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GEODETIC GRAPHS WHICH ARE HOMEOMORPHIC TO COMPLETE GRAPHS

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This paper is a contribution to the paper by J. Plesník [2]. Finite undirected graphs without loops and multiple edges are considered.

A geodetic graph [1] is a graph in which to any two vertices exactly one path of the minimal length connecting them exists; this path is called the geodesic of these two vertices.

A graph G homeomorphic to a complete graph with n vertices has the following structure. The vertex set V of G contains a subset V_0 of the cardinality n whose elements are called basic vertices of G. To any two vertices of V_0 there exists exactly one simple path which connects these two vertices and whose inner vertices (if any) belong to $V - V_0$; this path is called the segment connecting these vertices. Any vertex of $V - V_0$ and any edge of the graph G belongs exactly to one segment. This means that each vertex of $V - V_0$ has degree two. The class of all graphs homeomorphic to a complete graph with n vertices for a fixed n will be denoted by $\mathfrak{K}_n$.

J. Plesník has studied a certain subclass of $\mathfrak{K}_n$ which will be denoted here by $\mathfrak{K}_n^*$ (in [2] a different notation is used). Let $G \in \mathfrak{K}_n$, let the basic vertices of G be $v_1, \ldots, v_n$. By s_{ij} the length of the segment connecting v_i and v_j in G will be denoted, where $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i \neq j$. The graph G belongs to $\mathfrak{K}_n^*$, if and only if there exist non-negative integers $h_1, \ldots, h_n$ such that $s_{ij} = h_i + h_j + 1$ for each i and j such that $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i \neq j$.

J. Plesník has suggested the following conjecture:

If a geodetic graph is homeomorphic to a complete graph with n vertices for $n \geqq 4$, then it belongs to $\mathfrak{K}_n^*$.

He has proved this for $n = 4$.

We shall prove a weakened variant of this conjecture.

Theorem. *Let G be a geodetic graph from $\mathfrak{K}_n$ for some positive integer n. For any two basic vertices of G let the geodesic connecting them be the segment connecting them. Then $G \in \mathfrak{K}_n^*$.*

Before proving this theorem we shall state two lemmas.

Lemma 1. *Let s_{12}, s_{23}, s_{13} be three positive integers such that $s_{12} < s_{23} + s_{13}$, $s_{23} < s_{12} + s_{13}$, $s_{13} < s_{12} + s_{23}$. Then there exist non-negative numbers h_1, h_2, h_3 such* that

$$(1) \qquad \left.\begin{aligned} h_1 + h_2 + 1 &= s_{12}, \\ h_2 + h_3 + 1 &= s_{23}, \\ h_1 + h_3 + 1 &= s_{13}. \end{aligned}\right\}$$

These numbers h_1, h_2, h_3 are uniquely determined.

Proof. The system of equations (1) has the unique solution

$$(2) \qquad \left.\begin{aligned} h_1 &= \tfrac{1}{2}(s_{12} + s_{13} - s_{23} - 1), \\ h_2 &= \tfrac{1}{2}(s_{12} + s_{23} - s_{13} - 1), \\ h_3 &= \tfrac{1}{2}(s_{13} + s_{23} - s_{12} - 1). \end{aligned}\right\}$$

As $s_{23} < s_{12} + s_{13}$, the number $s_{12} + s_{13} - s_{23}$ is positive. As the numbers s_{12}, s_{23}, s_{13} are integers, we have $s_{12} + s_{13} - s_{23} \geqq 1$ and thus $s_{12} + s_{13} - s_{23} - 1 \geqq 0$ and $h_1 \geqq 0$. Analogously $h_2 \geqq 0$, $h_3 \geqq 0$.

Lemma 2. *Let G be a geodetic graph from $\mathfrak{K}_n$ for some positive integer n. For any two basic vertices of G let the geodesic connecting them be the segment connecting them. Let G' be the subgraph of G consisting of some m basic vertices of G, $m < n$, and all segments connecting pairs of these vertices. Then G' is a geodetic graph from $\mathfrak{K}_m$ and for any two basic vertices of G' the geodesic connecting them is the segment connecting them.*

Proof. It is easy to see that $G' \in \mathfrak{K}_m$. For proving the rest of the assertion it is sufficient to prove that for any two vertices of G' the geodesic connecting them in G is also a path of minimal length connecting them in G'. Let u, v be two vertices of G', let P be the geodesic connecting u and v in G. If P does not contain basic vertices of G, then u and v both lie on the same segment of G; as u and v are in G', also this segment is in G' and the path P is in G'. It is a geodesic connecting u and v in G', because no path in G' can have length smaller than or equal to the length of P; such a path would be contained also in G, which would be a contradiction. Now let P contain at least one basic vertex of G. Let u_0 (or v_0) be the basic vertex of G lying on P whose distance from u (or v respectively) is minimal. (We admit the cases $u = u_0$, $v = v_0$, $u_0 = v_0$.) If $u_0 \neq v_0$, then the geodesic connecting u_0 and v_0 in G is the segment connecting them. This segment must be contained in P; otherwise P would not be the geodesic connecting u and v. Thus P does not contain other basic vertices of G than u_0 and v_0. The vertex u lies on a segment whose end vertex is u_0, therefore u_0 must be in G'. Analogously v_0 must be in G' and also the segment connecting u_0 and v_0 must be in G'. The path P consists of one segment and two parts of segments which are in G', therefore it is in G'. If $u_0 = v_0$, the proof is analogous.

Proof of Theorem. For $n=1$ and $n=2$ the assertion is trivial; for $n=1$ the graph G consists of only one vertex, for $n=2$ the graph G is a simple path. For $n=3$ the graph G is a circuit. It is well-known that a circuit is a geodetic graph if and only if its length is odd. In such a graph we cannot distinguish basic vertices from the others, because all vertices have degree two. But suppose that we have a graph G satisfying the assumption of the theorem for $n=3$ and that its basic vertices are labelled in it. (This does not make our work easier by any way.) We have three basic vertices v_1, v_2, v_3. As for any two of them the geodesic connecting them is a segment, we must have $s_{12}<s_{13}+s_{23}$, $s_{23}<s_{12}+s_{13}$, $s_{13}<s_{12}+s_{23}$. By Lemma 1 there exist numbers h_1, h_2, h_3 satisfying (1); these numbers are given by (2) and are non-negative. It remains to determine when they are integers. The difference between the numbers $s_{12}+s_{13}+s_{23}$ and $s_{12}+s_{13}-s_{23}-1$ is $2s_{23}+1$, which is an odd number. Thus $s_{12}+s_{13}-s_{23}-1$ is even if and only if $s_{12}+s_{13}+s_{23}$ is odd; in this case h_1 is an integer. But $s_{12}+s_{13}+s_{23}$ is the length of the circuit G. Therefore if it is odd, the assertion is true; in the opposite case G is not geodetic. Analogously for h_2 and h_3. For $n=4$ the assertion was proved in [2]. Now for $n \geqq 5$ we can use the induction. Let $G \in \mathfrak{K}_m$ for $m \geqq 5$ and let G fulfill the assumption of the theorem. Let the basic vertices of G be $v_1, \ldots, v_m$. Suppose that the assertion is true for each $n \leqq m-1$. Let G_1 (or G_2, or G_3) be the subgraph of G obtained by deleting v_1 (or v_2, or v_3) and all sements connecting v_1 (or v_2, or v_3 respectively) with other basic vertices. According to Lemma 2 the graphs G_1, G_2, G_3 are in $\mathfrak{K}_{m-1}$ and fulfill the assumption of the theorem. According to the induction assumption the assertion of the theorem is true for G_1, G_2, G_3. For G_1 we can determine $h_2, h_3, \ldots, h_m$, for G_2 we can determine $h_1, h_3, \ldots, h_m$. If $3 \leqq i \leqq m$, then h_i is the same in G_1 and in G_2. In fact, the graph G has at least five vertices, therefore we can take some j and k from the numbers $3, \ldots, m$ such that $i \neq j \neq k \neq i$. The vertices v_i, v_j, v_k are in both G_1 and G_2, the lengths s_{ij}, s_{ik}, s_{jk} are the same in G_1 and G_2, thus by Lemma 1 so are h_i, h_j, h_k. Analogously we can determine $h_1, h_2, h_4, \ldots, h_m$ in G_3 and prove that they are the same as the corresponding numbers in G_1 and G_2. Each segment of G is contained at least in one of the graphs G_1, G_2, G_3. Thus the assertion is true also for G.

REFERENCES

[1] ORE, O.: Theory of Graphs. Providence 1962.

[2] PLESNÍK, J.: Two constructions of geodetic graphs. Math. Slovaca 27, 1977, 65—71.

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ГЕОДЕТИЧЕСКИЕ ГРАФЫ, КОТОРЫЕ ГОМЕОМОРФНЫ ПОЛНЫМ ГРАФАМ

Богдан Зелинка

Резюме

Статья изучает одну гипотезу Я. Плесника касаюшуюся геодетических графов, которые гомеоморфны полным графам. Доказан ослабленный вариант этой гипотезы.