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ASYMPTOTIC BEHAVIOUR
OF SOME MARKOV OPERATORS
APPEARING IN MATHEMATICAL MODELS
OF BIOLOGY

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ABSTRACT. A class of Markov operators satisfies the Foguel alternative if its members are either sweeping or have stationary densities. We show that this alternative holds for some integral Markov operators appearing in mathematical models of biology.

1. Introduction

Let $K: L_1(X) \rightarrow L_1(X)$ be an integral Markov operator of the form:

$$Kf(x) = \int_X K(x, y)f(y) \, dy,$$

where $K(x, y)$ defined on $X \times X$ is a kernel. Such operators were intensively studied. In [1], [4], [6], [7] some sufficient conditions for sweeping (see Definition 3.1) and asymptotical stability were given. It was proved in [4] that, under the assumption of having subinvariant locally integrable function, the alternative of sweeping or having stationary density holds. The condition without the assumption of the existence of a subinvariant locally integrable function for operators satisfying some property (P) was given in [3]. The main result of this paper is the proof of the Foguel alternative for operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} \left(-\frac{\partial}{\partial x}H(\lambda(x)) - Q(y)\right)f(y) \, dy,$$

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where $Q, \lambda, -H$ are nonnegative, nondecreasing, absolutely continuous functions on $\mathbb{R}^+$ satisfying:

$$H(0) = 1, \quad \lim_{x \to \infty} H(x) = 0,$$

$$Q(0) = \lambda(0) = 0, \quad \lim_{x \to \infty} Q(x) = \lim_{x \to \infty} \lambda(x) = \infty.$$ 

Operators of this type need not satisfy the property (P). The asymptotic behaviour of operators of the form (1.2) has many practical applications in biology.

In Section 2, some necessary results of [2] are presented. In Section 3, the main result (Theorem 3.2) is proved.

## 2. Some properties of Markov processes and integral Markov operators

Theorems 2.1–2.4 are proved in [2].

**Definition 2.1.** A Markov process is defined to be a quadruple $(X, \Sigma, m, P)$, where $(X, \Sigma, m)$ is a $\sigma$-finite measure space with positive measure and where $P$ is an operator on $L_1(X)$ satisfying

(i) $P$ is a contraction: $\|P\| \leq 1$,

(ii) $P$ is positive: if $0 \leq u \in L_1(X)$ then $Pu \geq 0$.

**Definition 2.2.** If $u$ is an arbitrary non-negative function, set $Pu := \lim_{k \to \infty} Pu_k$ for $0 \leq u_k \in L_1(X)$, $u_k \nearrow u$, where the symbol $\nearrow$ denotes monotone pointwise convergence almost everywhere.

The sequence $Pu_k$ is increasing so that $\lim Pu_k$ exists (it may be infinite). By [2] the definition of $Pu$ is independent of the particular sequence $u_k$.

**Definition 2.3.** Take $u_0 \in L_1(X)$ with $u_0 > 0$. Define

$$C = \left\{ x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty \right\}, \quad D = X \setminus C.$$

By [2] this definition is independent of the choice of $u_0$.

**Theorem 2.1.** If $0 \leq u \in L_1(X)$ then

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \quad \text{for } x \in D, \quad \sum_{k=0}^{\infty} P^k u(x) = 0 \text{ or } \infty \quad \text{for } x \in C.$$
**Definition 2.4.** A function $K(x, y) \geq 0$ defined on $X \times X$ which is jointly measurable with respect to its variables is called a kernel. If $\int_{X} K(x, y) \, dx = 1$, then $K$ is called a stochastic kernel.

Stochastic kernel defines an operator on $L_1(X)$:

$$
Kf(x) = \int_{X} K(x, y)f(y) \, dy
$$

with $\|K\| = 1$. So $(X, \Sigma, m, K)$ is a Markov process.

**Definition 2.5.** Let $P$ be an integral Markov operator, then $(X, \Sigma, m, P)$ is said to be a Harris process if $X = C$.

**Theorem 2.2.** Let $K$ be an integral Markov operator and a Harris process. Then there exists $0 < u < \infty$ such that $Ku = u$ (a $\sigma$-finite invariant measure).

**Theorem 2.3.** Let $P$ be a Markov process with $X = D$. Then there exists $0 < g < \infty$ such that $Pg \leq g$.

Proof. Let $0 < u_0 \in L_1(X)$. Set $g = \sum_{k=0}^{\infty} P^k u_0$.

**Definition 2.6.** Let $P$ be a Markov process. Define operators $P_C, P_D$:

$$
P_C : L_1(C) \to L_1(C), \quad P_C f = (P \tilde{f}) \upharpoonright C,
$$

where the symbol $\upharpoonright$ denotes the restriction to the set $C$, $\tilde{f}$ is the function $f$ extended by 0 on $D$,

$$
P_D : L_1(D) \to L_1(D), \quad P_D f = (P \tilde{f}) \upharpoonright D,
$$

where $\tilde{f}$ is the function $f$ extended by 0 on $C$.

**Theorem 2.4.** Let $P$ be a Markov process. If $f \subseteq C$, then $\text{supp } Pf \subseteq C$ ($\text{supp } f = \{x : f(x) \neq 0\}$).

**Corollary 2.1.** Let $K$ be an integral Markov operator. Then 

$$(C, \Sigma \upharpoonright C, m \upharpoonright C, K_C)$$

is a Harris process. $(\Sigma \upharpoonright C$ denotes the $\sigma$-algebra restricted to the space $C$, $m \upharpoonright C$ denotes the measure $m$ restricted to the space $\Sigma \upharpoonright C$).

Proof. By Theorem 2.4, $\text{supp } f \subseteq C$ implies $\text{supp } Kf \subseteq C$. By Theorem 2.1, for $u > 0$ on $C$, $u = 0$ on $D$:

$$
\infty = \sum_{k=0}^{\infty} K^k u(x) = \sum_{k=0}^{\infty} K_C^k (u \upharpoonright C)(x)
$$

for every $x \in C$. 

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**COROLLARY 2.2.** Let $P$ be a Markov process on $L_1(X)$. Then

$$P_D(f \mid D) = (Pf) \mid D.$$ 

**Proof.** $f = f_D + f_C$, where $f_C = f \cdot 1_C$, $f_D = f \cdot 1_D$. By Theorem 2.4, $(Pf_C) \mid D = 0$, hence

$$(Pf) \mid D = (Pf_D) \mid D = P_D(f \mid D).$$

□

**COROLLARY 2.3.** We have

$$P^n_D(f \mid D) = (P^n f) \mid D.$$ 

**COROLLARY 2.4.** Let $P$ be a Markov process on $X$, let $u > 0$ on $D$. Then

$$\sum_{n=0}^{\infty} P^n_D u < \infty.$$ 

**Proof.** Let $\tilde{u}$ be a function on $X$ such that $\tilde{u} \mid C = 0$, $\tilde{u} \mid D = u$. By Corollary 2.3,

$$\sum_{n=0}^{\infty} P^n_D u = \left( \sum_{n=0}^{\infty} P^n \tilde{u} \right) \mid D.$$ 

By Theorem 2.1, $\left( \sum_{n=0}^{\infty} P^n \tilde{u} \right) \mid D < \infty$. □

3. The Foguel alternative for integral Markov operators of the form (1.2)

**DEFINITION 3.1.** Let a family $A \subset \Sigma$ be given. A Markov process is called **sweeping with respect to** $A$, if

$$\lim_{n \to \infty} \int_A P^n f \ dm = 0$$

for $A \in A$ and $f \in D$ ($D = \{ f \in L_1(X), \ ||f|| = 1, \ f \geq 0 \}$)

In the sequel we shall assume that $A$ satisfies the following properties:

(i) $0 < m(A) < \infty$ for $A \in A$,

(ii) $A_1, A_2 \in A$ implies $A_1 \cup A_2 \in A$,

(iii) there exists a sequence $\{A_n\} \subseteq A$ such that $\bigcup A_n = X$.

A family satisfying (i)–(iii) will be called **admissible**.
**DEFINITION 3.2.** Let \((X, \Sigma, m)\) and an admissible family \(A \subseteq \Sigma\) be given. A measurable function \(f: X \to \mathbb{R}\) is called **locally integrable**, if

\[
\int_A |f| \, dm < \infty \quad \text{for} \quad A \in A.
\]

The following theorem is proved in [4].

**THEOREM 3.1.** Let a measure space \((X, \Sigma, m)\), an admissible family \(A\) and an integral Markov operator \(K\) be given. If \(K\) has no invariant density but there exists a positive locally integrable function \(f_*\) subinvariant with respect to \(K\), then \(K\) is sweeping.

**REMARK 3.1.** Theorem 3.1 was proved in [4] for stochastic kernel operators \((\int K(x,y) \, dx = 1)\). But the proof is exactly the same for integral Markov operators.

Let \(K\) be an integral Markov operator. Recall the definition of \(K_C\) and \(K_D\) (see Definition 2.6). By Corollary 2.1, \(K_C\) is a Harris process and by Corollary 2.4, \(K_D\) is dissipative \((X = D)\). By Theorem 2.2 and Theorem 2.3, there exist \(g_C, g_D\) such that \(K_C g_C = g_C\) and \(K_D g_D \leq g_D\). The following two lemmas (3.1 and 3.2) claim that \(g_C\), resp. \(g_D\) are locally integrable in all points \(y \in C\) (resp. \(y \in D\)) such that

\[
\int K_C(x,y) \, dm(x) > 0 \quad \text{(resp.} \int K_D(x,y) \, dm(x) > 0).\]

Denote by \(\mathbb{R}^+\) the set \([0, \infty)\) and by \(T\) the Euclidean metric topology on \(\mathbb{R}^+\).

**Lemma 3.1.** Let \(K\) be an integral Markov operator of the form (1.2), let \(y \in \mathbb{R}^+\). Let \(0 < g < \infty\) and \(K_C g \leq g\). Let

\[
\int C K_C(x,y) \, dm(x) > 0.
\]

Then there exists an open neighbourhood \(U_0\) of \(y\) such that

\[
\int \int \int g(z) \, dz < \infty.
\]

**Proof.** Let

\[
\forall U_y \in T, \quad y \in U_y \quad \int \int \int g(z) \, dz = \infty.
\]
Let $B = \{x \in C : K(x,y) > 0\}$. Let $E \subseteq B$ and $m(E) > 0$. Then
\[ \int_E g(x) \, dx \geq \int_E \int_{U_y \cap C} g(z) K(x,z) \, dz \, dx \]
\[ = \int_{U_y \cap C} g(z) \int_E K(x,z) \, dx \, dz. \tag{3.1} \]

Since
\[ K(x,y) = q(\lambda(x)) \cdot \lambda'(x) h(Q(\lambda(x)) - Q(y)) \]
and $Q(y)$ is absolutely continuous,
\[ \int_E K(x,z) \, dx = \int h(t - Q(z)) \, dt \]
is continuous with respect to $z$. By the assumption there exists $\varepsilon > 0$ such that
\[ \int_E K(x,y) \, dx > \varepsilon > 0. \]

Since $\int E K(x,z) \, dx$ is continuous with respect to $z$, there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and
\[ \forall z \in U_y \quad \int_E K(x,z) \, dx > \varepsilon. \]

Now (3.1) and $\int_{U_y \cap C} g(z) \, dz = \infty$ imply that
\[ \int_E g(x) \, dx = \infty. \]

$E \subseteq B$ was arbitrary, so $g(x) = \infty$ on the set $B$. But by the assumption $0 < g < \infty$. \hfill \Box

**Lemma 3.2.** Let $K$ be an integral Markov operator of the form (1.2), let $y \in \mathbb{R}^+$. Let $0 < g < \infty$ and $K_D g \leq g$. Let
\[ \int_D K_D(x,y) \, dm(x) > 0. \]

Then there exists an open neighbourhood $U_0$ of $y$ such that
\[ \int_{U_0 \cap D} g(z) \, dz < \infty. \]

The proof of Lemma 3.2 is the same as the proof of Lemma 3.1.
THEOREM 3.2. Let $K$ be an integral Markov operator of the form (1.2). Let $A$ be the family of compact subsets of $\mathbb{R}^+$ (with respect to the Euclidean metric topology). If $K$ has no stationary density, then $K$ is sweeping with respect to $A$.

Proof. Denote
\[
\tilde{K}_C f = (K f) \cdot 1_C, \quad \tilde{K}_D f = (K f) \cdot 1_D,
\]
\[
f_C = f \cdot 1_C, \quad f_D = f \cdot 1_D.
\]
Now
\[
\|\tilde{K}_D f_D\| = \|K \tilde{K}_D f_D\| = \|\tilde{K}_C \tilde{K}_D f_D\| + \|\tilde{K}_D f_D\|,
\]
hence
\[
\|\tilde{K}_C \tilde{K}_D f_D\| = \|\tilde{K}_D f_D\| - \|\tilde{K}_D f_D\|^1,
\]
\[
\sum_{l=k}^{n} \|\tilde{K}_C \tilde{K}_D f_D\| = \|\tilde{K}_D f_D\| - \|\tilde{K}_D f_D\|^n. \quad (3.2)
\]

LEMMA 1. Let $y \in \mathbb{R}^+$. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and
\[
\lim_{n \to \infty} \int_{U_y \cap D} K_D^n f \, dm = 0
\]
for every $f \in L_1(D)$.

Proof. By Corollary 2.4,
\[
0 < \sum_{n=0}^{\infty} K_D^n u(x) < \infty
\]
for $u > 0$, hence the process $K_D$ is dissipative. By Theorem 2.3, there exists a $\sigma$-finite subinvariant measure $\lambda$ equivalent to $m \upharpoonright D$.

Let $A_\lambda$ be the family of all sets of finite measure (with respect to $m$) such that
\[
\forall A \in A_\lambda \quad \int_A \frac{d\lambda}{dm} \, dm < \infty.
\]
Since $\frac{d\lambda}{dm} < \infty$, the family $A_\lambda$ is admissible. $K_D$ is dissipative, hence by Theorem 3.1, $K_D$ is sweeping with respect to $A_\lambda$. Let $y$ be such that for every neighbourhood $U \in \mathcal{T}$ of $y$ the set $D \cap U$ has positive measure. Denote $g = \frac{d\lambda}{dm}$. Let
\[
\int_D K(x, y) \, dx > 0.
\]
By Lemma 3.2, there exists $U_y \in \mathcal{T}$ such that 
\[ \int_{U_y \cap D} g(x) \, dx < \infty, \]

hence 
\[ U_y \cap D \in \mathcal{A}_\lambda, \quad \lim_{n \to \infty} \int_{U_y \cap D} K^n_D f \, dm = 0. \]

Let \( \int_D K(x, y) \, dx = 0 \). Let 
\[ \lim_{n \to \infty} \int_{U_y \cap D} K^n_D (f \upharpoonright D) \neq 0 \]

for all $U_y \in \mathcal{T}$ such that $y \in U_y$ and some $f \in L_1(\mathbb{R}^+)$. Now \( \int_C K(x, y) \, dx = 1 \). 

Since \( \int_C K(x, y) \, dx \) is continuous with respect to $y$ (see the proof of Lemma 3.1), there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and 
\[ \forall z \in U_y \quad \int_C K(x, z) \, dx > \varepsilon > 0. \]

By the assumption there exists $\delta > 0$ such that 
\[ \int_{U_y \cap D} K^n_D (f \upharpoonright D) > \delta \]

for infinitely many $n$. By Corollary 2.3, 
\[ K^n_D (f \upharpoonright D) = (\tilde{K}^n_D f_D) \upharpoonright D. \]

Then 
\[ \int_C \tilde{K}_C \tilde{K}^n_D f_D(x) \, dx \geq \int_{U_y \cap D} \int_C K(x, z) \tilde{K}^n_D f_D(z) \, dz \, dx \]
\[ = \int_{U_y \cap D} \tilde{K}^n_D f_D(z) \int_C K(x, z) \, dx \, dz \]
\[ \geq \varepsilon \int_{U_y \cap D} \tilde{K}^n_D f_D(z) \, dz \geq \varepsilon \cdot \delta \]

for infinitely many $n$. Hence 
\[ \sum_{n=0}^{\infty} \| \tilde{K}_C \tilde{K}^n_D f_D \| \geq \sum_{n=0}^{\infty} \int_C \tilde{K}_C \tilde{K}^n_D f_D(x) \, dx = \infty \]

which contradicts (3.2).
LEMMA 2. Let \( y \in \mathbb{R}^+ \), let \( K_C \) has no stationary density. Then there exists \( U_y \in \mathcal{T} \) such that \( y \in U_y \) and 
\[
\lim_{n \to \infty} \int_{U_y \cap C} K^n_C f \ dm = 0
\]
for every \( f \in L_1(C) \).

Proof. By Corollary 2.1 and Theorem 2.2, \( K_C \) is Harris and there exists a function \( g, 0 < g < \infty \) such that \( K_C g = g \).

Let \( y \) be such that for every neighbourhood \( U \in \mathcal{T} \) of \( y \) the set \( C \cap U \) has a positive measure. Since \( \int_{\mathbb{R}^+} K(x, y) \ dx = 1 \) and by Corollary 2.2 \( K(x, y) = 0 \) for \( x \in D, y \in C \),
\[
\int_C K(x, y) \ dx = 1.
\]
By Lemma 3.1, there exists \( U_y \in \mathcal{T} \) such that \( y \in U_y \) and
\[
\int_{U_y \cap C} g(x) \ dx < \infty. \tag{3.3}
\]
Let \( A_g \) be the family of all sets of finite measure such that
\[
\forall A \in A_g \quad \int_A g \ dm < \infty.
\]
Since \( g < \infty \), the family \( A_g \) is admissible. By (3.3) \( U_y \cap C \in A_g \) and by Theorem 3.1
\[
\forall f \in L_1(C) \quad \lim_{n \to \infty} \int_{U_y \cap C} K^n_C f \ dm \to 0.
\]
\[\square\]

LEMMA 3. Let \( K_C \) has no stationary density, let \( A \in A \). Then
\[
\lim_{n \to \infty} \int_{A \cap C} K^n_C f_1 \ dm = 0, \quad \lim_{n \to \infty} \int_{A \cap D} K^n_D f_2 \ dm = 0 \tag{3.4}
\]
for every \( f_1 \in L_1(C), f_2 \in L_1(D) \).

Proof. Let \( y \in \mathbb{R}^+ \). By Lemma 1, there exists \( U_1 \in \mathcal{T} \) such that \( y \in U_1 \) and
\[
\forall f_2 \in L_1(D) \quad \lim_{n \to \infty} \int_{U_1 \cap D} K^n_D f_2 \ dm = 0.
\]
By Lemma 2, there exists $U_2 \in \mathcal{T}$ such that $y \in U_2$ and
\[
\forall f_1 \in L_1(C) \quad \lim_{n \to \infty} \int_{U_2 \cap C} K_n^f f_1 \, dm = 0.
\]

Set $U_y = U_1 \cap U_2$. Then
\[
\lim_{n \to \infty} \int_{U_y \cap C} K_n^f f_1 \, dm = 0, \quad \lim_{n \to \infty} \int_{U_y \cap D} K_n^f f_2 \, dm = 0. \tag{3.5}
\]

Thus we have proved that for every $y \in \mathbb{R}^+$ there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and (3.5) holds. Finally (3.4) follows from compactness of $A$. \hfill \square

By Lemma 3, $K_D$ is sweeping, $K_C$ is sweeping or has a stationary density.

Let $K_C$ have a stationary density $\tilde{f}$. Let $f_\ast$ be a function on $\mathbb{R}^+$ such that $f_\ast \upharpoonright C = \tilde{f}$, $f_\ast \upharpoonright D = 0$. Then
\[
(Kf_\ast) \upharpoonright C = (K(f_\ast \cdot 1_C)) \upharpoonright C + (K(f_\ast \cdot 1_D)) \upharpoonright C = K_C \tilde{f} = \tilde{f}.
\]

By Corollary 2.2, $(Kf_\ast) \upharpoonright D = K_D(f_\ast \upharpoonright D) = 0$, hence $Kf_\ast = f_\ast$. Let $K_C$ be sweeping. We shall prove that $K$ is sweeping.

Let $f \in L_1(\mathbb{R}^+)$, then $f = f_C + f_D$, where $f_C = f \cdot 1_C$, $f_D = f \cdot 1_D$. By Corollary 2.3,
\[
(K^n f_C) \upharpoonright D = 0, \quad (K^n f) \upharpoonright D = K^n_D(f \upharpoonright D).
\]

By Lemma 3,
\[
\forall A \in \mathcal{A} \quad \int_{A \cap D} K^n f \, dm \to 0.
\]

Now it is enough to prove that
\[
\int_{A \cap C} K^n f \, dm \to 0 \quad \text{for} \quad A \in \mathcal{A}.
\]

Clearly
\[
\tilde{K}_C f = \tilde{K}_C(f_C + f_D), \quad K f = \tilde{K}_C f + \tilde{K}_D f,
\]
\[
\tilde{K}_C(Kf) = \tilde{K}_C^2 f_C + \tilde{K}_C^2 f_D + \tilde{K}_C \tilde{K}_D f_D,
\]
\[
\tilde{K}_C(K^2 f) = \tilde{K}_C^3 f_C + \tilde{K}_C^3 f_D + \tilde{K}_C^2 \tilde{K}_D f_D + \tilde{K}_C \tilde{K}_D^2 f_D,
\]
\[\vdots\]
\[
K^n f \cdot 1_C = \tilde{K}_C(K^{n-1} f) = \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \cdots
\]
\[
\cdots + \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \cdots + \tilde{K}_C \tilde{K}_D^{n-1} f_D.
\]

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Take $1 < k < n$ and define:

$$M_{k,n}f = \tilde{K}_C^n f_C + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \cdots + \tilde{K}_C^{n-k+1} \tilde{K}_D^{k-1} f_D,$$

$$R_{k,n}f_D = \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \cdots + \tilde{K}_C \tilde{K}_D^{n-1} f_D.$$ 

$\tilde{K}_C$ is contraction, hence

$$\|R_{k,n}f_D\| \leq \|\tilde{K}_C^{n-k} \tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\|.$$

By (3.2)

$$\|R_{k,n}f_D\| \leq \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D f_D\|.$$ 

The sequence $\{\|\tilde{K}_D^n f\|\}$ is nonincreasing for $\tilde{K}_D$ being contraction. Thus

$$\|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| \leq \frac{\varepsilon}{2} \quad \text{for } n, k \geq n_0(\varepsilon), \ n \geq k.$$ 

Now fix $k \geq n_0(\varepsilon), \ A \in \mathcal{A}$. $\tilde{K}_C$ be sweeping implies

$$\int_{A \cap C} M_{k,n}f \, dm < \frac{\varepsilon}{2}$$

for $n$ sufficiently large, hence

$$\int_{A \cap C} K^n f \, dm \to 0 \quad \text{for } A \in \mathcal{A}.$$

$\square$

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