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MAĽCEV TYPE CONDITIONS FOR TWO VARIETIES

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Preliminaries. W. Taylor [5] suggests to consider the properties of n-tuples of varieties which can be characterized via the existence of polynomial symbols (terms). An example of such a situation is the independence of varieties. The varieties $K_0, K_1, \ldots, K_{n-1}$ (of the same type) of algebras are *independent* (cf. [2]) if there is a polynomial symbol p such that for each $i \in \{0,1, \ldots, n-1\}$ the identity $p(x_0, \ldots, x_{n-1}) = x_i$ holds in K_i. In the present note characterizations of this kind of further properties are given for the case $n=2$ (see the theorems below). To simplify notation we use the same symbol for the polynomial symbol and for its induced polynomials. Let K_0, K_1 be varieties of the same type. The smallest variety K containing K_0 and K_1 will be denoted by $K_0 \vee K_1$. $\mathscr{C}(\mathfrak{A})$ will denote the lattice of all congruence relations on the algebra $\mathfrak{A} = \langle A ; F \rangle$. Given elements a, b of an algebra $\mathfrak{A}$, $\Theta(a, b)$ will denote the smallest congruence relation of $\mathfrak{A}$ containing (a, b). If $\mathfrak{A} \in K_0 \vee K_1$ and a, b are elements of $\mathfrak{A}$, $\Theta^i(a, b)$ $(i=0,1)$ will denote the smallest congruence relation Φ of $\mathfrak{A}$ such that $(a, b) \in \Phi$ and $\mathfrak{A}/\Phi \in K_i$.

Statement of the results

Theorem 1. *Let K_0, K_1 be varieties of the same type. The following conditions are equivalent.*

(1) *For each $\mathfrak{A} \in K_0 \vee K_1$ and each $\beta^0, \beta^1 \in \mathscr{C}(\mathfrak{A})$ such that $\mathfrak{A}/\beta^i \in K_i$ $(i=0,1)$, $\beta^0\beta^1 = \beta^1\beta^0$.*

(2) *There is a ternary polynomial symbol p such that*
   (i) *$p(x, x, y) = y$ is an identity of K_0,*
   (ii) *$p(x, y, y) = x$ is an identity of K_1.*

Remark 1. Let us observe that in the case $K_0 = K_1$ we get the known Maľcev's result [3].

Theorem 2. *Let K_0, K_1 be varieties of the same type. The following conditions are equivalent.*

(3) *For each $\mathfrak{A} \in K_0 \vee K_1$ and each $\alpha, \beta^0, \beta^1 \in \mathscr{C}(\mathfrak{A})$ such that $\mathfrak{A}/\beta^i \in K_i$ $(i=0,1)$, $\beta^0\beta^1 = \beta^1\beta^0$ and $\alpha \wedge (\beta^0\beta^1) = (\alpha \wedge \beta^0)(\alpha \wedge \beta^1)$.*

(4) *There is a ternary polynomial symbol* q *such that*

(iii) $q(x, x, y) = y = q(y, x, y)$ *hold in* K_0,

(iv) $q(x, y, y) = x = q(x, y, x)$ *hold in* K_1.

Remark 2. In case $K_0 = K_1$ Theorem 2 yields the result of A. F. Pixley [4, Lemma 2.3].

Theorem 3. *Let* K_0, K_1 *be varieties of the same type. The following conditions are equivalent.*

(5) *The variety* $K_0 \wedge K_1$ *consists of one-element algebras only.*

(6) *There exist binary polynomial symbols* p_k, $k = 0, 1, \dots, n$ *such that*

(v) $p_0(x, y) = x$ and $p_n(x, y) = y$,

(vi) $p_k(x, y) = p_{k+1}(x, y)$ *holds in* K_0 *for* k *even,*

(vii) $p_k(x, y) = p_{k+1}(x, y)$ *holds in* K_1 *for* k *odd.*

As an application of Theorem 1 and Theorem 3 we get a simple proof of the following theorem.

Theorem W [1, Theorem 1]. *Let* K_0, K_1 *be varieties of the same type.* K_0, K_1 *are independent if and only if the conditions* (1) *and* (5) *hold.*

Proofs of the theorems

The proofs of theorems 1 and 2 are similar to the known proofs of the special case of the theorems $K_0 = K_1$.

Proof of Theorem 1. Let the condition (2) be satisfied. Let $\mathfrak{A} = \langle A; F\rangle \in K_0 \vee K_1$, $\beta^i \in \mathscr{C}(\mathfrak{A})$ be such that $\mathfrak{A}/\beta^i \in K_i$ $(i = 0,1)$. It suffices to show that $\beta^0\beta^1 \leqq \beta^1\beta^0$. Let $a, b \in A$, $a\beta^0\beta^1 b$. Then there exists $c \in A$ such that $a\beta^0 c$ and $c\beta^1 b$. It follows that $a\beta^1 p(a, c, b)$, $p(a, c, b)\beta^0 b$, hence $a\beta^1\beta^0 b$ and (1) holds. Conversely, let the condition (1) be satisfied. Denote by $\mathfrak{F}$ the free algebra over $K_0 \vee K_1$ with three generators x, y, z. Take $\Theta^0(x, y)$, $\Theta^1(y, z) \in \mathscr{C}(\mathfrak{F})$. Since $(x, z) \in \Theta^0(x, y)\Theta^1(y, z) = \Theta^1(y, z)\Theta^0(x, y)$ there exists $p(x, y, z)$ in $\mathfrak{F}$ such that $x\Theta^1(y, z)p(x, y, z)$ and $p(x, y, z)\Theta^0(x, y)z$. Since $\mathfrak{F}/\Theta^0(x, y)$ $(\mathfrak{F}/\Theta^1(y, z))$ is the free algebra over $K_0(K_1)$ with two generators, we get the validity of (i) and (ii), q.e.d.

Proof of Theorem 2. Let the condition (3) be satisfied. Let $\mathfrak{F}$ be the free algebra over $K_0 \vee K_1$ with three generators x, y, z. Take $\Theta(x, z)$, $\Theta^0(x, y)$, $\Theta^1(y, z) \in \mathscr{C}(\mathfrak{F})$. Since $(z, x) \in \Theta(x, z) \wedge (\Theta^0(x, y)\Theta^1(y, z)) = (\Theta(x, z) \wedge \Theta^0(x, y))(\Theta(x, z) \wedge \Theta^1(y, z))$, there exists $q(x, y, z)$ in $\mathfrak{F}$ such that $z(\Theta(x, z) \wedge \Theta^0(x, y))q(x, y, z)$ and $q(x, y, z)(\Theta(x, z) \wedge \Theta^1(y, z))x$. Since $\mathfrak{F}/\Theta^0(x, y)(\mathfrak{F}/\Theta^1(y, z))$ is the free algebra over K_0 (over K_1) with two generators, we get that the identity $q(x, x, y) = y$ $(q(x, y, y) = x)$ holds in K_0 (in K_1). Since $\mathfrak{F}/\Theta(x, z)$ is the free algebra over $K_0 \vee K_1$ with two generators, we get that the identity $x=q(x,y,x)$ holds in $K_0\vee K_1$, hence it holds in K_0 and in K_1 too, i.e. (4) is satisfied. Conversely, let the condition (4) be satisfied. Let $\mathfrak{A}\in K_0\vee K_1$, take α, β^0, $\beta^1\in\mathscr{C}(\mathfrak{A})$ such that $\mathfrak{A}/\beta^i\in K_i$, $i=0,1$. The condition (4) implies (2), hence using Theorem 1 we get $\beta^0\beta^1=\beta^1\beta^0$. Let a, $c\in\mathfrak{A}$ and $(a,c)\in\alpha\wedge(\beta^0\beta^1)$. Then $a\alpha c$ and there exists $b\in\mathfrak{A}$ such that $a\beta^0 b$, $b\beta^1 c$. Using (iv) we get $a\beta^1 q(a,b,c)$, $a=q(a,b,a)\alpha q(a,b,c)$, hence $a(\alpha\wedge\beta^1)q(a,b,c)$. Similarly we get $q(a,b,c)(\alpha\wedge\beta^0)c$. Hence $\alpha\wedge(\beta^0\beta^1)\leqq(\alpha\wedge\beta^1)(\alpha\wedge\beta^0)$, which implies $\alpha\wedge(\beta^0\beta^1)=(\alpha\wedge\beta^0)(\alpha\wedge\beta^1)$.

Proof of Theorem 3. Let the condition (5) be satisfied. Let $\mathfrak{F}$ be the free algebra over $K_0\vee K_1$ with two generators x, y and Θ^i the smallest congruence relations on $\mathfrak{F}$ such that $\mathfrak{F}/\Theta^i\in K_i$, $i=0,1$. $\mathfrak{F}/\Theta^0\vee\Theta^1\in K_0\wedge K_1$, hence $\Theta^0\vee\Theta^1$ is the greatest congruence relation of $\mathfrak{F}$, i.e. $x(\Theta^0\vee\Theta^1)y$ holds for arbitrary elements x, y of $\mathfrak{F}$. It follows that there exists a natural number n and $p_0(x,y)$, $p_1(x,y)$, ..., $p_n(x,y)$ in $\mathfrak{F}$ satisfying $x=p_0(x,y)$, $y=p_n(x,y)$, $p_k(x,y)\ \Theta^0\ p_{k+1}(x,y)$ for k even and $p_k(x,y)\ \Theta^1\ p_{k+1}(x,y)$ for k odd. Since $\mathfrak{F}/\Theta^i$ is the free algebra over K_i $(i=0,1)$ with two generators, the identity $p_k(x,y)=p_{k+1}(x,y)$ holds in K_0 for k even and $p_k(x,y)=p_{k+1}(x,y)$ holds in K_1 for k odd, i.e. the condition (6) is satisfied. The converse assertion is obvious.

Proof of Theorem W. Let the conditions (1) and (5) be satisfied. With respect to Theorem 1, there exists a ternary polynomial symbol p satisfying (i) and (ii). According to Theorem 3 there exist binary polynomial symbols p_0, ..., p_n satisfying the conditions (v), (vi), (vii). We shall show by induction on n that K_0, K_1 are independent, i.e. there exists a binary polynomial symbol t such that $t(x,y)=x$ holds in K_0 and $t(x,y)=y$ holds in K_1. The case $n=2$ is trivial. For $n=3$ define $t(x,y)=p(y,p_2(x,y),p_1(x,y))$. To finish the proof it suffices to show that if p_0, ..., p_n $(n\geqq 4)$ are binary polynomial symbols satisfying (v), (vi) and (vii), then there exist binary polynomial symbols s_0, s_1, ..., s_{n-2} satisfying the conditions (v), (vi), (vii) for $k\leqq n-2$ (i.e. $s_0(x,y)=p_0(x,y)$, $s_{n-2}(x,y)=p_n(x,y)$ and for $k<n-2$ $s_k(x,y)=s_{k+1}(x,y)$ holds in K_0 for k even and $s_k(x,y)=s_{k+1}(x,y)$ holds in K_1 for k odd). Such polynomial symbols can be defined as follows: $s_0(x,y)=p_0(x,y)$, $s_1(x,y)=p(p_3(x,y),p_2(x,y),p_1(x,y))$ and for $1<k\leqq n-2$ $s_k(x,y)=p_{k+2}(x,y)$. Hence K_0 and K_1 are independent. Conversely, let K_0, K_1 be independent and let t be a binary polynomial symbol such that $t(x,y)=x$ holds in K_0 and $t(x,y)=y$ holds in K_1. Then (5) trivially holds. According to Theorem 1 to show (1) it suffices to check that the polynomial symbol $p(x,y,z)=t(t(z,y),x)$ satisfies (i) and (ii).

REFERENCES

[1] DRAŠKOVIČOVÁ, H.: Independence of equational classes. Mat. Čas. 23, 1973, 125—135.
[2] GRÄTZER, G.—LAKSER, H.—PŁONKA, J.: Joins and direct products of equational classes. Canad. Math. Bull. 12, 1969, 741—744.
[3] MAĽCEV, A. I.: On the general theory of algebraic systems (Russian). Mat. Sb. N.S. 35 (77), 1954, 3—20.
[4] PIXLEY, A. F.: The ternary discriminator function in universal algebra. Math. Ann. 191, 1971, 167—180.
[5] TAYLOR, W.: Characterizing Maľcev conditions. Algebra Universalis 3, 1973, 351—397.

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УСЛОВИЯ ТИПА МАЛЬЦЕВА ДЛЯ ДВУХ МНОГООБРАЗИЙ

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Резюме

Пусть K_0, K_1 многообразия алгебр одинакового типа. Для $k \in \{1, 2, 3\}$ условия (ka) $(k\sigma)$ эквивалентны, где

(1a) Для всякой алгебры $\mathfrak{A} \in K_0 \vee K_1$ конгруэнции β^0, β^1 на $\mathfrak{A}$ такие, что $\mathfrak{A}/\beta^i \in K_i$, $i=0,1$, перестановочны.

(1б) Существует полиномиальный символ p так, что $p(x, x, y) = y$ в K_0 и $p(x, y, y) = x$ в K_1.

(2a) Для всякой алгебры $\mathfrak{A} \in K_0 \vee K_1$ и всяких конгруэнций α, β^0, β^1 на $\mathfrak{A}$, таких, что $A/\beta^i \in K_i$ $(i=0,1)$, имеет место $\beta^0\beta^1 = \beta^1\beta^0$ и $\alpha \wedge (\beta^0\beta^1) = (\alpha \wedge \beta^0)(\alpha \wedge \beta^1)$.

(2б) Существует полиномиальный символ q так, что
$q(x, x, y) = y = q(y, x, y)$ в K_0 и
$q(x, y, y) = y = q(x, y, x)$ в K_1.

(3a) Многообразие $K_0 \wedge K_1$ содержит только одноэлементные алгебры.

(3б) Существуют бинарные полиномиальные символы $p_0, \ldots, p_n$ такие, что $p_0(x, y) = x$ и $p_n(x, y) = y$ и тождество $p_k(x, y) = p_{k+1}(x, y)$ имеет место в K_0 для к-чётных и в K_1 для к-нечётных.