MALCEV TYPE CONDITIONS FOR TWO VARIETIES

HILDA DRAŠKOVIČOVÁ

Preliminaries. W. Taylor [5] suggests to consider the properties of $n$-tuples of varieties which can be characterized via the existence of polynomial symbols (terms). An example of such a situation is the independence of varieties. The varieties $K_0, K_1, ..., K_{n-1}$ (of the same type) of algebras are independent (cf. [2]) if there is a polynomial symbol $p$ such that for each $i \in \{0,1, ..., n-1\}$ the identity $p(x_0, ..., x_{n-1}) = x_i$ holds in $K_i$. In the present note characterizations of this kind of further properties are given for the case $n=2$ (see the theorems below). To simplify notation we use the same symbol for the polynomial symbol and for its induced polynomials. Let $K_0, K_1$ be varieties of the same type. The smallest variety $K$ containing $K_0$ and $K_1$ will be denoted by $K_0 \vee K_1$. $\mathcal{C}(\mathfrak{A})$ will denote the lattice of all congruence relations on the algebra $\mathfrak{A} = \langle A ; F \rangle$. Given elements $a, b$ of an algebra $\mathfrak{A}$, $\Theta(a, b)$ will denote the smallest congruence relation of $\mathfrak{A}$ containing $(a, b)$. If $\mathfrak{A} \in K_0 \vee K_1$ and $a, b$ are elements of $\mathfrak{A}$, $\Theta'(a, b) (i = 0,1)$ will denote the smallest congruence relation $\Phi$ of $\mathfrak{A}$ such that $(a, b) \in \Phi$ and $\mathfrak{A} / \Phi \in K_i$.

Statement of the results

Theorem 1. Let $K_0, K_1$ be varieties of the same type. The following conditions are equivalent.

1. For each $\mathfrak{A} \in K_0 \vee K_1$ and each $\beta^0, \beta^1 \in \mathcal{C}(\mathfrak{A})$ such that $\mathfrak{A} / \beta^i \in K_i$ ($i = 0, 1$), $\beta^0 \beta^1 = \beta^1 \beta^0$.

2. There is a ternary polynomial symbol $p$ such that
   i. $p(x, x, y) = y$ is an identity of $K_0$,
   ii. $p(x, y, y) = x$ is an identity of $K_1$.

Remark 1. Let us observe that in the case $K_0 = K_1$ we get the known Malcev's result [3].

Theorem 2. Let $K_0, K_1$ be varieties of the same type. The following conditions are equivalent.

3. For each $\mathfrak{A} \in K_0 \vee K_1$ and each $\alpha, \beta^0, \beta^1 \in \mathcal{C}(\mathfrak{A})$ such that $\mathfrak{A} / \beta^i \in K_i$ ($i = 0, 1$), $\beta^0 \beta^1 = \beta^1 \beta^0$ and $\alpha \land (\beta^0 \beta^1) = (\alpha \land \beta^0)(\alpha \land \beta^1)$. 

177
There is a ternary polynomial symbol $q$ such that

(iii) $q(x, x, y) = y = q(y, x, y)$ hold in $K_0$,

(iv) $q(x, y, y) = x = q(x, y, x)$ hold in $K_1$.

Remark 2. In case $K_0 = K_1$ Theorem 2 yields the result of A. F. Pixley [4, Lemma 2.3].

**Theorem 3.** Let $K_0$, $K_1$ be varieties of the same type. The following conditions are equivalent.

(5) The variety $K_0 \cap K_1$ consists of one-element algebras only.

(6) There exist binary polynomial symbols $p_k$, $k = 0, 1, \ldots, n$ such that

(v) $p_0(x, y) = x$ and $p_n(x, y) = y$, 

(vi) $p_k(x, y) = p_{k+1}(x, y)$ holds in $K_0$ for $k$ even,

(vii) $p_k(x, y) = p_{k+x}(x, y)$ holds in $K_1$ for $k$ odd.

As an application of Theorem 1 and Theorem 3 we get a simple proof of the following theorem.

**Theorem W [1, Theorem 1].** Let $K_0$, $K_1$ be varieties of the same type. $K_0$, $K_1$ are independent if and only if the conditions (1) and (5) hold.

### Proofs of the theorems

The proofs of theorems 1 and 2 are similar to the known proofs of the special case of the theorems $K_0 = K_1$.

**Proof of Theorem 1.** Let the condition (2) be satisfied. Let $A = (A; F) \in K_0 \cap K_1$, $b' \in \mathcal{C}(A)$ be such that $A/b' \in K_i$ ($i = 0,1$). It suffices to show that $b'^0 b' \leq b'^i b'$. Let $a, b \in A$, $a b'^0 b$. Then there exists $c \in A$ such that $a b'^0 c$ and $c b'^i b'$. It follows that $a b'^0 p(a, c, b), p(a, c, b) b'^0 b$, hence $a b'^0 b$ and (1) holds. Conversely, let the condition (1) be satisfied. Denote by $\mathcal{F}$ the free algebra over $K_0 \cap K_1$ with three generators $x$, $y$, $z$. Take $\Theta^0(x, y)$, $\Theta^1(y, z) \in \mathcal{C}(\mathcal{F})$. Since $(x, z) \in \Theta^0(x, y) \Theta^1(y, z) = \Theta^1(y, z) \Theta^0(x, y)$ there exists $p(x, y, z)$ in $\mathcal{F}$ such that $x \Theta^1(y, z) p(x, y, z)$ and $p(x, y, z) \Theta^0(x, y, z)$. Since $\mathcal{F}/\Theta^0(x, y)$ ($\mathcal{F}/\Theta^1(y, z)$) is the free algebra over $K_0(K_1)$ with two generators, we get the validity of (i) and (ii), q.e.d.

**Proof of Theorem 2.** Let the condition (3) be satisfied. Let $\mathcal{F}$ be the free algebra over $K_0 \cap K_1$ with three generators $x$, $y$, $z$. Take $\Theta(x, z)$, $\Theta^0(x, y)$, $\Theta^1(y, z) \in \mathcal{C}(\mathcal{F})$. Since $(x, z) \in \theta(x, z) \wedge (\Theta^0(x, y) \Theta^1(y, z)) = (\Theta(x, z) \wedge \Theta^0(x, y)) \Theta^1(y, z)$, there exists $q(x, y, z)$ in $\mathcal{F}$ such that $z \Theta(x, z) \wedge \Theta^0(x, y) q(x, y, z) and q(x, y, z) \Theta^1(y, z)$ holds in $K_0$ (in $K_1$). Since $\mathcal{F}/\Theta(x, z)$ is the free algebra over $K_0 \cap K_1$ with two generators, we get that the
identity \( x = q(x, y, z) \) holds in \( K_i \lor K_i \), hence it holds in \( K_{ii} \) and in \( K_i \) too, i.e. (4) is satisfied. Conversely, let the condition (4) be satisfied. Let \( \mathfrak{A} \in K_i \lor K_i \), take \( \alpha, \beta', \beta' \in \mathcal{C}(\mathfrak{A}) \) such that \( \mathfrak{A}/\beta' \in K_i \), \( i = 0, 1 \). The condition (4) implies (2), hence using Theorem 1 we get \( \beta'' \beta' = \beta' \beta'' \). Let \( a, c \in \mathfrak{A} \) and \( (a, c) \in \alpha \land (\beta'' \beta') \). Then \( \alpha ac \) and there exists \( b \in \mathfrak{A} \) such that \( \alpha b'' b, \beta' c. \) Using (iv) we get \( \alpha \beta'' q(a, b, c), a = q(a, b, a) \alpha q(a, b, c) \), hence \( \alpha (\alpha \land \beta') q(a, b, c) \). Similarly we get \( q(a, b, c) (\alpha \land \beta'') c. \) Hence \( \alpha \land (\beta'' \beta') \subseteq (\alpha \land \beta') (\alpha \land \beta'') \), which implies \( \alpha \land (\beta'' \beta') = (\alpha \land \beta'') (\alpha \land \beta') \).

**Proof of Theorem 3.** Let the condition (5) be satisfied. Let \( \mathfrak{A} \) be the free algebra over \( K_i \lor K_i \) with two generators \( x, y \) and \( \Theta' \) the smallest congruence relations on \( \mathfrak{A} \) such that \( \mathfrak{A}/\Theta' \in K_i \), \( i = 0, 1 \). \( \mathfrak{A}/\Theta' \lor \Theta' \in K_i \lor K_i \), hence \( \Theta' \lor \Theta' \) is the greatest congruence relation of \( \mathfrak{A} \), i.e. \( x(\Theta' \lor \Theta') y \) holds for arbitrary elements \( x, y \) of \( \mathfrak{A} \). It follows that there exists a natural number \( n \) and \( p_n(x, y), p_1(x, y), \ldots, p_n(x, y) \) in \( \mathfrak{A} \) satisfying \( x = p_0(x, y), y = p_n(x, y), p_k(x, y) \Theta' p_{k+1}(x, y) \) for \( k \) even and \( p_0(x, y) \Theta' p_k(x, y) \) for \( k \) odd. Since \( \mathfrak{A}/\Theta' \) is the free algebra over \( K_i \) (\( i = 0, 1 \)) with two generators, the identity \( p_k(x, y) = p_{k+1}(x, y) \) holds in \( K_i \) for \( k \) even and \( p_k(x, y) = p_{k+1}(x, y) \) holds in \( K_i \) for \( k \) odd, i.e. the condition (6) is satisfied. The converse assertion is obvious.

**Proof of Theorem W.** Let the conditions (1) and (5) be satisfied. With respect to Theorem 1, there exists a ternary polynomial symbol \( p \) satisfying (i) and (ii). According to Theorem 3 there exist binary polynomial symbols \( p_0, \ldots, p_n \) satisfying the conditions (v), (vi), (vii). We shall show by induction on \( n \) that \( K_n \) are independent, i.e. there exists a binary polynomial symbol \( \tau \) such that \( \tau(x, y) = x \) holds in \( K_0 \) and \( \tau(x, y) = y \) holds in \( K_1 \). The case \( n = 2 \) is trivial. For \( n = 3 \) define \( \tau(x, y) = p(y, p(x, y), p_1(x, y)) \). To finish the proof it suffices to show that if \( p_0, \ldots, p_n \) \( (n \geq 4) \) are binary polynomial symbols satisfying (v), (vi) and (vii), then there exist binary polynomial symbols \( s_0, s_1, \ldots, s_{n-2} \) satisfying the conditions (v), (vi), (vii) for \( k \leq n - 2 \) (i.e. \( s_0(x, y) = p_0(x, y), s_{n-2}(x, y) = p_n(x, y) \) and for \( k < n - 2 \) \( s_k(x, y) = s_{k+1}(x, y) \) holds in \( K_0 \) for \( k \) even and \( s_k(x, y) = s_{k+1}(x, y) \) holds in \( K_1 \) for \( k \) odd). Such polynomial symbols can be defined as follows: \( s_0(x, y) = p_0(x, y), s_1(x, y) = p(p_1(x, y), p_2(x, y), p_1(x, y)) \) and for \( 1 < k \leq n - 2 \) \( s_k(x, y) = p_{k+1}(x, y) \). Hence \( K_0 \) and \( K_1 \) are independent. Conversely, let \( K_0, K_1 \) be independent and let \( \tau \) be a binary polynomial symbol such that \( \tau(x, y) = x \) holds in \( K_0 \) and \( \tau(x, y) = y \) holds in \( K_1 \). Then (5) trivially holds. According to Theorem 1 to show (1) it suffices to check that the polynomial symbol \( p(x, y, z) = \tau(y, z, x) \) satisfies (i) and (ii).
УСЛОВИЯ ТИПА МАЛЬЦЕВА ДЛЯ ДВУХ МНОГООБРАЗИЙ

Гильда Драшковичова

Резюме

Пусть $K_0$, $K_1$ многообразия алгебр одинакового типа. Для $k \in \{1, 2, 3\}$ условия $(ka)$ ($ko$) эквивалентны, где

(1a) Для всякой алгебры $A \in K_0 \vee K_1$ конгруэнции $\beta^i$, $\beta'$ на $A$ такие, что $A/\beta^i \in K_i$, $i = 0, 1$, перестановочны.

(16) Существует полиномиальный символ $p$ так, что $p(x, x, y) = y$ в $K_0$ и $p(x, y, y) = x$ в $K_1$.

(2a) Для всякой алгебры $A \in K_0 \vee K_1$ и всяких конгруэнций $\alpha$, $\beta^i$, $\beta'$ на $A$, таких, что $A/\beta^i \in K_i$ ($i = 0, 1$), имеет место $\beta^i \beta' = \beta' \beta^i$ и $\alpha \wedge (\beta^i \beta') = (\alpha \wedge \beta')(\alpha \wedge \beta')$.

(26) Существует полиномиальный символ $q$ так, что $q(x, x, y) = y = q(y, x, y)$ в $K_0$ и $q(x, y, y) = y = q(x, y, x)$ в $K_1$.

(3а) Многообразие $K_0 \wedge K_1$ содержит только одноэлементные алгебры.

(36) Существуют бинарные полиномиальные символы $p_0$, ..., $p_3$ такие, что $p_0(x, y) = x$ и $p_3(x, y) = y$ и тождество $p_k(x, y) = p_k(x, y)$ имеет место в $K_0$ для $k$-чётных и в $K_1$ для $k$-нечётных.