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TRANSMISSION IN GRAPHS:
A BOUND AND VERTEX REMOVING

LUBOMÍR ŠOLTÉS

ABSTRACT. The transmission of a graph $G$ is the sum of all distances in $G$. Strict upper bound on the transmission of a connected graph with a given number of vertices and edges is provided. Changes of the transmission caused by removing a vertex are studied.

1. Introduction

All graphs considered in this paper are undirected without loops and multiple edges. For all terminology on graphs not explained here we refer to [1].

If $S$ is set, then $|S|$ denotes the cardinality of $S$. Given a graph $G$, $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. The cardinalities $|v(G)|$ and $|E(G)|$ are often denoted $n$ and $m$, respectively. If $v$ and $w$ are the vertices of $G$, then $d_G(v,w)$ or, briefly, $d(v,w)$ denotes the distance from $v$ to $w$ in $G$, $ec(v)$ or $ec(v)$ denotes the eccentricity of $v$.

The transmission of a vertex $v$ of a graph $G$ is defined by

$$\sigma_G(v) = \sum_{w \in V(G)} d_G(v,w).$$

The transmission $\sigma(G)$ of a graph $G$ is the sum of the transmissions of all its vertices.

The main subject of this paper is the transmission. Several results on this notion are surveyed in [5]. The strict upper bound on the transmission of a connected graph with a given number of vertices and edges is provided in this paper. Changes of the transmission caused by removing a vertex are studied.

2. An upper bound for transmission

Entringer, Jackson and Snyder [1] have given some upper bounds for transmission of a connected graph with $n$ vertices and $m$ edges. But they are
not sharp for each \( m \). Now we are going to establish the sharp upper bound. Let \( u \) be an isolated vertex or one endvertex of a path. Let us join \( u \) with at least one vertex of a complete graph. This new graph is called a path-complete graph and denoted by \( PK_{n,m} \), where \( n \) and \( m \) are the cardinalities of its vertex-set and edge-set, respectively (see. Fig. 1.). One can verify that there is exactly one path-complete graph \( Pk_{n,m} \) for all \( 1 \leq n - 1 \leq m \leq \binom{n}{2} \).

![Fig. 1](image)

The maximal distance in \( G \) is the diameter of \( G \), \( \text{diam} (G) \). The following upper bound on the diameter, depending on the number of vertices and edges, was given by Harary [4].

**Lemma 1** ([4]). Let \( G \) be a connected graph with \( n \geq 2 \) vertices and \( m \) edges. Then we have \( \text{diam} (G) \leq \text{diam} (PK_{n,m}) \).

If \( R \subseteq V(G) \), then \( G(R) \) is the induced subgraph of \( G \) with the vertex-set \( R \). For a graph \( G \) and integer \( k \geq 1 \) let \( S_k(G) \) be the set of all unordered pairs of such not adjacent vertices in \( G \) that their distance does not exceed \( k \). Hence \( S_1(G) = \emptyset \) holds. The following lemma gives the sharp lower bound for the cardinality of the set \( S_k(G) \) with respect to the order and the diameter of a graph \( G \).

**Lemma 2.** Let \( G \) be a connected graph with \( n \geq 2 \) vertices and diameter \( d \geq 3 \). Then for any integer \( k, 2 \leq k \leq d - 1 \), we have

\[
|S_k(G)| \geq \sum_{i=2}^{k} (n - i) = (k - 1)n - k(k + 1)/2 + 1.
\]

Moreover, the equality occurs if \( G \) is a path-complete graph.

**Proof.** Let \( G_0 \) be a shortest path in \( G \) joining two vertices with distance \( d \). Then we can denote the vertices not lying in \( G_0 \) by the symbols \( v_1, v_2, \ldots, v_{n-d-1} \) in such a way that the graphs \( G_j := G(V(G_0) \cup \{v_1, v_2, \ldots, v_j\}) \) are connected for all \( j \leq n - d - 1 \). Let \( k \) be a fixed integer, \( 2 \leq k \leq d - 1 \). Obviously the equality occurs in (1) for \( G = G_0 \). Clearly, \( S_k(G_j) \) contains \( S_k(G_{j-1}) \) for all \( 1 \leq j \leq n - d - 1 \). Thereby Lemma 2 will be established if we show that...
the set $S_k(G_j) - S_k(G_{j-1})$ has at least $k - 1$ elements. Now we distinguish two cases.

Case 1. Let $e_{G_j}(v_j) \geq k$. Obviously, for at least $k - 1$ vertices $z$ in $G_{j-1}$ we have $2 \leq d_{G_j}(v_j, z) \leq k$.

Case 2. Let $e_{G_j}(v_j) < k$. Note that the vertex $v_j$ is adjacent to at most 3 vertices from $G_0$. That is why there are at least $d - 2$ vertices $z$ such that $(v_j, z) \in S_k(G_j)$. Clearly, $d - 2 \geq k - 1$ holds.

One can directly verify that the equality occurs in (1) if $G$ is a path-complete graph with diameter at least 3.

\textbf{Theorem 1.} Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Then $\sigma(G) \geq \sigma(PK_{n,m})$ holds.

\textbf{Proof.} Let $D$ and $d$ be the diameters of the graphs $PK_{n,m}$ and $G$, respectively. If $d \leq 2$ holds, then we have $\sigma(G) = 2n(n - 1) - 2m \leq \sigma(PK_{n,m})$. Next we shall suppose that $d \geq 3$ holds. Let $s_i$ be the number of unordered pairs of vertices in $G$ with distance $i$, for integer $i \geq 0$. Note that

$$s_1 = m \text{ and } s_1 + s_2 + \ldots + s_d = m + |S_d(G)| = \binom{n}{2}$$

holds. A little calculation gives

$$\sigma(G)/2 = \sum_{i=1}^{d} is_i = s_1 + |S_d(G)| + \sum_{i=1}^{d-1} (|S_d(G)| - |S_i(G)|) =$$

$$= \binom{n}{2} + \sum_{i=1}^{d-1} \left( \binom{n}{2} - m - |S_i(G)| \right).$$

Now Lemma 1 gives $D \geq d$ and from Lemma 2 the inequality

$$\sigma(G)/2 \leq \binom{n}{2} + \sum_{i=1}^{d-1} \left( \binom{n}{2} - m - |S_i(PK_{n,m})| \right) = \sigma(PK_{n,m})/2$$

follows. \qed

\section*{3. The removal of a vertex}

Now we shall study how the transmission will change if we remove a vertex from a graph. We shall obtain the graphs $G - e$, $G - v$ if we remove from $G$ the edge $e$ or the vertex $v$, respectively. Favaron, Kouider and Maheo in [3] solved a certain problem suggested by Plesnik in [5]. They have found the maximum value of $\sigma(G - e) - \sigma(G)$, as a function of $n$, where $e$ is an edge of the graph $G$ and $G - e$ is connected. Next we shall study a similar problem for removing a vertex.
Let \( f \) be a real function of two integer variables. Then we define the real function \( I_y \) such that for a connected graph \( G \) and such its vertex \( v \) that \( G - v \) is connected we have \( I_y(G, v) = f(\sigma(G - v), \sigma(G)) \).

Next we shall consider the following three properties of a function \( f \):

(\( \text{Dj} \)): the function \( f(i, j) \) is decreasing with respect to \( j \)

(\( \text{Ii} \)): the function \( f(i, j) \) is increasing with respect to \( i \)

(\( \text{Ih} \)): the function \( g(h) = f(h, h + t) \) is increasing for any fixed integer \( t > 0 \).

Finally, by \( w + PK_{n, m} \) we mean the graph obtained from \( PK_{n, m} \) in such a way that we join the new vertex \( w \) to every vertex of \( PK_{n, m} \) by an edge. Next we shall study the extremal values of a function \( F_f \).

**Theorem 2.** Let \( v \) be a vertex of a graph \( G \) with \( n \geq 2 \) vertices and \( m \geq 2n - 3 \) edges and both \( G \) and \( G - v \) be connected. If a function \( f(i, j) \) fulfils (\( \text{Dj} \)) and (\( \text{Ii} \)) then we have

\[
F_f(G, v) < F_f(w + PK_{n-1, m-n+1}, w).
\]

**Proof.** Note that \( \sigma(G) \geq 2n(n - 1) - 2m = \sigma(w + PK_{n-1, m-n+1}, w) \) holds. Further, for the graph \( G - v \) with \( n - 1 \) vertices and \( m' \) edges, \( m - (n - 1) \leq m' \leq m \), we get \( \sigma(G - v) \leq \sigma(PK_{n-1, m-n+1}) \) from Theorem 1. Using the properties (\( \text{Dj} \)) and (\( \text{Ii} \)) we complete this proof.

**Theorem 3.** Let \( G \) be a connected graph of order \( n \geq 2 \), \( v \in V(G) \) and the graph \( G - v \) be connected. If the function \( f(i, j) \) fulfils (\( \text{Dj} \)) and (\( \text{Ii} \)) then we have

\[
F_f(G, v) < \max_{2n - 3 \leq m \leq n(n - 1)/2} F_f(w + PK_{n-1, m-n+1}, w).
\]

**Proof.** If we add to \( G \) an edge incident to \( v \), then the value of \( F_f \) increases. That is why we can restrict ourselves to graphs with at least \( 2n - 3 \) edges. The rest follows from Theorem 2.

Let \( T_{n, t} \), be the set of all connected graphs of the order \( n \) which contain a vertex having the transmission \( t \). The following lemma shows that the path-complete graph has the maximal transmission of all the graphs from \( T_{n, t} \).

**Lemma 3.** Let two integers \( n \geq 2 \) and \( t, n - 1 \leq t \leq \left( \binom{n}{2} \right) \) be given. Then for any graph \( G' \in T_{n, t} \), we have \( \sigma(G') \geq \sigma(PK_{n, m}) \), where \( m = (n + 2)(n - 1)/2 - t \). Moreover, the equality occurs if and only if \( G \cong PK_{n, m} \).

**Proof.** Let \( G \) be the graph from \( T_{n, t} \), having the minimal transmission, \( v \) be its vertex with the transmission \( t \), \( r \) be the eccentricity of \( v \) and \( N_i \) be the set of such vertices \( u \) that \( d(v, u) = i \), for any integer \( i \).

The minimality of the transmission gives that

\[
G(N_i \cup N_{i+1}) \text{ are the complete graphs for all } i \leq r - 1.
\]
If $r = 1$, then $G = G(N_0 \cup N_1)$ is the complete graph, hence it is the path-complete graph on $n$ vertices and $\binom{n}{2}$ edges.

Now we can suppose that $r \geq 2$ holds. Here it is sufficient to prove that the set $N_i$ contains just one element for each $0 \leq i \leq r - 2$. This together with (2) gives that $G$ is a path-complete graph.

We prove it indirectly. Suppose that $i$ is the smallest number such that $N_i$ has at least two elements and $i \leq r - 2$. Clearly $N_0 = \{v\}$, hence $i \geq 1$ holds. Let $v_i \in N_i$, $v_r \in N_r$. Now we shall construct a graph $H$ such that we “move $v_i$ from $N_i$ to $N_{i+1}$ and move $v_r$ from $N_r$ to $N_{r-1}$”. More formally, we omit the edge $v_{i-1}v_i$ where $N_{i-1} = \{v_{i-1}\}$, we add the edges $v_i v_{i+2}$, $v_r v_{r-2}$ for all $v_{i+2} \in N_{i+2}$, $v_{r-2} \in N_{r-2}$. Finally we shall add the edge $v_i v_r$ if $r - i \leq 3$ holds.

Note that the distance of any vertices $u, z$ from $V(G) - \{v_i, v_r\}$ unchanged. Further the sum $d(z, v_i) + d(z, v_r)$ did not change or decreased. The last term unchanged for $z = v_i$ hence $\sigma_H(v_i) = t$ holds and so $H \in T_{n,r}$. Finally $d(v_i, v_r)$ decreased, which gives $\sigma(H) < \sigma(G)$, a contradiction. Thus $G$ is a path-complete graph.

Note that for the vertex $u$ of $PK_{n,m}$ with the smallest degree we have $\sigma(u) + m = \binom{n}{2} + n - 1 = (n + 2)(n - 1)/2$. For $m = n - 1$ this equality holds and if we alter $PK_{n,m}$ to $PK_{n,m-1}$, then we omit one edge and $\sigma(u)$ increases by one. This completes the proof.  

Theorem 4. Let $v$ be a vertex of a graph $G$ on $n \geq 2$ vertices and both $G$ and $G - v$ be connected. If a function $f(i, j)$ fulfils (Dj) and (Ih), then we have

$$\min_{n - 1 \leq m \leq \frac{n}{2} - (n - 2)} F_f(PK_{n,m}, u_{n,m}) \leq F_f(G, v),$$

where $u_{n,m}$ is the endvertex of the graph $PK_{n,m}$.

Proof. The property (Dj) means that if we omit from $G$ an edge incident to $v$, then the value of $F_f$ decreases. So we can restrict it to the case when $v$ is an endvertex of $G$. Therefore

$$2n - 3 \leq \sigma(v) \leq \binom{n}{2} \quad (3)$$

holds. Moreover, we have

$$\sigma(G) = 2\sigma(v) + \sigma(G - v) \quad (4)$$

and so

$$F_f(G, v) = f(\sigma(G - v), 2\sigma(v) + \sigma(G - v)) \quad (5)$$
holds. Let us put \( t' = \sigma(v) \). Then Lemma 3, the equality (5) and the property (Ih) together give
\[
F_f(P_{K_{n,m}}, u_{n,m}) < F_f(G, v) \quad \text{where} \quad \sigma(u_{n,m}) = t' \quad \text{and so} \quad m = (n + 2)(n - 1)/2 - t'.
\]
Further, the inequalities (3) give
\[
n - 1 \leq m \leq \left(\frac{n}{2}\right) - (n - 2).
\]
This establishes the theorem.

Remark. Now we shall consider two special choices of the function \( f \).
Note that the function \( f(i, j) = i/j \) fulfils (Dj), (Ii) and (Ih). So we can apply
Theorems 2, 3, 4 to the ratio \( \sigma(G - v)/\sigma(G) \).

Next we shall study the extremes of the function \( a\sigma(G - v) + b\sigma(G) \) where
\( a, b \) are real. The case \( ab \geq 0 \) is trivial. The other cases can be reduced to the
form \( \sigma(G - v) - q\sigma(G) \) with \( q > 0 \). The function \( i - qj \) fulfils (Dj), (Ii) and also
(Ih) if \( 0 < q < 1 \). But if we want to find the minimal value of \( f \) as a function
of \( n \) for \( q \geq 1 \), then we can restrict ourselves to the case when \( v \) is an endvertex
(it follows from (Dj)). Hence (4) holds and we immediately get
\[
F_f(G, v) = -(2q\sigma(v) + (q - 1)\sigma(G - v)),
\]
which is minimal if and only if \( G \) is the path on \( n \) vertices. We will not deal here
with further technical details.

Eventually the following unsolved problem is presented.

Problem. Find all such graphs \( G \) that the equality \( \sigma(G) = \sigma(G - v) \) holds for
all their vertices \( v \). We know just one such graph — the cycle on 11 vertices.

REFERENCES

[3] FAVARON, O.—KOUIDER, M. MAHEO, M.: Edge-vulnerability and mean distance, pre-
print.
1142—1146.

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