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# ON THE MAYER PROBLEM II. EXAMPLES

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ABSTRACT. Given an underdetermined system of ordinary differential equations, extremals of all possible variational problems relevant to the system together with the corresponding Poincaré-Cartan forms were characterized in geometrical terms in previous Part I of this article. The present Part II demonstrates the utility of this approach: it enables a deep insight into the structure of Euler-Lagrange and Hamilton-Jacobi equations not available by other methods and provides the sufficient extremality conditions without uncertain multipliers similar to the common Hilbert-Weierstrass theory. Degenerate variational problems are in principle not excluded and, like in the "royal road" by Carathéodory, no subtle investigation of admissible variations satisfying the boundary conditions is needed.

## Introduction

Let us overview the main achievements of this paper by using the common terminology. Two underdetermined systems of differential equations are discussed here: the case of a single equation $y'_m = f(x, y_1, \dots, y_m, y'_1, \dots, y'_{m-1})$ and the case of two equations $y'_m = f(x, y_1, \dots, y_m, y'_1, \dots, y'_{m-2})$, $y'_{m-1} = g(x, y_1, \dots, y_m, y'_1, \dots, y'_{m-2})$, where $m \geq 2$ or $m \geq 3$, respectively. For technical reasons, some results are stated only for the particular value $m = 3$.

In the first case, the nondegenerate subcase $\det(\partial^2 f / \partial y'_i \partial y'_j) \neq 0$ $(i, j = 1, \dots, m-1)$ leads to extremals given by a second order system. They are identical with the characteristics of the Pfaffian equation $\omega = 0$, where

$$h\omega = h\Big(\mathrm{d}y_m - f\mathrm{d}x - \sum \frac{\partial f}{\partial y'_i}(\mathrm{d}y_i - y'_i\,\mathrm{d}x)\Big) = h\Big(H\mathrm{d}x - \sum \frac{\partial f}{\partial y'_i}\,\mathrm{d}y_i + \mathrm{d}y_m\Big)$$

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may be regarded as a generalized Poincaré-Cartan ($\mathcal{PC}$) form. Here $h$ is a parameter and $H = -f + \sum y_i' \, \partial f / \partial y_i'$ stands for the generalized Hamilton function. Various extremality problems can be easily discussed in a uniform manner and the Hilbert-Weierstrass criterion is obtained: the existence of appropriate solutions of the Hamilton-Jacobi ($\mathcal{HJ}$) equation together with the convexity of $f$ in $y_1', \dots, y_{m-1}'$ ensures the global extremum. We moreover deal with the generic degenerate case assuming $m = 3$ for better clarity. Then the extremals are given by a first order system. (It seems that this result cannot be easily verified without the use of the $\mathcal{PC}$ forms.) Alas, the usual $\mathcal{HJ}$ equation is insufficient to cope with the extremality problems. (We intend to discuss this remarkable and old-standing problem in future and refer to [1; Parts II, III, IV] dealing with analogous tasks for this time.) Finally the particular degenerate case when the extremals satisfy an underdetermined system (namely $y_3' = f$ and a single second order equation) is mentioned. It is similar to (but more general than) the so called "parametrical problem" of the classical theory (i.e., if the variational integral is independent of the parametrization). It does not cause much difficulties in spite of the fact that the extremals are depending on the choice of one arbitrary function and a $\mathcal{HJ}$ *involutive system* (instead of a single equation) is obtained.

In the case of two differential equations, the results are more instructive if compared with the common point of view. For simplicity, let us refer only to the case $m = 3$ here. Then the functions

$$F = \frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_1'}\frac{\partial f}{\partial y_3} + \frac{\partial g}{\partial y_1'}\frac{\partial f}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_1'}\,, \qquad G = \frac{\partial g}{\partial y_1} + \frac{\partial f}{\partial y_1'}\frac{\partial g}{\partial y_3} + \frac{\partial g}{\partial y_1'}\frac{\partial g}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial y_1'}$$

play the crucial role. If $G\partial^2 f / \partial {y_1'}^2 \neq F \partial^2 g / \partial {y_1'}^2$, the extremals are given by the additional third order equation

$$F\frac{\mathrm{d}G}{\mathrm{d}x} - G\frac{\mathrm{d}F}{\mathrm{d}x} + G^2\frac{\partial f}{\partial y_2} + FG\left(\frac{\partial f}{\partial y_3} - \frac{\partial g}{\partial y_2}\right) - F^2\frac{\partial g}{\partial y_3} = 0$$

(equivalent to $y_1''' +$ lower order terms $= 0$). The generalized $\mathcal{PC}$ form

$$\begin{aligned} &h\Big(G\Big(\mathrm{d}y_3 - f\,\mathrm{d}x - \frac{\partial f}{\partial y_1'}(\mathrm{d}y_1 - y_1'\,\mathrm{d}x)\Big) - F\Big(\mathrm{d}y_2 - g\,\mathrm{d}x - \frac{\partial g}{\partial y_1'}(\mathrm{d}y_1 - y_1'\,\mathrm{d}x)\Big)\Big) \\ &= h\Big(H\,\mathrm{d}x - \Big(G\frac{\partial f}{\partial y_1'} - F\frac{\partial g}{\partial y_1'}\Big)\,\mathrm{d}y_1 - F\,\mathrm{d}y_2 - G\,\mathrm{d}y_3\Big) \end{aligned}$$

with a parameter $h$ and the generalized Hamilton function $H$ provide the $\mathcal{HJ}$ equation and the Hilbert-Weierstrass extremality criterion. The convexity of the function $Gf - Fg$ in the variable $y_1'$ (which is freezed in coefficients $F$, $G$) is needful. This corresponds to the common "nondegenerate case". We also deal with the "degenerate case" assuming that both $f$, $g$ are linear in $y_1'$.

The simplest result is obtained if the functions $f$, $g$ are independent of $y_1'$ and the inequality

$$\frac{\partial f}{\partial y_1}\frac{\partial^2 g}{\partial y_1^2} \neq \frac{\partial g}{\partial y_1}\frac{\partial^2 f}{\partial y_1^2}$$

holds true. Then the extremals are given by an additional first order equation (briefly: $y_1' + \cdots = 0$), the $\mathcal{PC}$ form is retained (it simplifies considerably), a reasonable $\mathcal{HJ}$ equation appears, but the relevant Weierstrass function is a multiple of the function

$$\frac{\partial \bar{g}}{\partial y_1}(r)\big(\bar{f}(y_1) - \bar{f}(r)\big) - \frac{\partial \bar{f}}{\partial y_1}(r)\big(\bar{g}(y_1) - \bar{g}(r)\big)$$

(where $\bar{f}(\cdot) = f(x,\cdot\,,y_2,y_3)$, $\bar{g}(\cdot) = g(x,\cdot\,,y_2,y_3)$ and $x$, $y_2$, $y_3$ are regarded as parameters) which looks very strange.

*A few concluding remarks.*

The nondegenerate variational problems are frequently investigated in actual textbooks, however, the common presence of uncertain coefficients essentially obscures the sense of the results. The degenerate problems were ignored as yet: even the true identification of extremals is hardly possible by the common direct calculations (cf. the surprising identities (20), (21)). The only exception represents the top linear problems appearing in sub-Riemannian geometries [4], [5]. It should be however noted that the classical definition of degenerate variational problems does not make a good sense since it may be highly affected by the choice of coordinates (e.g., the problem of Section 14 can be seen equivalent to the theory of the simplest nondegenerate variational integral $\int f(x,y,y')\,\mathrm{d}x$) by appropriate adaptation of variables.

Some results of this article were already referred to without proofs in [2].

## Preliminaries

### 1. Contact diffieties.

In order to enter into the applications easily, we begin with a short review of classical concepts suitably adapted for our future aims. Let $\mathbf{M}(m)$ be the space (isomorphic to $\mathbb{R}^\infty$, but moreover) equipped with (infinite order) *jet coordinates* $x, w_s^i \in \mathcal{F}\big(\mathbf{M}(m)\big)$ $(i = 1,\dots,m,\ s = 0,1,\dots)$, *contact forms* $\omega_s^i = \mathrm{d}w_s^i - w_{s+1}^i\,\mathrm{d}x \in \Phi\big(\mathbf{M}(m)\big)$, and the *formal derivative vector field*

$$X = \partial/\partial x + \sum w_{s+1}^i \partial/\partial w_s^i \in \mathcal{T}\big(\mathbf{M}(m)\big)\,. \tag{1}$$

Then the submodule $\Omega(m) = \{\omega_s^i\} \subset \Phi\big(\mathbf{M}(m)\big)$ generated by all contact forms clearly is a diffiety with the slope $\mathcal{H}(m) = \Omega(m)^\perp = \{X\} \subset \mathcal{T}\big(\mathbf{M}(m)\big)$ generated by the single vector field (1).

*Hint.*
We refer to [3; Sect. 3]: $\mathcal{L}_X\omega_s^i = \omega_{s+1}^i$, hence

$$\Omega(m)_*\colon \Omega(m)_0 \subset \Omega(m)_1 \subset \cdots \subset \Omega(m) = \bigcup \Omega(m)_l \qquad (\Omega(m)_l = \{\omega_s^i : s \le l\}) \tag{2}$$

is a good filtration.

The solutions $P(t) \in \mathbf{M}(m)$, $0 \le t \le 1$, of $\Omega(m)$ given by the formulae $x = x(t)$, $w_s^i = w_s^i(t)$ satisfy $w_{s+1}^i\,\mathrm{d}x/\,\mathrm{d}t = \mathrm{d}w_s^i/\,\mathrm{d}t$ (equivalently $w_{s+1}^i = \mathrm{d}w_s^i/\,\mathrm{d}x$ if $x'(t) \ne 0$) and it follows that the diffiety $\Omega(m)$ corresponds to the empty system of ordinary differential equations for $m$ unknown functions $w_0^1, \dots, w_0^m$ of one independent variable $x$.

General systems of ordinary differential equations are realized as certain subdiffieties of $\Omega(m)$. More explicitly, let $f^1, \dots, f^c \in \mathcal{F}(\mathbf{M}(m))$ be functions such that the equations $X^l f^j = 0$ ($j = 1, \dots, c$, $l = 0, 1, \dots$) determine a subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(m)$. Then we obtain the induced diffiety $\Omega = \mathbf{m}^*\Omega(m) \subset \Phi(\mathbf{M}(m))$ with the good filtration

$$\Omega_* = \mathbf{m}^*\Omega(m)_*\colon \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \bigcup \Omega_l \qquad (\Omega_l \equiv \mathbf{m}^*\Omega(m)_l) \tag{3}$$

and the slope $\mathcal{H} = \Omega^\perp \subset \mathcal{T}(\mathbf{M}(m))$ generated by the same vector field (1) restricted to $\mathbf{M}$. Denoting a little symbolically $f^j \equiv f^j(x, \dots, w_s^i, \dots)$, the diffiety $\Omega$ corresponds to the system of differential equations $f^j(x, \dots, \mathrm{d}^s w_0^i/\,\mathrm{d}x^s, \dots) \equiv 0$ or, better, to the infinite prolongation $X^l f^j \equiv 0$ with derivatives $\mathrm{d}^s w_0^i/\,\mathrm{d}x^s$ substituted for the jet variables $w_s^i$.

On this occasion, it is necessary to recall the common abbreviations of restrictions like $f = \mathbf{m}^* f$, $\varphi = \mathbf{m}^*\varphi$, especially

$$x = \mathbf{m}^* x\,, \qquad w_s^i = \mathbf{m}^* w_s^i\,, \qquad \omega_s^i = \mathbf{m}^*\omega_s^i \tag{4}$$

and of partial derivatives like

$$f_x = \partial f/\partial x\,, \qquad f_s^i = \partial f/\partial w_s^i\,, \qquad f_{rs}^{ij} = \partial^2 f/\partial w_r^i \partial w_s^j\,.$$

We believe that no confusions will arise.

The contact forms $\omega_s^i \in \Phi(\mathbf{M}(m))$ provide a basis of $\Omega(m)$. The restrictions $\omega_s^i \equiv \mathbf{m}^*\omega_s^i \in \Phi(\mathbf{M})$ (abbreviation (4)) generate $\Omega$ but they are not linearly independent. In more detail: the expansions

$$\mathrm{d}F = F_x\,\mathrm{d}x + \sum F_s^i\,\mathrm{d}w_s^i = XF\,\mathrm{d}x + \sum F_s^i\omega_s^i \in \Phi(\mathbf{M}(m)) \qquad (F \in \mathcal{F}(\mathbf{M}(m)))$$

are clearly true, hence the identity $\mathbf{m}^*F = 0$ (i.e., $F$ vanishing on $\mathbf{M}$ which also implies $\mathbf{m}^*XF = 0$) leads to the linear dependence

$$0 = \mathrm{d}\mathbf{m}^*F = \mathbf{m}^*\,\mathrm{d}F = \sum F_s^i\omega_s^i \in \Phi(\mathbf{M})\,, \tag{5}$$

where $F_s^i = \mathbf{m}^*F_s^i$, $\omega_s^i = \mathbf{m}^*\omega_s^i$ (abbreviations (4)). In particular, $F = X^l f^j$ ($j = 1, \dots, c$, $l = 0, 1, \dots$) may be substituted here which provides a large family of identities (5).

**2. A survey of algorithms.**

The main result of Part I was as follows: for a given diffiety $\Omega \subset \Phi(\mathbf{M})$, the *Euler-Lagrange* ($\mathcal{EL}$) *subspaces* $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ together with *Poincaré-Cartan* ($\mathcal{PC}$) *submodules* $\breve{\Omega} \subset \Omega$ can be explicitly calculated by a certain algorithm. Recall that the restriction $\mathbf{e}^*\Omega \subset \Phi(\mathbf{E})$ is a diffiety (the $\mathcal{EL}$ *diffiety*, a subdiffiety of $\Omega$) and $\breve{\Omega} \subset \Omega$ is a finite-dimensional submodule satisfying the equivalent conditions

$$\mathrm{d}\breve{\Omega} \cong 0 \quad (\mathrm{mod}\, \breve{\Omega},\, \Omega \wedge \Omega), \qquad \mathcal{L}_Z \breve{\Omega} \subset \breve{\Omega} \quad (\, Z \in \Omega^{\perp}\,) \tag{6}$$

*along* $\mathbf{E}$. Since $\mathbf{E}$, $\breve{\Omega}$ can be interpreted in terms of certain variational problems, the solutions of $\mathcal{EL}$ diffieties are called *extremals*. (Alternatively: *extremals* are solutions of $\Omega$ that lie in $\mathbf{E}$.)

The algorithm consists of two steps. First, certain maximal possible subspaces $\tilde{\mathbf{M}} \subset \mathbf{M}$ and submodules $\tilde{\Omega} \subset \Omega$ were determined such that (6) holds true for $\tilde{\Omega}$ (instead of $\breve{\Omega}$) along $\tilde{\mathbf{M}}$. Second, this $\tilde{\mathbf{M}}$ was adapted to obtain the maximal subspace $\mathbf{E} \subset \tilde{\mathbf{M}}$ (hence $\mathbf{E} \subset \mathbf{M}$) such that the restriction of $\Omega$ to $\mathbf{E}$ is a diffiety (the $\mathcal{EL}$ diffiety) and the modules $\breve{\Omega} = \tilde{\Omega}$ were identified along $\mathbf{E}$.

In principle, the second step is quite clear (cf. also [3; Sect. 4] and any of the examples below). Concerning the first step, let us state the main idea. If $\mathbf{N} \subset \mathbf{M}$ is a subset and $\Theta \subset \Omega$ a submodule, we introduce the submodule $\mathrm{Ker}_{\mathbf{N}}\, \Theta \subset \Theta$ of all $\vartheta \in \Theta$ satisfying $\mathcal{L}_Z \vartheta \in \Theta$ ($Z \in \Omega^{\perp}$) along $\mathbf{N}$. This operation $\mathrm{Ker}_{\mathbf{N}}$ is repeatedly applied to certain terms of the filtration (3) which gives the desired subspace $\tilde{\mathbf{M}}$ (which is equal to the last stationary term $\mathbf{N}$) and the desired module $\tilde{\Omega}$ (the least value of the module $\mathrm{Ker}_{\mathbf{N}}$).

**3. Adjoint modules.**

For the convenience of reader, some well-known results concerning the existence of *special* (local) *bases* of certain modules will be briefly stated. (Note that the existence of *some* local bases of all modules under consideration is always tacitly supposed.)

*Let* $\vartheta \in \Phi(\mathbf{M})$ *and consider the submodule*

$$\mathrm{Adj}\, \mathrm{d}\vartheta = \{ Z \rfloor \mathrm{d}\vartheta :\ Z \in \mathcal{T}(\mathbf{M}) \} \subset \Phi(\mathbf{M})\,. \tag{7}$$

*This submodule has a special basis* $\mathrm{d}u^1, \mathrm{d}v^1, \ldots, \mathrm{d}u^c, \mathrm{d}v^c$ ($c \geq 0$) *such that* $\mathrm{d}\vartheta = \mathrm{d}u^1 \wedge \mathrm{d}v^1 + \cdots + \mathrm{d}u^c \wedge \mathrm{d}v^c$. *In particular* $c = \frac{1}{2} \dim \mathrm{Adj}\, \mathrm{d}\vartheta$. *If* $\vartheta \notin \mathrm{Adj}\, \mathrm{d}\vartheta$, *then* $\vartheta = \mathrm{d}u^0 + v^1\, \mathrm{d}u^1 + \cdots + v^c\, \mathrm{d}u^c$ *for appropriate* $u^0 \in \mathcal{F}(\mathbf{M})$.

*Let* $\omega \in \Phi(\mathbf{M})$ *be nonvanishing and consider the submodule*

$$\mathrm{Adj}\{\omega\} = \{ \omega, Z \rfloor d\omega :\ Z \in \mathcal{T}(\mathbf{M}),\ \omega(Z) = 0 \} \subset \Phi(\mathbf{M})\,. \tag{8}$$

*This submodule has a special basis* $\mathrm{d}u^0, \mathrm{d}u^1, \mathrm{d}v^1, \ldots, \mathrm{d}u^c, \mathrm{d}v^c$ ($c \geq 0$) *such that* $h\omega = \mathrm{d}u^0 + v^1\, \mathrm{d}u^1 + \cdots + v^c\, \mathrm{d}u^c$ *for appropriate (nonvanishing) factor* $h \in \mathcal{F}(\mathbf{M})$. *In particular* $c = \frac{1}{2}(\dim \mathrm{Adj}\{\omega\} - 1)$.

By using these results, it follows that the codimension of the greatest subspace $\mathbf{l}\colon \mathbf{L} \subset \mathbf{M}$, the so called *Hamilton-Jacobi* ($\mathcal{HJ}$) *subspace*, satisfying the requirements $\mathbf{l}^* \mathrm{d}\vartheta = 0$, $\mathbf{l}^* \mathrm{d}(h\omega) = 0$, respectively, is clearly equal to $c$. On a simply connected domain, these requirements read

$$\mathbf{l}^*(\vartheta - \mathrm{d}W) = 0\,, \qquad \mathbf{l}^*(h\omega - \mathrm{d}W) = 0\,, \tag{9}$$

respectively, where $W \in \mathcal{F}(\mathbf{M})$ is an unknown function. In the following applications, $\mathbf{E}$ will stand for the space $\mathbf{M}$ and $\vartheta, \omega \in \mathbf{e}^*\breve{\Omega}$ will be the generalized $\mathcal{PC}$ forms. Then the conditions (9) turn into a generalization of the familiar classical $\mathcal{HJ}$ *equation*.

## First example

### 4. One differential equation.

In order to deal with the equation

$$w_1^m = f(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^{m-1})\,,$$

we introduce the subdiffiety $\Omega = \mathbf{m}^*\Omega(m) \subset \Phi(\mathbf{M})$ of the diffiety $\Omega(m)$ on the subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(m)$ of all points that satisfy the equations $X^l(w_1^m - f) \equiv 0$ ($l = 0, 1, \dots$). Clearly

$$X^l(w_1^m - f) = w_{1+l}^m - \sum f_1^j w_{1+l}^j - \cdots$$

as the top order terms are concerned, therefore the functions

$$x\,,\ w_0^i\,,\ w_s^j \qquad (i = 1, \dots, m\,,\ j = 1, \dots, m-1\,,\ s = 1, 2, \dots)$$

provide the coordinate system on $\mathbf{M}$. Then

$$X = \partial/\partial x + \sum w_1^i \partial/\partial w_0^i + \sum w_{s+1}^j \partial/\partial w_s^j \in \Omega^\perp \subset \mathcal{T}(\mathbf{M}) \qquad (w_1^m = f)$$

in terms of these coordinates and the forms

$$\omega_0^i = \mathbf{m}^*\omega_0^i\,, \quad \omega_s^j = \mathbf{m}^*\omega_s^j \qquad (i = 1, \dots, m\,,\ j = 1, \dots, m-1\,,\ s = 0, 1, \dots)$$

provide a basis of $\Omega$ (abbreviations (4) of the notation). Filtration (3) consists of submodules

$$\Omega_l = \{\omega_0^i, \omega_s^j :\ i = 1, \dots, m\,,\ j = 1, \dots, m-1\,,\ s \le l\} \subset \Omega$$

and the obvious formula $\mathcal{L}_X \omega_s^i = \omega_{s+1}^i$ implies $\operatorname{Ker} \Omega_l \equiv \Omega_{l-1}$ for every $l \ge 1$ (abbreviation $\operatorname{Ker} = \operatorname{Ker}_{\mathbf{M}}$).

The identity (5) applied to $F = w_1^m - f$ turns into the linear dependence

$$\omega_1^m - \sum f_0^i \omega_0^i - \sum f_1^j \omega_1^j = 0$$

in the module $\Phi(\mathbf{M})$ and therefore

$$\mathcal{L}_X\left(\omega_0^m - \sum f_1^j \omega_0^j\right) \cong \omega_1^m - \sum f_1^j \omega_1^j \cong 0 \pmod{\Omega_0}.$$

It follows that $\operatorname{Ker}\Omega_0 = \{\omega\} \subset \Omega_0$ is the one-dimensional submodule which is generated by the form $\omega = \omega_0^m - \sum f_1^j \omega_0^j$. In more detail

$$\mathcal{L}_X \omega = \sum e^j \omega_0^j + f_0^m \omega \qquad (\, e^j \equiv f_0^j + f_1^j f_0^m - X f_1^j \,) \tag{10}$$

by easy calculation. In accordance with the algorithm, the subset $\tilde{\mathbf{M}} \subset \mathbf{M}$ is defined by the equations $e^j \equiv 0$ $(j = 1, \ldots, m-1)$ which imply $(6_2)$ along $\tilde{\mathbf{M}}$. Then the $\mathcal{EL}$ subspace $\mathbf{E} \subset \tilde{\mathbf{M}}$ should be defined by the requirements $X^l e^j \equiv 0$ $(j = 1, \ldots, m-1,\ l = 0, 1, \ldots)$ and we may choose $\breve{\Omega} = \tilde{\Omega} = \{\omega\}$ as the $\mathcal{PC}$ module is concerned.

The conditions ensuring that we indeed have a subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ (not a mere subset) and hence the reasonable $\mathcal{EL}$ diffiety $\mathbf{e}^*\Omega \subset \Phi(\mathbf{E})$ are not quite clear yet. We shall mention three "generic" cases.

**5. The nondegenerate case.**

Assuming $\det(f_{11}^{jk}) \neq 0$, the equations

$$X^l e^j = -X^l(X f_1^j) + \cdots = -\sum f_{11}^{jk} w_{2+l}^k + \cdots$$
$$(j, k = 1, \ldots, m-1,\ \ l = 0, 1, \ldots)$$

are equivalent to certain conditions $w_{2+l}^k + \cdots \equiv 0$ and it follows that they indeed determine a certain subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ equipped with coordinates $x$, $w_0^i$, $w_1^j$ $(i = 1, \ldots, m,\ j = 1, \ldots, m-1)$.

In order to determine the Lagrangian subspaces, the formulae

$$\mathrm{d}\omega = \mathrm{d}x \wedge \sum e^j \omega_0^j + \sum a^{jk} \omega_0^j \wedge \omega_0^k - \sum f_{11}^{jk} \omega_1^j \wedge \omega_0^k + \xi \wedge \omega\,, \tag{11}$$

where $j, k = 1, \ldots, m-1$ and

$$2a^{jk} = f_{10}^{jk} - f_{10}^{kj} + f_{10}^{jm} f_1^k - f_{10}^{km} f_1^j\,, \qquad \xi = f_0^m\,\mathrm{d}x + \sum f_{10}^{jm} \omega_0^j \tag{12}$$

are needful.

It follows easily that

$$\operatorname{Adj}\{\omega\} = \{\omega, \omega_0^i, \omega_1^j :\ i = 1, \ldots, m,\ \ j = 1, \ldots, m-1\}$$

on $\mathbf{E}$ (where $e^j \equiv 0$ identically). Therefore $\dim \mathrm{Adj}\{\omega\} = 2m - 1$ and the Lagrangian subspaces $\mathbf{l}\colon \mathbf{L} \subset \mathbf{E}$ defined by $(9_2)$ are of the codimension $c = m - 1$, hence $\dim \mathbf{L} = 2m - (m-1) = m+1$. We shall be interested in such $\mathbf{L}$ that the functions $x, w_0^1, \dots, w_0^m$ provide the coordinate system on $\mathbf{L}$. It is necessary to solve $(9_2)$ with unknown functions $h = \mathbf{l}^*h$ and $W = \mathbf{l}^*W$ of the coordinates mentioned above. This is expressed by the system

$$h\Big(\sum f_1^j w_1^j - f\Big) = W_x\,, \quad -hf_1^j \equiv W_0^j \quad (j = 1, \dots, m-1), \quad h = W_0^m\,. \tag{13}$$

Functions $x$, $w_0^1, \dots, w_0^m$, $f_1^1, \dots, f_1^{m-1}$ can be taken for alternative (local) coordinates on $\mathbf{E}$. In terms of these coordinates, we may introduce the *Hamilton function* $H$ such that

$$H(\dots, f_1^1, \dots, f_1^{m-1}) = \sum f_1^j w_1^j - f(\dots, w_1^1, \dots, w_1^{m-1})$$

(where $\dots = x, w_0^1, \dots, w_0^m$) and then $(13_1)$ yields the $\mathcal{HJ}$ equation

$$W_x / W_0^m = H(\dots, -W_0^1/W_0^m, \dots, -W_0^{m-1}/W_0^m) \tag{14}$$

quite analogously as in the common classical theory. Remaining equations (13) may be regarded as the embedding equations $W_0^j/W_0^m = -f_1^j$ of the subspace $\mathbf{L}$ into $\mathbf{E}$. (In classical terms, the inclusion $\mathrm{Adj}\{\omega\} \subset \Omega$ along $\mathbf{E}$ means that the extremals are identical with Cauchy characteristics of the $\mathcal{HJ}$ equation and the space $\mathbf{L}$ represents the generalized *Mayer field* of extremals.)

Let us eventually mention the increment formula [3; Part I, (30)]. It will be applied to the form $h\omega$ (instead of $\breve{\omega}$ appearing in [3; (27), (29), (30)]) and to certain curves $P(t), R(t) \in \mathbf{L}$ and $Q(t) \in \mathbf{M}$ $(0 \le t \le 1)$. More explicitly, let us denote

$$\begin{aligned} Q(t) &= \big(x(t), w_0^1(t), \dots, w_0^m(t), w_1^1(t), \dots, w_1^{m-1}(t), \dots\big) \in \mathbf{M}\,, \\ R(t) &= \big(x(t), w_0^1(t), \dots, w_0^m(t), r_1^1(t), \dots, r_1^{m-1}(t), \dots\big) \in \mathbf{L}\,, \end{aligned} \tag{15}$$

where coordinates of $P(t) \in \mathbf{L}$ need not be explicitly stated. We shall suppose that both $P(t)$, $Q(t)$ are solutions of $\Omega$ (in particular $P(t) \in \mathbf{L} \subset \mathbf{E}$ is an extremal and $Q^*\omega_0^i \equiv 0$ whence $\mathrm{d}w_0^i(t) = w_1^i(t)\,\mathrm{d}x(t)$) and moreover

$$W\big(R(0)\big) = W\big(P(0)\big) \tag{16}$$

in accordance with [3; $(28_1)$]. Thanks to (15) and the choice of variables in $W = W(x, w_0^1, \dots, w_0^m)$, condition $(28_2)$ is trivial. Then [3; (29)] reads

$$W\big(Q(1)\big) - W\big(P(1)\big) = \int R^*(h\omega) = \int_0^1 \mathcal{E}\ \mathrm{d}x(t) \tag{17}$$

with the *Weierstrass function*

$$\mathcal{E} = h(\ldots)\Big(f(\ldots, w_1^1, \ldots, w_1^{m-1}) - f(\ldots, r_1^1, \ldots, r_1^{m-1}) \\ - \sum f_1^j(\ldots, r_1^1, \ldots, r_1^{m-1})(w_1^1 - r_1^1)\Big)$$

(abbreviation $\cdots = x, w_0^1, \ldots, w_0^m$) evaluated on the curve $w_1^j = w_1^j(t)$, $r_1^j = r_1^j(t)$. We shall suppose $h = W_0^m > 0$ (cf. $(13_3)$) and $x'(t) > 0$. Then (e.g.) *the convexity of $f$ in the variables $w_1^1, \ldots, w_1^{m-1}$ ensures the inequality $W(Q(1)) \geq W(P(1))$*. This is a preliminary result: as yet we do not deal with any extremality problem. The point lies in the fact that a large spectrum of such problems can be resolved by means of appropriate choice of the function $W$.

**6. Continuation: Some extremality problems.**

Only a few simple possibilities will be mentioned. Recall for clarity that we shall deal with extremals $P(t) \in \mathbf{L}$, $0 \leq t \leq 1$, embedded into a Mayer field and such solutions $Q(t) \in \mathbf{M}$, $0 \leq t \leq 1$, of $\Omega$ given by $(15_1)$ that the "projections $R(t) \in \mathbf{L}$" given by $(15_2)$ make a good sense. (The functions $r_1^j(t)$ result from the embedding equations of $\mathbf{L}$ into $\mathbf{E}$.) The increment formula (17) will be applied, however, some additional requirements for the solution $W$ of the $\mathcal{HJ}$ equations may appear if we wish to resolve certain particular extremality problem.

($\iota$) *The classical Mayer problem.*

Assuming the boundary conditions

$$\begin{aligned} x(P(0)) &= x(Q(0)), & w_0^i(P(0)) &\equiv w_0^i(Q(0)), \quad i = 1, \ldots, m, \\ x(P(1)) &= x(Q(1)), & w_0^j(P(1)) &\equiv w_0^j(Q(1)), \quad j = 1, \ldots, m-1 \end{aligned}$$

(the functions $x, w_0^i \in \mathcal{F}(\mathbf{M})$ that should not be confused with arguments in (15)), we have (16) is a triviality hence (17) may be applied. Assuming $W_0^m > 0$, the inequalities $W(Q(1)) \geq W(P(1))$ and $w_0^m(Q(1)) \geq w_0^m(P(1))$ are equivalent. So we have the *sufficient criterion ensuring the extremality of the function $g = w_0^m$ at the right-hand end points of the curves $Q(t)$*. One can observe that the solution $W$ of the $\mathcal{HJ}$ equation need not satisfy any additional boundary conditions in this case: all Mayer fields are appropriate to solve the extremality. (See also [3; 13($\iota$)] with a special choice of data.)

($\iota\iota$) *The terminal problem.*

Assuming the boundary conditions

$$\begin{aligned} x(P(0)) &= x(Q(0)), & w_0^i(P(0)) &\equiv w_0^i(Q(0)), \quad i = 1, \ldots, m, \\ x(P(1)) &= x(Q(1)) = c, \end{aligned}$$

(17) may be again applied and it ensures the extremality of function $W(c, w_0^1, \dots, w_0^m)$ at the right-hand end points. *The extremality of a prescribed function* $g = g(w_0^1, \dots, w_0^m)$ *is ensured if the solution* $W$ *of the* $\mathcal{HJ}$ *equation is chosen such that* (e.g.) $g = W(c, w_0^1, \dots, w_0^m)$. (Certain transversality conditions are tacitly involved: $(13_{2,3})$ together with $\mathrm{d}g = \mathrm{d}W(c, w_0^1, \dots, w_0^m)$ imply the conditions $g_0^j + g_0^m f_1^j \equiv 0$ $(j = 1, \dots, m-1)$ valid at the right-hand end points. One can observe that then the stationarity requirement [3; $(15_2)$] expressed by the inclusion $\mathrm{d}g \in \{\omega, \mathrm{d}x\}$ is satisfied.)

($\iota\iota\iota$) *The free time problem.*

Assuming the boundary conditions

$$w_0^i(P(0)) \equiv w_0^i(Q(0)) \equiv a^i\,, \qquad i = 1, \dots, m\,,$$
$$w_0^j(P(1)) \equiv w_0^j(Q(1)) \equiv b^j\,, \qquad j = 1, \dots, m-1\,,$$

(17) is ensured if $W(x, a^1, \dots, a^m) = \mathrm{const}$. Applying (17), we obtain the sufficient extremality condition for the function $W(x, b^1, \dots, b^{m-1}, w_0^m)$ at the right-hand end points. Then the extremality of a prescribed function $g = g(x, w_0^m)$ is ensured if (e.g.) $W$ is chosen such that $W(\,\cdot\,, b^1, \dots, b^{m-1}, \cdot\,) = g$. Altogether taken, *we have the peculiar boundary conditions*

$$W(\,\cdot\,, a^1, \dots, a^m) = \mathrm{const}\,, \qquad W(\,\cdot\,, b^1, \dots, b^{m-1}, \cdot\,) = g$$

*for the solution* $W$ *of the* $\mathcal{HJ}$ *equation* (and one can verify that they imply the relevant stationarity conditions [3; (15)].

($\iota\nu$) *The classical variational problems.*

If $f_0^m = 0$ is identically vanishing, all the above formulae are much simplified, e.g., $2a^{jk} \equiv f_{10}^{jk} - f_{10}^{kj}$, $\xi = 0$, $e^j \equiv f_0^j - X f_1^j$ in (12). Moreover, $\omega \notin \mathrm{Adj}\,\mathrm{d}\omega$ and $\dim \mathrm{Adj}\,\mathrm{d}\omega = 2(m-1)$ in this case, therefore the $\mathcal{HJ}$ equations $(9_1)$ are sufficient. (In other words, we may suppose $h = W_0^m = 1$ without the loss of generality.) Recalling the above Mayer problem ($\iota$) that concerns the extremality of the value

$$w_0^m(Q(1)) = w_0^m(Q(0)) + \int_0^1 f\big(x, w_0^1(t), \dots, w_0^{m-1}(t), w_1^1(t), \dots, w_1^{m-1}(t)\big)\,\mathrm{d}x(t)$$

(hence of the integral), we have the familiar classical problem.

## 7. The degenerate case.

Assuming $\det(f_{11}^{jk}) = 0$ from now on, we enter a huge realm which deserves a whole book. So we restrict ourselves to certain generic situations with $m = 3$ in this article.

Assuming moreover $f^{11}_{11} \neq 0$ for technical reasons, $f^{22}_{11} \approx b f^{12}_{11}$ where $b = f^{12}_{11}/f^{11}_{11}$ and it follows

$$e^1 = -f^{11}_{11}(w^1_2 + b w^2_2) + \cdots , \qquad e^2 = -b f^{11}_{11}(w^1_2 + b w^2_2) + \cdots , \tag{18}$$

consequently $e = e^2 - b e^1$ is a function of the order one at most. One can then verify the formula

$$\mathrm{d}\omega = e\,\mathrm{d}x \wedge \omega^2_0 + \left(e^1 \mathrm{d}x + a\omega^2_0 - f^{11}_{11}\omega_1\right) \wedge \omega_0 + \zeta \wedge \omega , \tag{19}$$

where $a = 2a^{12}$, $\zeta = f^3_0\,\mathrm{d}x + f^{13}_{10}\omega_0 + \left(f^{23}_{10} - f^{13}_{10} b\right)\omega^2_0$, $\omega_s \equiv \omega^1_s + b\omega^2_s$ by using (12), (13), and the identities

$$e^2_1 = b e^1_1 , \qquad b^2_1 = b b^1_1 , \qquad e^1_1 + e^1 b^1_1 + a + f^{11}_{11} X b = 0 \tag{20}$$

follow by identifying the coefficients in the congruence

$$0 = \mathrm{d}^2\omega \cong \mathrm{d}e \wedge \mathrm{d}x \wedge \omega^2_0 - \left(e^1 \mathrm{d}x + a\omega^2_0 - f^{11}_{11}\omega_1\right) \wedge \mathrm{d}\omega_0 \quad (\mathrm{mod}\ \omega, \omega_0) .$$

*Hint.*
Use also the formula $\mathrm{d}\omega_s = \mathrm{d}x \wedge \omega_{s+1} + \mathrm{d}b \wedge \omega^2_s$ and the congruences

$$\mathrm{d}F \cong XF\mathrm{d}x + F^1_1\omega_1 + \left(F^2_1 - bF^1_1\right)\omega^2_1 \quad (\mathrm{mod}\ \omega_0, \omega^2_0, \omega, \omega^i_s) \qquad (s \geq 2,\ i = 1, 2)$$

with $F = e, b$.

We suppose $e^1_1 \neq 0$ in this section. (This is a generic case, see $(20_3)$.) Then, thanks to $(18_1)$ and $(20_1)$, the function

$$\tilde{e} = e^1 + \frac{f^{11}_{11}}{e^1_1} X e = f^1_0 + f^1_1 f^3_0 - f^1_{1x} - \sum w^i_1 f^{1i}_{10} + \frac{f^{11}_{11}}{e^1_1}\left(e_x + \sum w^i_1 e^i_0\right)$$

(where $i = 1, 2, 3$ and $w^3_1 = f$) is again of the order at most one. The coefficient $e^1$ occurring in (19) can be replaced by this function $\tilde{e}$: we substitute $e^1\,\mathrm{d}x = \left(\tilde{e} - f^{11}_{11} Xe/e^1_1\right)\mathrm{d}x$, where moreover

$$Xe\,\mathrm{d}x = \mathrm{d}e - \left(e^1_0 + f^1_1 e^3_0\right)\omega_0 - c\omega^2_0 - e^3_0\omega - e^1_1\omega_1$$

$\left(c = e^2_0 + f^2_1 e^3_0 - b(e^1_0 + f^1_1 e^3_0)\right.$ by direct verification) to obtain

$$\mathrm{d}\omega = e\,\mathrm{d}x \wedge \omega^2_0 + \left(\tilde{e}\,\mathrm{d}x - \frac{f^{11}_{11}}{e^1_1}\,\mathrm{d}e + A\omega^2_0\right) \wedge \omega_0 + \left(\zeta - e^3_0 \frac{f^{11}_{11}}{e^1_1}\omega_0\right) \wedge \omega ,$$

where $A = a + c f^{11}_{11}/e^1_1$. It simplifies considerably on the subspace defined by $e = 0$ (hence $\mathrm{d}e = 0$):

$$\mathrm{d}\omega = \left(\tilde{e}\,\mathrm{d}x + A\omega^2_0\right) \wedge \omega_0 + \left(\zeta - e^3_0 \frac{f^{11}_{11}}{e^1_1}\omega_0\right) \wedge \omega .$$

Then the remarkable identity

$$\tilde{e}_1^2 - b\tilde{e}_1^1 = A \qquad (\text{valid at } e = 0) \tag{21}$$

follows by identifying the summand $\omega_1^2 \wedge \mathrm{d}x \wedge \omega_0$ in the identically vanishing form $\mathrm{d}^2\omega$ (at $e = 0$). Our calculations are done.

Assume $A \neq 0$. (This is a generic case.) The $\mathcal{EL}$ subspace $\mathbf{E} \subset \mathbf{M}$ defined by the equations $X^l e^1 = X^l e^2 \equiv 0$ ($l = 0, 1, \dots$) can be equivalently defined by $X^l e = X^l \tilde{e} \equiv 0$. Clearly

$$X^l e = e_1^1 w_{1+l}^1 + e_1^2 w_{1+l}^2 + \cdots, \qquad X^l \tilde{e} = \tilde{e}_1^1 w_{1+l}^1 + \tilde{e}_1^2 w_{1+l}^2 + \cdots,$$

where $e_1^1 \tilde{e}_1^2 - e_1^2 \tilde{e}_1^1 = e_1^1 A \neq 0$ in virtue of $(20_1)$, (21). *It follows that functions* $x$, $w_0^1$, $w_0^2$, $w_0^3$ *provide coordinates on* $\mathbf{E}$, *hence* $\dim \mathbf{E} = 4$.

With these results, various reasonable stationarity problems can be formulated, however, the relevant $\mathcal{HJ}$ concepts and the increment formula are insufficient to establish the extremality properties.

**8. The case of underdetermined extremals.**

Continuing the notation, let us assume $e = e^2 - be^1 = 0$ identically vanishing from now on. Then the $\mathcal{EL}$ system reduces to the equations $X^l e^1 \equiv 0$ ($l = 0, 1, \dots$) and, assuming again $f_{11}^{11} \neq 0$, the *infinite family of functions* $x$, $w_0^1$, $w_0^2$, $w_0^3$, $w_1^1$, $w_1^2$, $w_s^2$, $s \geq 2$, provide coordinates on the $\mathcal{EL}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$. Formula (19) simplifies a little:

$$\mathrm{d}\omega = \left(e^1 \mathrm{d}x + a\omega_0^2 - f_{11}^{11}\omega_1\right) \wedge \omega_0 + \zeta \wedge \omega. \tag{22}$$

Clearly $\operatorname{Adj}\{\omega\} = \{\omega, \omega_0, e^1\mathrm{d}x + a\omega_0^2 - f_{11}^{11}\omega_1\}$ and the Lagrange subspaces $\mathbf{l}\colon \mathbf{L} \subset \mathbf{E}$ are of the codimension $c = 1$. We shall be interested in the infinite-dimensional Lagrange subspaces where the functions $x, w_0^1, w_0^3, w_s^2$ ($s \neq 1$) provide the coordinate system. It is again necessary to resolve the equation $(9_2)$ with unknown functions $h = \mathbf{l}^*h$ and $W = \mathbf{l}^*W$. This leads to equations (13) with $m = 3$,

$$h\left(f_1^1 w_1^1 + f_1^2 w_1^2 - f\right) = W_x, \quad -hf_1^1 = W_0^1, \quad -hf_1^2 = W_0^2, \quad h = W_0^3, \tag{23}$$

however, they are *not equivalent* to a single $\mathcal{HJ}$ equation of the kind (14). In fact, by virtue of the degeneracy, there exist functions $G = G\left(x, w_0^1, w_0^2, w_0^3, z\right)$, $H = H\left(x, w_0^1, w_0^2, w_0^3, z\right)$ such that

$$f_1^2 = G\left(\dots, f_1^1\right), \qquad f_1^1 w_1^1 + f_1^2 w_1^2 - f = H\left(\dots, f_1^1\right)$$

(at least locally) and (23) turns into the (involutive) $\mathcal{HJ}$ system

$$\frac{W_x}{W_0^3} = H\left(\dots, -\frac{W_0^1}{W_0^3}\right), \qquad -\frac{W_0^2}{W_0^3} = G\left(\dots, -\frac{W_0^1}{W_0^3}\right)$$

instead of a single equation.

Passing to the increment formula and extremality properties, the curves $Q(t)$ given by $(15_1)$ with $m = 3$ are retained, but quite another curve

$$R(t) = \big(x(t), w_0^1(t), w_0^2(t), w_0^3(t), r_1^1(t), w_1^2(t), w_2^2(t), r_2^1(t), \ldots\big) \in \mathbf{L}$$

(distinctions at the places $w_s^2$, $s \geq 2$) stands for $(15_2)$. Fortunately, this change *does not affect* the Weierstrass function $\mathcal{E}$ and the main result that the *convexity of $f$ in the variable $w_1^1$ is enough to ensure the extremum.*

Analogous extremality problems as in Section 6 may be introduced without any difficulties. They do not have unique solutions, however, the existence of the relevant function $W$ (i.e., the embedding into the Mayer field and the convexity) provides the sufficient extremality condition as before.

## 9. A note on the Jacobi principle.

We shall deal with two equations $w_1^3 = f$ and $w_1^3 = F(f)$ simultaneously and then the more precise notation like $e^j[f]$, $b[f]$, $e[f]$ will clarify the exposition. Recalling $(10_2)$, we have $e^j[f] = f_0^j + f_1^j f_0^3 - X f_1^j$ and therefore

$$e^j\big[F(f)\big] = F'(f)e^j[f] + f_1^j\big(F'(f)\big(F'(f) - 1\big)f_0^3 - XF'(f)\big)$$

after a short calculation. Then, assuming $e[f] = e^2[f] - b[f]e^1[f] = 0$ identically vanishing, clearly

$$e^2\big[F(f)\big] - b[f]e^1\big[F(f)\big] = \big(f_1^2 - b[f]f_1^1\big)\big(F'(f)\big(F'(f) - 1\big)f_0^3 - XF'(f)\big)$$

is true and finally the formula

$$\det\big(F(f)_{11}^{jk}\big) = F'(f)F''(f)f_{11}^{11}\big(f_1^2 - b[f]f_1^1\big)^2$$

can be verified by using the identity $f_{11}^{22} = b[f]f_{11}^{12} = b[f]^2 f_{11}^{11}$.

Several conclusions follow.

($\iota$) Either of the identities $F'(f) = 0$, $F''(f) = 0$, $f_1^2 f_{11}^{11} = f_1^1 f_{11}^{12}$ ensures the degeneracy $\det\big(F(f)_{11}^{jk}\big) = 0$ and the converse is also true. Especially the last identity ensures moreover $b[F(f)] = b[f]$.

($\iota\iota$) If $f_1^2 f_{11}^{11} \neq f_1^1 f_{11}^{12}$, then the equations $e^1[F(f)] = e^2[F(f)] = 0$ imply

$$XF'(f) = F'(f)\big(F'(f) - 1\big)f_0^3,$$

hence $e^1[f] = e^2[f] = 0$ in the case when $F'(f) \neq 0$. Moreover $XF'(f) = 0$ (hence $F'(f) = \text{const} \neq 0$) if $f_0^3 = 0$ identically.

($\iota\iota\iota$) Conversely, the equation $e^1[f] = XF'(f) - F'(f)\big(F'(f) - 1\big)f_0^3 = 0$ implies $e^2[f] = 0$ hence $e^1\big[F(f)\big] = e^2\big[F(f)\big] = 0$.

The classical Jacobi least action principle for the "parametrical variational problems" appears as a very particular case of points ($\iota\iota$), ($\iota\iota\iota$) when $f_0^3 = 0$ is assumed. Then the equation $F'(f) = \text{const} \neq 0$ may be interpreted as a "normalization" of the independent variable on the *underdetermined* extremals (solutions of $e^1[f] = e^2[f] = 0$) to obtain *determined* extremals (solutions of $e^1[F(f)] = e^2[F(f)] = 0$).

**10. On the realization problem.**

Using the previous simple notation, we shall be interested in the overview of all cases when $e = 0$ is identically vanishing. So we have to resolve the identity $e^2 = be^1$ regarded as a differential equation for the unknown function $f = f(x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2)$. The $\mathcal{PC}$ form $\omega$ provides a useful tool.

Assuming $e = 0$ identically, $\dim \mathrm{Adj}\{\omega\} = 3$ by virtue of (22), therefore $h\omega = U\,\mathrm{d}V - \mathrm{d}W$ for appropriate functions $h$, $U$, $V$, $W$ of variables $x$, $w_0^i$, $w_1^j$ ($i = 1,2,3$, $j = 1,2$), see Section 3. We shall however begin with the congruence

$$\mathrm{d}(h\omega) = \mathrm{d}U \wedge \mathrm{d}V \cong 0 \quad (\mathrm{mod}\ \omega_0^1, \omega_0^2, \omega_0^3)$$

easily following from either of the formulae (11), (19), (22). In more detail

$$\mathrm{d}U \wedge \mathrm{d}V \cong (XU\,\mathrm{d}x + U_1^1\omega_1^1 + U_1^2\omega_1^2) \wedge (XV\,\mathrm{d}x + V_1^1\omega_1^1 + V_1^2\omega_1^2) \cong 0\,, \qquad (24)$$

whence $U_1^j XV \equiv V_1^j XU$ and moreover $U_1^1 V_1^2 = V_1^1 U_1^2$. We may assume $U = U(x, w_0^1, w_0^2, w_0^3, V)$ and then (24) is satisfied if and only if

$$U_x + U_0^1 w_1^1 + U_0^2 w_1^2 + U_0^3 w_1^3 = 0 \qquad (w_1^3 = f) \qquad (25)$$

(direct verification). With this particular result, let us consider the obvious congruence $h\omega = U\,\mathrm{d}V - \mathrm{d}W \cong 0$ (mod $\omega_0^1, \omega_0^2, \omega_0^3$). In more detail

$$U(XV\mathrm{d}x + V_1^1\omega_1^1 + V_1^2\omega_1^2) \cong XW\mathrm{d}x + W_1^1\omega_1^1 + W_1^2\omega_1^2\,, \qquad (26)$$

whence $UV_1^j \equiv W_1^j$ and we may assume $W = W(x, w_0^1, w_0^2, w_0^3, V)$. Then (26) is satisfied if and only if

$$W' = U\,, \qquad W_x + W_0^1 w_1^1 + W_0^2 w_1^2 + W_0^3 w_1^3 = 0\,, \quad (w_1^3 = f,\ \ ' = \frac{\partial}{\partial V}) \quad (27)$$

(direct verification).

With these preparatory results, let us turn to the construction of the function $f$. Abbreviating $\dots = x, w_0^1, w_0^2, w_0^3$, we may choose a function $W = W(\dots, V)$ and assuming $W_0^3 \neq 0$, $(27_2)$ determines the function

$$F(\dots, w_1^1, w_1^2, V) = -\frac{1}{W_0^3}(W_x + W_0^1 w_1^1 + W_0^2 w_1^2)\,, \qquad (28)$$

a temporary substitute for $f$. On the other hand, in accordance with $(27_1)$, we have to put $U = W'(\dots, V)$ and then

$$U_x + U_0^1 w_1^1 + U_0^2 w_1^2 + U_0^3 w_1^3 + W_0^3 F' = 0 \tag{29}$$

by using (27) and (28). In order to ensure (25), we must introduce the implicit equation

$$F'(\dots, w_1^1, w_1^2, V) = 0 \tag{30}$$

for the function $V$. Our calculations are done: assuming $F'' \neq 0$, (30) determines $V = V(\dots, w_1^1, w_1^2)$ which, substituted into (28), provides the sought function $f = F(\dots, w_1^1, w_1^2, V(\dots, w_1^1, w_1^2))$ such that $e = 0$ identically. (An alternative indirect verification: our solution ensures the identities (24)–(27), hence $h\omega = U\,\mathrm{d}V - \mathrm{d}W$, $\dim \mathrm{Adj}\{h\omega\} = 3$ and $e = 0$ by virtue of formula (19).)

In particular, assume $W = \bar{W}(V, w_0^1, w_0^2) + w_0^3$. Then $U = \bar{W}'$ and conditions (27), (29) are simplified as

$$U_0^1 w_1^1 + U_0^2 w_1^2 = 0\,, \qquad \bar{W}_0^1 w_1^1 + \bar{W}_0^2 w_1^2 + F = 0\,. \tag{31}$$

The function $V$ is clearly homogeneous of the order zero in $w_1^1$, $w_1^2$ (by virtue of $(31_1)$) and

$$F(\dots, w_1^1, w_1^2, V(\dots, w_1^1, w_1^2)) = f(\dots, w_1^1, w_1^2)$$

is homogeneous of the order one (by virtue of $(31_2)$). So we obtain the common familiar "parametrical" integral $\int f \,\mathrm{d}x$ in this very particular case.

## Second example

### 11. Two differential equations.

We deal with the equations

$$w_1^m = f(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^{m-2})\,,$$
$$w_1^{m-1} = g(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^{m-2})\,,$$

the relevant subdiffiety $\Omega = \mathbf{m}^*\Omega(m) \subset \Phi(\mathbf{M})$ is defined on the subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(m)$ of all points that satisfy the equations $X^l(w_1^m - f) = X^l(w_1^{m-1} - g) \equiv 0$ $(l = 0, 1, \dots)$. One can see that the functions

$$x\,,\ w_0^i\,,\ w_s^j \qquad (i = 1, \dots, m\,,\ j = 1, \dots, m-2\,,\ s = 0, 1, \dots)$$

provide coordinates on $\mathbf{M}$. Then the vector field

$$X = \partial/\partial x + \sum w_1^i \partial/\partial w_0^i + \sum w_{s+1}^j \partial/\partial w_s^j \in \Omega^\perp \subset \mathcal{T}(\mathbf{M})$$
$$(w_1^m = f\,,\ w_1^{m-1} = g)$$

generates $\Omega^{\perp}$ and the forms

$$\omega_0^i\,,\ \omega_s^j \qquad (i=1,\dots,m\,,\ j=1,\dots,m-2\,,\ s=1,2,\dots)$$

constitute a basis of $\Omega$. Filtration (3) consists of submodules

$$\Omega_l=\{\omega_0^i,\omega_s^j:\ i=1,\dots,m\,,\ j=1,\dots,m-2\,,\ s\le l\}\subset\Omega\,,$$

where $\operatorname{Ker}\Omega_l\equiv\Omega_{l-1}$ $(l\ge 1)$. Denoting

$$\omega=\omega_0^m-\sum f_1^j\omega_0^j\,,\qquad \bar\omega=\omega_0^{m-1}-\sum g_1^j\omega_0^j\,,$$

we obtain

$$\mathcal{L}_X\omega=\sum F^j\omega_0^j+f_0^m\omega+f_0^{m-1}\bar\omega\,,\qquad \mathcal{L}_X\bar\omega=\sum G^j\omega_0^j+g_0^m\omega+g_0^{m-1}\bar\omega\,,\ (32)$$

where

$$F^j\equiv f_0^j+f_1^jf_0^m+g_1^jf_0^{m-1}-Xf_1^j\,,\qquad G^j\equiv g_0^j+f_1^jg_0^m+g_1^jg_0^{m-1}-Xg_1^j$$

with $j=1,\dots,m-2$, and it follows that $\operatorname{Ker}\Omega_0=\{\omega,\bar\omega\}$. To calculate $\operatorname{Ker}^2\Omega_0$, the formula

$$\mathcal{L}_X(A\omega-B\bar\omega)=\sum(AF^j-BG^j)\omega_0^j+C\omega+D\bar\omega \qquad (33)$$

with coefficients

$$C=Af_0^m-Bg_0^m+XA\,,\qquad D=Af_0^{m-1}-Bg_0^{m-1}-XB$$

will be useful.

An exhaustive discussion of all interesting subcases which may occur is hardly possible at this place. We restrict ourselves to a few indications assuming moreover $m=3$, hence $j=1$ in the above formulae.

**12. The most peculiar case.**

If $F^1=G^1=0$ identically, then $\operatorname{Ker}\Omega_0=\mathcal{R}(\Omega)\subset\Omega$ is the maximal completely integrable submodule of $\Omega$ (see [3; Sect. 7], especially [3; 13($\nu$)]), explicitly $\operatorname{Ker}\Omega_0=\{\omega,\bar\omega\}=\{\mathrm{d}F,\mathrm{d}G\}$. The functions $F$, $G$ of variables $x$, $w_0^1$, $w_0^2$, $w_0^3$ can be determined by solving certain ordinary differential equations. It follows that the original system is equivalent to the conditions $F=\text{const}$, $G=\text{const}$, this should be viewed as a very peculiar and rare case.

**13. The nondegenerate case.**

Assuming (e.g.) $F^1\ne 0$, we have $\operatorname{Ker}^2\Omega_0$ is generated by the single form $A\omega-B\bar\omega\in\operatorname{Ker}\Omega_0$ along the subset $\mathbf{N}\subset\mathbf{M}$ of all points satisfying $AF^1=BG^1$. We may assume $B=1$ for better clarity, hence $A=G^1/F^1$ along $\mathbf{N}$.

Passing to the submodule $\mathrm{Ker}^3\,\Omega_0 \subset \mathrm{Ker}^2\,\Omega_0$, it is nontrivial if and only if the form $\mathcal{L}_X(A\omega - \bar{\omega})$ is a multiple of $A\omega - \bar{\omega}$ (and then $\mathrm{Ker}^3\,\Omega_0 = \mathrm{Ker}^2\,\Omega_0$) which is expressed by $AD + C = 0$. Since $A = G^1/F^1$ along the subset $\mathbf{N} \subset \mathbf{M}$, we obtain the condition

$$(e=)\ (G^1)^2 f_0^2 + F^1 G^1 (f_0^3 - g_0^2) - (F^1)^2 g_0^3 + F^1 X G^1 - G^1 X F^1 = 0\,. \qquad (34)$$

We are done: the subset $\tilde{\mathbf{M}} = \mathbf{N} \subset \mathbf{M}$ is defined by the equation $e = 0$ and $\tilde{\Omega} = \{A\omega - \bar{\omega}\} = \{G^1\omega - F^1\bar{\omega}\}$. It follows that the $\mathcal{PC}$ module $\breve{\Omega}$ is generated by the single form $G^1\omega - F^1\bar{\omega}$ along the $\mathcal{EL}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ of such points that satisfy the system $X^l e \equiv 0$ $(l = 0, 1, \dots)$. More precisely: clearly

$$e = F^1 X G^1 - G^1 X F^1 + \cdots = \left(G^1 f_{11}^{11} - F^1 g_{11}^{11}\right) w_3^1 + \cdots \qquad (35)$$

as the top order terms are concerned. We moreover suppose $G^1 f_{11}^{11} \neq F^1 g_{11}^{11}$ to ensure that $\mathbf{E} \subset \mathbf{M}$ is indeed a subspace and the functions $x$, $w_0^1$, $w_0^2$, $w_0^3$, $w_1^1$, $w_1^2$ provide the coordinate system on $\mathbf{E}$.

One can easily infer that the Lagrangian subspaces $\mathbf{l}\colon \mathbf{L} \subset \mathbf{E}$ are of the codimension two. We shall be interested in the case when $x$, $w_0^1$, $w_0^2$, $w_0^3$ are coordinates on $\mathbf{L}$. Then $(9_2)$ is expressed by

$$h\left(G^1\omega - F^1\bar{\omega}\right) = \mathrm{d}W\left(x, w_0^1, w_0^2, w_0^3\right)$$

which is equivalent to the system

$$-W_x = W_0^3 f + W_0^2 g + W_0^1 w_1^1\,, \quad W_0^1 + W_0^2 g_1^1 + W_0^3 f_1^1 = 0\,, \quad F^1 W_0^3 + G^1 W_0^2 = 0 \qquad (36)$$

(and $h = -W_0^2/F^1$, hence $W_0^2 \neq 0$ is supposed). The system (36) gives a single equation for the unknown function $W$: the equations $(36_{2,3})$ can be solved in terms of the parameters $w_1^1$, $w_2^1$ and these can be inserted into $(36_1)$ with the final result of the classical kind $W_x + H(x, w_0^1, w_0^2, w_0^3, W_0^1, W_0^2, W_0^3) = 0$ in terms of the corresponding Hamilton function $H$. (In more detail, $(36_3)$ rewritten as $W_0^3/W_0^2 + G^1/F^1 = 0$ can be resolved with respect to $w_2^1$ and then $(36_2)$ yields the variable $w_1^1$. These $w_2^1$ and $w_1^1$ inserted into the right-hand side $(36_1)$ provide the function $H$. The condition $G^1 f_{11}^{11} \neq F^1 g_{11}^{11}$ with $F^1 = -W_0^2/h$, $G^1 = W_0^3/h$ was employed here.) The increment formula (17) with the form $h\left(G^1\omega - F^1\bar{\omega}\right)$ at the place of $h\omega$ easily follows. The curves (15) with $m = 3$ lead to very simple Weierstrass function

$$\mathcal{E} = h(\dots, r_1^1)\left(F(\dots, w_1^1) - F(\dots, r_1^1) - F_1^1(\dots, r_1^1)(w_1^1 - r_1^1)\right),$$

where the nonvanishing factor $h$ may be in principle omitted, $F = G^1 f - F^1 g$ (alternatively $hF = W_0^3 f + W_0^2 g$) and the dots stand for the parameters $x$, $w_0^1$, $w_0^2$, $w_0^3$. Various extremality problems can be discussed without much difficulties analogously as in Section 6.

**14. A strange "degenerate" case.**

We shall suppose $f = f(x, w_0^1, w_0^2, w_0^3)$, $g = g(x, w_0^1, w_0^2, w_0^3)$, hence $f_1^1 = g_1^1 = 0$ vanishing identically for a moment. The above formulae simplify very much:

$$\omega = \omega_0^3\,, \quad \bar\omega = \omega_0^2\,, \quad F^1 = f_0^1\ (\neq 0)\,, \quad G^1 = g_0^1\,, \quad G^1\omega - F^1\bar\omega = g_0^1\omega_0^3 - f_0^1\omega_0^2\,,$$

and the latter form generates the $\mathcal{PC}$ submodule $\tilde\Omega \subset \Omega$. The relevant $\mathcal{EL}$ subspace $\mathbf{E} \subset \mathbf{M}$ is defined by the equations $X^l e \equiv 0$ $(l = 0, 1, \dots)$, where

$$e = (g_0^1)^2 f_0^2 + f_0^1 g_0^1 (f_0^3 - g_0^2) - (f_0^1)^2 g_0^3 + f_0^1 X g_0^1 - g_0^1 X f_0^1$$

is a particular case of (34). Clearly $e = (f_0^1 g_{00}^{11} - g_0^1 f_{00}^{11}) w_1^1 + \cdots$ as the top order summands are concerned. Assuming $f_0^1 g_{00}^{11} \neq g_0^1 f_{00}^{11}$ from now on (the *nondegenerate-degenerate case*), functions $x$, $w_0^1$, $w_0^2$, $w_0^3$ may be used for coordinates on $\mathbf{E}$.

One can see that there are one-codimensional Lagrangian subspaces $\mathrm{l}\colon \mathbf{L} \subset \mathbf{E}$. If $x$, $w_0^2$, $w_0^3$ are taken for coordinates on $\mathbf{L}$, then the condition $(9_2)$ reads $h(g_0^1\omega_0^3 - f_0^1\omega_0^2) = \mathrm{d}W(x, w_0^2, w_0^3)$, that is

$$-W_x = W_0^3 f + W_0^2 g\,, \qquad W_0^2 = -h f_0^1\,, \qquad W_0^3 = h g_0^1\,. \tag{37}$$

The system (37) leads to a single $\mathcal{HJ}$ equation $W_x + H(\dots, W_0^2, W_0^3) = 0$ for the unknown function $W$. (Indeed, the parameter $w_0^1$ can be determined by using $W_0^3/W_0^2 + g_0^1/f_0^1 = 0$ and, inserted into $(37_1)$, we obtain the desired result.) Passing to the increment formula for the curves

$$\begin{aligned} Q(t) &= \big(x(t), w_0^1(t), w_0^2(t), w_0^3(t), w_1^1(t), w_1^2(t), \dots\big) \in \mathbf{M}\,, \qquad 0 \le t \le 1\,,\\ R(t) &= \big(x(t), r_0^1(t), w_0^2(t), w_0^3(t), r_1^1(t), r_1^2(t), \dots\big) \in \mathbf{L}\,, \qquad 0 \le t \le 1\,, \end{aligned} \tag{38}$$

and $P(t) \in \mathbf{L}$ $(0 \le t \le 1)$ satisfying (16), then $(9_2)$ applied to the form $h(g_0^1\omega - f_0^1\bar\omega)$ instead of $h\omega$ leads to the very unorthodox Weierstrass function

$$\mathcal{E} = h(r_0^1)\big(g_0^1(r_0^1)\big(f(w_0^1) - f(r_0^1)\big) - f_0^1(r_0^1)\big(g(w_0^1) - g(r_0^1)\big)\big)\,, \tag{39}$$

where the parameters $x$, $w_0^2$, $w_0^3$ are omitted for brevity. The inequalities $\mathcal{E} \ge 0$, $\mathcal{E} \le 0$ admit a geometrical interpretation. We shall not discuss the self-evident results for various extremality problems, they do not bring any novelty.

**15. The top linear case.**

Slightly generalizing the previous Section 14, we shall eventually conclude with the quasilinear system

$$w_1^2 = K w_1^1 + L\,, \qquad w_1^3 = M w_1^1 + N\,,$$

where $K$, $L$, $M$, $N$ are given functions of variables $x$, $w_0^1$, $w_0^2$, $w_0^3$. Retaining the notation of Sections 11 and 12, we have the forms

$$\omega = \omega_0^3 - M\omega_0^1 = \mathrm{d}w_0^3 - M\mathrm{d}w_0^1 - N\mathrm{d}x\,, \qquad \bar{\omega} = \omega_0^2 - K\omega_0^1 = \mathrm{d}w_0^2 - K\mathrm{d}w_0^1 - L\mathrm{d}x$$

such that (32) holds true with the coefficients

$$\begin{aligned} F^1 &= N_0^1 - M_x + KN_0^2 - LM_0^2 + MN_0^3 - NM_0^3\,,\\ G^1 &= L_0^1 - K_x + ML_0^3 - NK_0^3 + KL_0^2 - LK_0^2\,, \end{aligned}$$

$$f_0^k \equiv M_0^k w_1^1 + N_0^k\,, \quad g_0^k \equiv K_0^k w_1^1 + L_0^k \qquad (k = 2, 3)\,.$$

Assuming $F^1 \neq 0$, the form

$$G^1\omega - F^1\bar{\omega} = G^1\,\mathrm{d}w_0^3 - F^1\,\mathrm{d}w_0^2 + (F^1K - G^1M)\,\mathrm{d}w_0^1 + (F^1L - G^1N)\,\mathrm{d}x \quad (40)$$

generates the $\mathcal{PC}$ module $\tilde{\Omega} \subset \Omega$ along the relevant $\mathcal{EL}$ subspace $\mathbf{E} \subset \mathbf{M}$ defined by $X^l e \equiv 0$ $(l = 0, 1, \dots)$. The function $e$ was stated in (34), however, quite another top order terms than in (35) appear:

$$\begin{aligned} e &= Ew_1^1 + \dots\,,\\ E &= F^1(G^1)_0^1 + F^1(G^1)_0^2K + F^1(G^1)_0^3M\\ &\quad - G^1(F^1)_0^1 - G^1(F^1)_0^2K - G^1(F^1)_0^3M\,. \end{aligned}$$

From now on, we suppose $E \neq 0$, then the functions $x$, $w_0^1$, $w_0^2$, $w_0^3$ provide coordinates on $\mathbf{E}$.

One can see that the Lagrangian subspaces $\mathbf{L} \subset \mathbf{E}$ are of the codimension one and we shall be interested in the case when $x$, $w_0^2$, $w_0^3$ may be chosen for coordinates on $\mathbf{L}$, hence $W = W(x, w_0^2, w_0^3)$ in equation $(9_2)$. A certain inconvenience appears since the form (40), which should be substituted into the left-hand side of $(9_2)$, involves the abundant summand $\mathrm{d}w_0^1$.

To cope with this problem, we introduce the equation $w_0^1 = r(x, w_0^2, w_0^3)$ valid along the subspace $\mathbf{L} \subset \mathbf{E}$, where $r$ is regarded as an additional unknown function. In other words, the equation $(9_2)$ is completed to the system

$$h(G^1\omega - F^1\bar{\omega}) = \mathrm{d}W(x, w_0^2, w_0^3)\,, \qquad w_0^1 = r(x, w_0^2, w_0^3)\,. \qquad (41)$$

Then, by using (40) and $\mathrm{d}w_0^1 = r_x\,\mathrm{d}x + r_0^2\,\mathrm{d}w_0^2 + r_0^3\,\mathrm{d}w_0^3$, one can obtain the requirements

$$h(T + Sr_x) = W_x\,, \qquad h(-F^1 + Sr_0^2) = W_0^2\,, \qquad h(G^1 + Sr_0^3) = W_0^3$$

for three unknown functions $h$, $r$, $W$ of variables $x$, $w_0^2$, $w_0^3$ with abbreviations $S = F^1K - G^1M$, $T = F^1L - G^1N$. So we have a strange modification of the common $\mathcal{HJ}$ equation

$$(F^1 - Sr_0^2)W_0^3 + (G^1 + Sr_0^3)W_0^2 = 0\,, \qquad (G^1 + Sr_0^3)W_x = (T + Sr_x)W_0^3\,, \ (42)$$

where the "parameter" function $w_0^1 = r$ cannot be algebraically deleted. According to the sense of the system (42), the method of Cauchy characteristics may be applied and therefore the solution can be reduced to mere *ordinary* differential equations.

The increment formula for the curves (38) and an extremal $P(t) \in \mathbf{L}$ $(0 \leq t \leq 1)$ immediately follows. One can obtain the Weierstrass function

$$\begin{aligned}\mathcal{E} = h(r_0^1)\{G^1(r_0^1)\big[\big(M(w_0^1) - M(r_0^1)\big)w_1^1 + N(w_0^1) - N(r_0^1)\big] - \\ -F^1(r_0^1)\big[\big(K(w_0^1) - K(r_0^1)\big)w_1^1 + L(w_0^1) - L(r_0^1)\big]\},\end{aligned} \tag{43}$$

where the parameters $x$, $w_0^2$, $w_0^3$ were omitted and the variable $w_1^1$ may be regarded as an independent parameter. This is clearly a generalization of the previous result (39).

We again omit the discussion of extremality problems. In principle, it should not cause much difficulties.

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