b_1 < \ldots < b_n = v, v > b_{k+1} > \ldots > b_{k+n} = b)\), and a one-to-one mapping of the set of all prime intervals of line \( A \) onto the set of all prime intervals of the line \( B \) such that the corresponding prime intervals are projective and the middle intervals \( \langle x, y \rangle \) satisfy \( ax, b, ay, b \).

Proof. We can restrict our consideration to the following line \( A: a = a_0 < a_1 < \ldots < a_{r_1} > a_{r_1+1} > \ldots > a_{r_2} a_{r_2} < a_{r_2+1} < \ldots < a_{r_3} \ldots, a_{r_n} a_{r_n} < a_{r_n+1} < \ldots < a_{r_{n+1}} > \ldots > a_{r_{m-1}} = b \). (The proof is analogous, if the line \( A \) has another form.) The elements \( a_{r_1}, a_{r_2}, \ldots, a_{r_n} \) are called edges. We prove the assertion of the lemma by induction with respect to the number of the edges in \( A \). Let \( s = 1 \), then the assertion follows by Lemma 8. Now we assume that the assertion holds for \( s = m - 1 \) and we prove it for \( s = m \). The elements \( a = a_0 < a_1 < \ldots < a_{r_1} a_{r_1} > a_{r_1+1} > \ldots > a_{r_2} \ldots, a_{r_n} a_{r_n} < a_{r_n+1} < \ldots < a_{r_{n+1}} > \ldots > a_{r_{m-1}} = b \) form a connected line \( C \), which has \( m - 1 \) edges \( a_{r_1}, a_{r_2}, \ldots, a_{r_{m-1}} \). Evidently, \( C \subset \langle u, v \rangle \). From this it follows that there exist \( w \in (a \lor a_{r_1}), z \in (a \land a_{r_1}) \) such that \( C \subset \langle z, w \rangle \). Since \( C \) has \( m - 1 \) edges, there exists a finite connected line \( D: a = d_0 > d_1 > \ldots > d_1 = z, z < d_{n+1} < \ldots < d_{n+n} = a_{r_{m-1}} \) and a one-to-one mapping of the set of all prime intervals of the line \( C \) onto the set of all prime intervals of the line \( D \) such that the corresponding prime intervals are projective and the middle intervals \( \langle x, y \rangle \) satisfy \( ax, b, ay, b \).
Theorem 4. Let $A$, $B$ be finite connected lines with end elements $a$, $b$ in a directed distributive multilattice. Then the lines $A$, $B$ have the same length and there exists a one-to-one mapping of the set of all prime intervals of the line $A$ onto the set of all prime intervals of the line $B$ such that the corresponding prime intervals are projective and the middle intervals $\langle x, y \rangle$ satisfy $axb$, $ayb$.

Proof. If $a$, $b$ are comparable, then the assertion is true by Theorem D. Let $a$, $b$ be incomparable. Let $A$, $B$ be finite connected lines with end elements $a$, $b$. By Lemma 5 there exist $u$, $u' \in a \land b$, $v$, $v' \in a \lor b$ such that $A = \langle u, v \rangle$ and $B = \langle u', v' \rangle$. By Lemma 9 there exists a finite connected line $C$ with end elements $a$, $b$, which has elements $a = c_0 > c_1 > \ldots > c_s = u$, $u < c_{k+1} < \ldots < c_{k+n} = b$, and a one-to-one mapping $q_1$ of the set of all prime intervals of the line $A$ onto the set of all prime intervals of the line $C$ such that the corresponding intervals are projective and the middle intervals $\langle x, y \rangle$ satisfy $axb$, $ayb$. Analogously there exist a finite connected line $D$ with end elements $a$, $b$, which has elements $a = d_0 < d_1 < \ldots < d_r = v'$, $v' > d_{r+1} > \ldots > d_{r+m} = b$, and a one-to-one mapping $q_2$ of the set of all prime intervals of the line $B$ onto the set of all prime intervals of the line $D$ such that the corresponding intervals are projective and the middle intervals $\langle x, y \rangle$ satisfy $axb$, $ayb$. By Lemma 8 there exists a finite connected line $E$, which has elements $a = e_0 > e_1 > \ldots > e_m = u$, $u < e_{m+1} < \ldots < e_{m+r} = b$, such that the intervals $\langle d_p, d_{p+1} \rangle$ and $\langle e_{m+p}, e_{m+p+1} \rangle$ are transposes, $\langle d_{r+q}, d_{r+q+1} \rangle$ and $\langle e_q, e_{q+1} \rangle$ are transposes for $p = 0, 1, \ldots, r-1$, $q = 0, 1, \ldots, m-1$. Hence there exists a one-to-one mapping $q_3$ of the set of all prime intervals of the line $D$ onto the set of all prime intervals of the line $E$ such that the corresponding intervals are transposes. The elements $a = e_0 > e_1 > \ldots > e_m = u$ form a finite connected chain $E_1$. The elements $a = c_0 > c_1 > \ldots > c_s = u$ form a finite connected chain $C_1$. The chains $C_1$, $E_1$ have the same end elements $a$, $u$. By Theorem D $k = m$ and there exists a one-to-one mapping of the set of all prime intervals of the chain $C_1$ onto the set of all prime intervals of the chain $E_1$ such that the corresponding prime intervals are projective and the middle intervals $\langle x, y \rangle$ satisfy $\langle x, y \rangle \subset \langle u, a \rangle$. Analogously we get the chain $C_2$ with the elements $u < c_{k+1} < \ldots < c_{k+n} = b$ and the chain $E_2$ with the elements $u < e_{k+1} < \ldots < e_{k+r} = b$. By Theorem D $r = n$ and there exists a one-to-one mapping of the set of all prime intervals of the chain $C_2$ onto the set of all prime intervals of the chain $E_2$ such that the corresponding prime intervals are projective and the middle intervals $\langle x, y \rangle$ satisfy $\langle x, y \rangle \subset \langle u, b \rangle$. Consequently, the line $C$ and the line $E$ have the same length and there exists a one-to-one mapping $q_4$ of the set of all prime intervals of the line $C$ onto the set of all prime intervals of the line $E$ such that the corresponding prime intervals are projective. The middle intervals of the corresponding prime intervals under $q_4$ are the intervals $\langle x, y \rangle$ and the intervals $\langle x, y \rangle$. Since $\langle x, y \rangle \subset \langle u, a \rangle$ and $\langle x, y \rangle \subset \langle u, b \rangle$ we get $axb$, $ayb$, $axb$, $ayb$ by Theorem B. From these
considerations it follows that the line $A$ and the line $B$ have the same length and the mapping $q = q_2^{-1}q_3^{-1}q_4q_1$ is a one-to-one mapping of the set of all prime intervals of the line $A$ onto the set of all prime intervals of the line $B$ such that the corresponding intervals are projective. The middle intervals $(x_i, y_i)$ of the corresponding prime intervals under $q$ are the intervals $(x_i, y_i), (c_i, c_{i+1}), (x_i, y_i), (x_{i+1}, y_{i+1}), (e_i, e_{i+1}), (d_i, d_{i+1}), (x_d, y_d)$, which have the demanded property. This completes the proof.

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ЛИНИИ В'НАПРАВЛЕННЫХ ДИСТРИБУТИВНЫХ МУЛЬТИСТРУКТУРАХ

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Резюме

Понятие дистрибутивной мультиструктуры, которым мы пользуемся в этой работе, совпадает с понятием, введенным М. Бенадо [2]. В работе определяется понятие линий в направленной дистрибутивной мультиструктуре при помощи отношения «между». Исследуются некоторые свойства линий (теорема 1, теорема 2, теорема 3). Показывается далее, что для линий в направленной дистрибутивной мультиструктуре справедлива теорема Жордана-Гёльдера.