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ESTIMATION OF THE VARIANCE COMPONENTS IN THE LINEAR REGRESSION MODEL

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ABSTRACT. In the paper we derive the explicit expressions for the Bayes invariant quadratic unbiased estimators of the variance components in the regression model with two unhomogeneous variances. Furthermore it is shown from the bayesian viewpoint that such estimator of one variance component is simultaneously optimal in the normal case.

1. Introduction

Let us consider the regression model

$$\boldsymbol{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad E(\boldsymbol{y}) = \mathbf{X}\boldsymbol{\beta}, \quad \operatorname{Var}(\boldsymbol{y}) = \sum_{i=1}^{p} \phi_i \mathbf{V}_i = \mathbf{V}(\boldsymbol{\phi}), \tag{1}$$

where $\boldsymbol{y}$ is an n-dimensional normally distributed random vector, $\mathbf{X}$ is a known $n \times m$ matrix of rank $r(\mathbf{X}) = m \le n$, $\boldsymbol{\beta} \in \mathbb{R}^m$ is a vector of unknown parameters, $\mathbf{V}_1, \dots, \mathbf{V}_p$ are known symmetric matrices and $\boldsymbol{\phi} = (\phi_1, \dots, \phi_p)'$ is a vector of unknown variance components, $\boldsymbol{\phi} \in \Phi$, where $\Phi = \{\boldsymbol{\phi} \in \mathbb{R}^p : \mathbf{V}(\boldsymbol{\phi})$ is a positive definite (p.d.) matrix$\}$.

In the following section we derive the explicit expressions for the Bayes invariant quadratic unbiased estimator (BAIQUE) [1] of the variance components in the regression model with two unhomogeneous variances. The result is applied to the case of the directly measured parameter with two different measuring instruments and to the case of the linear regression function measured with two instruments with different dispersion characteristics.

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We usually get the estimators for the regression parameters and variance components by LSM, i.e. by minimizing the sum of squares

$$S^2 = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}).$$

Let now $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, $\mathbf{V}$ be a p.d. matrix. Then the most used estimator for the variance component σ^2 is

$$\hat{\sigma}^2 = S_0^2/(n-m), \tag{2}$$

where $S_0^2 = \mathbf{y}'\big[\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\big]\mathbf{y}$ is the minimum of the sum of squares S^2 (residual sum of squares). Using the results of [3] we show in section 3 that this estimator is optimal (i.e. quadratic unbiased with minimum variance) in the normal model with one variance component. We shall get this well-known result in a very simple way from the bayesian viewpoint.

2. BAIQUE in the regression models with the unhomogeneous variances

We considered in [3] a linear mixed model with two unknown variance components. It is the model (1) with $p = 2$. The prior distribution of the vector component $\boldsymbol{\phi} = (\phi_1, \phi_2)'$ was characterized by the matrix

$$\mathbf{C} = \big[E(\phi_i\phi_j)\big] = \begin{bmatrix} 1 & u \\ u & u^2 + v^2 \end{bmatrix}.$$

We have shown that BAIQUE for the parametric function $\tau = \mathbf{f}'\boldsymbol{\phi} = f_1\phi_1 + f_2\phi_2$ exists if $\mathcal{M}(\mathbf{V}_2) \subset \mathcal{M}(\mathbf{V}_1 + u\mathbf{V}_2)$ for all $u \geq 0$ and $\mathbf{f} \in \mathcal{M}(\mathbf{R})$, where $\mathcal{M}(\mathbf{V})$ is a vector space generated by the columns of $\mathbf{V}$, $\mathbf{R} = \big[\operatorname{tr}(\mathbf{MN}_i\mathbf{MV}_j)\big]$, $\mathbf{N}_j = \big[(\mathbf{MWM})^+\mathbf{V}_j(\mathbf{MKM})^+ + (\mathbf{MKM})^+\mathbf{V}_j(\mathbf{MWM})^+\big]/2$, $j = 1, 2$, $\mathbf{W} = \mathbf{V}_1 + u\mathbf{V}_2$, $\mathbf{K} = \mathbf{W} + v^2\mathbf{V}_2\mathbf{W}^+\mathbf{V}_2$.

The BAIQUE has the form

$$\hat{\tau} = \mu_1\mathbf{y}'\mathbf{MN}_1\mathbf{My} + \mu_2\mathbf{y}'\mathbf{MN}_2\mathbf{My},$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ satisfies the condition $\mathbf{R}\boldsymbol{\mu} = \mathbf{f}$.

We apply those results to a regression model in which two groups of mutually independent measurings are included. The different precision of the measurings is characterized by the variances σ_1^2, σ_2^2. It can be described by the model (1) with

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}, \quad \mathbf{V}_1 = \begin{bmatrix} \mathbf{I}_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I}_{n_2} \end{bmatrix},$$

$n = n_1 + n_2$, $p = 2$, $\phi_1 = \sigma_1^2$, $\phi_2 = \sigma_2^2$. For $\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ we have

$$\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}'.$$

Using the well-known formel

$$(\mathbf{A} + \mathbf{BDB}')^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{B}'\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}^{-1})^{-1}\mathbf{B}'\mathbf{A}^{-1},$$

we get

$$\mathbf{M} = \mathbf{I} - \mathbf{X}\{(\mathbf{X}_1'\mathbf{X}_1)^{-1} - (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_2'\left[\mathbf{X}_2(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_2' + \mathbf{I}\right]^{-1}\mathbf{X}_2(\mathbf{X}_1'\mathbf{X}_1)^{-1}\}\mathbf{X}'.$$

Now we use the notation

$$\mathbf{P}_{11} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1', \qquad \mathbf{P}_{22} = \mathbf{X}_2(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_2',$$
$$\mathbf{P}_{12} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_2' = \mathbf{P}_{21}'.$$

Then

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix},$$

where

$$\mathbf{M}_{11} = \mathbf{I} - \mathbf{P}_{11} + \mathbf{P}_{12}(\mathbf{P}_{22} + \mathbf{I})^{-1}\mathbf{P}_{21},$$
$$\mathbf{M}_{22} = \mathbf{I} - \mathbf{P}_{22} + \mathbf{P}_{22}(\mathbf{P}_{22} + \mathbf{I})^{-1}\mathbf{P}_{22} = (\mathbf{P}_{22} + \mathbf{I})^{-1},$$
$$\mathbf{M}_{12} = -\mathbf{P}_{12} + \mathbf{P}_{12}(\mathbf{P}_{22} + \mathbf{I})^{-1}\mathbf{P}_{22} = -\mathbf{P}_{12}(\mathbf{P}_{22} + \mathbf{I})^{-1} = \mathbf{M}_{21}'.$$

To check BAIQUE we use the formel

$$(\mathbf{MWM})^{+} = \mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}^{-1}$$

(see [2, Lemma 1.3, p. 384]). In our case (for $u \neq 0$) is

$$\mathbf{W} = \begin{bmatrix} \mathbf{I}_{n_1} & 0 \\ 0 & u\,\mathbf{I}_{n_2} \end{bmatrix}, \qquad \mathbf{W}^{+} = \begin{bmatrix} \mathbf{I}_{n_1} & 0 \\ 0 & u^{-1}\mathbf{I}_{n_2} \end{bmatrix},$$
$$\mathbf{K} = \begin{bmatrix} \mathbf{I}_{n_1} & 0 \\ 0 & r\,\mathbf{I}_{n_2} \end{bmatrix}, \qquad \mathbf{K}^{+} = \begin{bmatrix} \mathbf{I}_{n_1} & 0 \\ 0 & r^{-1}\mathbf{I}_{n_2} \end{bmatrix},$$

where $r = (u^2 + v^2)/u$. By routine computation, we get

$$(\mathbf{MWM})^{+} = \begin{bmatrix} \mathbf{M}_{11}(u_1) & \mathbf{M}_{12}(u_1) \\ \mathbf{M}_{21}(u_1) & \mathbf{M}_{22}(u_1) \end{bmatrix}, \qquad (\mathbf{MKM})^{+} = \begin{bmatrix} \mathbf{M}_{11}(u_2) & \mathbf{M}_{12}(u_2) \\ \mathbf{M}_{21}(u_2) & \mathbf{M}_{22}(u_2) \end{bmatrix}$$

with

$$\mathbf{M}_{11}(u_i) = \mathbf{I} - \mathbf{P}_{11} + \mathbf{P}_{12}(\mathbf{P}_{22} + u_i\,\mathbf{I})^{-1}\mathbf{P}_{21}\,,$$
$$\mathbf{M}_{22}(u_i) = (\mathbf{P}_{22} + u_i\,\mathbf{I})^{-1}\,,$$
$$\mathbf{M}_{12}(u_i) = -\mathbf{P}_{12}(\mathbf{P}_{22} + u_i\,\mathbf{I})^{-1} = \mathbf{M}'_{21}(u_i)\,,$$

$i = 1, 2$, $u_1 = u$, $u_2 = r$, and

$$\mathbf{N}_1 = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{M}\mathbf{N}_1\mathbf{M}\,, \qquad \mathbf{N}_2 = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \mathbf{M}\mathbf{N}_2\mathbf{M}\,, \tag{3}$$

with

$$\mathbf{A}_{11} = \mathbf{I} - \mathbf{P}_{11} + \mathbf{P}_{12}(\mathbf{P}_{22} + u\,\mathbf{I})^{-1}\mathbf{P}_{22}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1}\mathbf{P}_{21} = \mathbf{A}'_{11}\,,$$
$$\mathbf{A}_{22} = (\mathbf{P}_{22} + u\,\mathbf{I})^{-1}\mathbf{P}_{22}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1} = \mathbf{A}'_{22}\,,$$
$$\mathbf{A}_{12} = -\mathbf{P}_{12}(\mathbf{P}_{22} + u\,\mathbf{I})^{-1}\mathbf{P}_{22}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1} = \mathbf{A}'_{21}\,,$$
$$\mathbf{B}_{11} = \mathbf{P}_{12}(\mathbf{P}_{22} + u\,\mathbf{I})^{-1}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1}\mathbf{P}_{21} = \mathbf{B}'_{11}\,,$$
$$\mathbf{B}_{22} = (\mathbf{P}_{22} + u\,\mathbf{I})^{-1}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1} = \mathbf{B}'_{22}\,,$$
$$\mathbf{B}_{12} = -\mathbf{P}_{12}(\mathbf{P}_{22} + u\,\mathbf{I})^{-1}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1} = \mathbf{B}'_{21}\,,$$

since

$$\begin{aligned}
&ur(\mathbf{P}_{22} + u\,\mathbf{I})^{-1}\mathbf{P}_{22}(\mathbf{P}_{22} + r\,\mathbf{I})^{-1} \\
&= \big[\mathbf{I} - \mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2 + u\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_2\big]\mathbf{X}_2(\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_2\big[\mathbf{I} - \mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2 + r\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_2\big] \\
&= ur\mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2 + u\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_1\mathbf{X}_1(\mathbf{X}'_2\mathbf{X}_2 + r\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_2 \\
&= ur\mathbf{X}_2\big[\mathbf{X}'_2\mathbf{X}_2(\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_2\mathbf{X}_2 + u\mathbf{X}'_2\mathbf{X}_2 + r\mathbf{X}'_2\mathbf{X}_2 + ur\mathbf{X}'_1\mathbf{X}_1\big]^{-1}\mathbf{X}'_2\,.
\end{aligned}$$

The BAIQUE of the parametric function $f_1\sigma_1^2 + f_2\sigma_2^2$ exists if and only if

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{M}\begin{bmatrix} n_1 - m_1 + k & g \\ g & h \end{bmatrix},$$

where $m_1 = \operatorname{tr}(\mathbf{X}_1)$, $k = \operatorname{tr}(\mathbf{P}_{22}\mathbf{A}_{22})$, $g = \operatorname{tr}(\mathbf{A}_{22})$, $h = \operatorname{tr}(\mathbf{B}_{22})$.

The BAIQUE is

$$\hat{\tau} = \mu_1\boldsymbol{y}'\mathbf{N}_1\boldsymbol{y} + \mu_2\boldsymbol{y}'\mathbf{N}_2\boldsymbol{y}\,, \tag{4}$$

where μ_1, μ_2 are solutions of the system

$$\begin{aligned} \mu_1(n_1 - m_1 + k) + \mu_2 g &= f_1 , \\ \mu_1 g \qquad\qquad + \mu_2 h &= f_2 . \end{aligned} \tag{5}$$

We get the most simple situation if $\mathbf{X}_1 = \mathbf{1}_{n_1}$, $\mathbf{X}_2 = \mathbf{1}_{n_2}$, where $\mathbf{1}_n = (1,\dots,1)'$. This case is known as a *model of the direct measurings of the scalar parameter* β. Then

$$\mathbf{P}_{11} = \mathbf{J}_{11}/n_1 , \qquad \mathbf{P}_{22} = \mathbf{J}_{22}/n_1 , \qquad \mathbf{P}_{12} = \mathbf{J}_{12}/n_1 ,$$

where

$$\mathbf{J}_{11} = \mathbf{1}_{n_1}\mathbf{1}'_{n_1} , \qquad \mathbf{J}_{22} = \mathbf{1}_{n_2}\mathbf{1}'_{n_2} , \qquad \mathbf{J}_{12} = \mathbf{1}_{n_1}\mathbf{1}'_{n_2} ,$$

$$\begin{aligned} (\mathbf{P}_{22} + u\,\mathbf{I})^{-1} &= (\mathbf{I} - u'\mathbf{J}_{22})/u \qquad \text{with} \quad u' = 1/(n_1 u + n_2) , \\ (\mathbf{P}_{22} + r\,\mathbf{I})^{-1} &= (\mathbf{I} - r'\mathbf{J}_{22})/r \qquad \text{with} \quad r' = 1/(n_1 r + n_2) . \end{aligned}$$

The BAIQUE is given by (3) – (5), where

$$\begin{aligned} \mathbf{A}_{11} &= \mathbf{I} - (n_2 u + n_2 r + n_1 u r) u' r' \mathbf{J}_{11} , \\ \mathbf{A}_{22} &= n_1 u' r' \mathbf{J}_{22} , \\ \mathbf{A}_{12} &= -n_2 u' r' \mathbf{J}_{12} = \mathbf{A}'_{21} , \\ \mathbf{B}_{11} &= n_2 u' r' \mathbf{J}_{11} , \\ \mathbf{B}_{22} &= \left[\mathbf{I} - (u' + r' - u' r' n_2)\mathbf{J}_{22}\right]/(ur) , \\ \mathbf{B}_{12} &= -n_1 u' r' \mathbf{J}_{12} = \mathbf{B}'_{21} . \end{aligned}$$

We get the coefficients μ_1, μ_2 as solution of (5) with

$$\begin{aligned} k &= n_2^2 u' r' , \\ g &= n_1 n_2 u' r' , \\ h &= n_2(1 - u' - r' + u' r' n_2)/(ur) . \end{aligned}$$

Now we derive the explicit expressions for the BAIQUE in the case of the *regression line with two unhomogeneous variances.* Then we have

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{x}_1 \\ \mathbf{1}_{n_2} & \mathbf{x}_2 \end{bmatrix} ,$$

where $\mathbf{x}_1 = (x_1, \dots, x_{n_1})'$, $\mathbf{x}_2 = (x'_1, \dots, x'_{n_2})'$. Let us denote

$$\mathbf{Z}_1 = \begin{bmatrix} n_1 & \sum_{i=1}^{n_1} x_i \\ \sum_{i=1}^{n_1} x_i & \sum_{i=1}^{n_1} x_i^2 \end{bmatrix}, \qquad \mathbf{Z}_2 = \begin{bmatrix} n_2 & \sum_{i=1}^{n_2} x'_i \\ \sum_{i=1}^{n_2} x'_i & \sum_{i=1}^{n_2} {x'_i}^2 \end{bmatrix},$$

$$\mathbf{Z}_{u_i} = \begin{bmatrix} u_i n_1 + n_2 & u_i \sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} x'_i \\ u_i \sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} x'_i & u_i \sum_{i=1}^{n_1} x_i^2 + \sum_{i=1}^{n_2} {x'_i}^2 \end{bmatrix},$$

$i = 1, 2$, $u_1 = u$, $u_2 = r$. Then

$$\mathbf{P}_{11} = (\mathbf{1}_{n_1}, \mathbf{x}_1)\mathbf{Z}_1^{-1}(\mathbf{1}_{n_1}, \mathbf{x}_1)', \qquad \mathbf{P}_{22} = (\mathbf{1}_{n_2}, \mathbf{x}_2)\mathbf{Z}_1^{-1}(\mathbf{1}_{n_2}, \mathbf{x}_2)',$$
$$\mathbf{P}_{12} = (\mathbf{1}_{n_1}, \mathbf{x}_1)\mathbf{Z}_1^{-1}(\mathbf{1}_{n_2}, \mathbf{x}_2)'.$$

The BAIQUE has the form (3) - (4) with

$$\begin{aligned}
\mathbf{A}_{11} &= \mathbf{I}_{n_1} - (\mathbf{1}_{n_1}, \mathbf{x}_1)\big[\mathbf{Z}_1^{-1} - \mathbf{Z}_1^{-1}\mathbf{Z}_2\mathbf{Z}_u^{-1}\mathbf{Z}_1\mathbf{Z}_r^{-1}\mathbf{Z}_2\mathbf{Z}_1^{-1}\big](\mathbf{1}_{n_1}, \mathbf{x}_1)', \\
\mathbf{A}_{22} &= (\mathbf{1}_{n_2}, \mathbf{x}_2)\mathbf{Z}_u^{-1}\mathbf{Z}_1\mathbf{Z}_r^{-1}(\mathbf{1}_{n_2}, \mathbf{x}_2)', \\
\mathbf{A}_{12} &= -(\mathbf{1}_{n_1}, \mathbf{x}_1)\mathbf{Z}_1^{-1}\mathbf{Z}_2\mathbf{Z}_u^{-1}\mathbf{Z}_1\mathbf{Z}_r^{-1}(\mathbf{1}_{n_2}, \mathbf{x}_2)' = \mathbf{A}'_{21}, \\
\mathbf{B}_{11} &= (\mathbf{1}_{n_1}, \mathbf{x}_1)\big[\mathbf{Z}_1^{-1}\mathbf{Z}_2\mathbf{Z}_1^{-1} \\
&\qquad - \mathbf{Z}_1^{-1}\mathbf{Z}_2(\mathbf{Z}_u^{-1} + \mathbf{Z}_r^{-1} - \mathbf{Z}_u^{-1}\mathbf{Z}_2\mathbf{Z}_r^{-1})\mathbf{Z}_2\mathbf{Z}_1^{-1}\big](\mathbf{1}_{n_1}, \mathbf{x}_1)'/(ur), \\
\mathbf{B}_{22} &= \big[(\mathbf{I}_{n_2} - (\mathbf{1}_{n_2}, \mathbf{x}_2)(\mathbf{Z}_u^{-1} + \mathbf{Z}_r^{-1} - \mathbf{Z}_u^{-1}\mathbf{Z}_2\mathbf{Z}_r^{-1})(\mathbf{1}_{n_2}, \mathbf{x}_2)'\big]/(ur), \\
\mathbf{B}_{12} &= -(\mathbf{1}_{n_1}, \mathbf{x}_1)\big[\mathbf{Z}_1^{-1} - \mathbf{Z}_1^{-1}\mathbf{Z}_2(\mathbf{Z}_u^{-1} + \mathbf{Z}_r^{-1} - \mathbf{Z}_u^{-1}\mathbf{Z}_2\mathbf{Z}_r^{-1})\big](\mathbf{1}_{n_2}, \mathbf{x}_2)'/(ur) \\
&= \mathbf{B}'_{21},
\end{aligned}$$

where μ_1, μ_2 are solutions of the system (5) and

$$k = \operatorname{tr}(\mathbf{Z}_2\mathbf{Z}_1^{-1}\mathbf{Z}_2\mathbf{Z}_u^{-1}\mathbf{Z}_1\mathbf{Z}_r^{-1}).$$

Using the results of [4] we can get expressions for the BAIQUE in the model with p unhomogeneous variances $\sigma_1^2, \dots, \sigma_p^2$, but the computations are here even more complicated.

3. A note to the optimality of the BAIQUE in the linear normal model with one variance component

We shall use the following estimators: invariant quadratic unbiased estimator (IQUE) and the best invariant quadratic unbiased estimator (BIQUE) in the same sense as in [1, p. 56].

Let us denote by Γ the class of parametric functions $\tau = \boldsymbol{f}'\boldsymbol{\phi}$, $\boldsymbol{f} = (f_1, \dots, f_p)'$, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_p)'$ for which an IQUE exists, i.e.:

$$\Gamma = \left\{\boldsymbol{f}'\boldsymbol{\phi} :\ \boldsymbol{f} \in \mathcal{M}(\mathbf{Q}),\quad \mathbf{Q} = \left[\operatorname{tr}(\mathbf{MV}_i, \mathbf{MV}_j)\right],\quad \mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right\},$$

$$T = \left\{\mathbf{V}(\boldsymbol{\phi}) :\ \boldsymbol{\phi} \in \mathbb{R}^p\right\},$$

$$Z = \{\mathbf{MVM} :\ \mathbf{V} \in T\}.$$

Now we characterize the optimality of the IQUE in the considered normal case as follows.

Lemma 1. *Let* $\mathbf{I} \in T$. *Then*

a) A BIQUE exists for all $\tau \in \Gamma$ if and only if Z forms a quadratic subspace (i.e. $\mathbf{A} \in Z \implies \mathbf{A}^2 \in Z$).

b) If a BIQUE of $\tau = \boldsymbol{f}'\boldsymbol{\phi}$ exists it is uniquely determined by $\hat{\tau} = \boldsymbol{y}'\mathbf{A}\boldsymbol{y}$, $\mathbf{A} \in Z$, $\operatorname{tr}(\mathbf{AV}_k) = f_k$, $k = 1, \dots, p$.

P r o o f. See [1, Theorems 6–7, p. 56–57].

Suppose that $\mathbf{I} \notin T$, but there exists a p.d. matrix $\mathbf{V}$ such that $\mathbf{V} \in T$. It is equivalent to the condition $\mathbf{I} \in \left\{\sum_{i=1}^{p} \phi_i \mathbf{V}^{-1/2}\mathbf{V}_i\mathbf{V}^{-1/2} :\ \boldsymbol{\phi} \in \mathbb{R}^p\right\}$. Therefore using the regular transformation $\underline{\boldsymbol{y}} = \mathbf{V}^{-1/2}\boldsymbol{y}$ we get a new model with transformed variables

$$\underline{\mathbf{V}}_i = \mathbf{V}^{-1/2}\mathbf{V}_i\mathbf{V}^{-1/2}, \qquad i = 1, \dots, p,$$

$$\underline{\mathbf{X}} = \mathbf{V}^{-1/2}\mathbf{X},$$

$$\underline{\mathbf{M}} = \mathbf{I} - \underline{\mathbf{X}}(\underline{\mathbf{X}}'\underline{\mathbf{X}})^{-1}\underline{\mathbf{X}}'$$

for which Lemma 1 is valid.

Therefore we get:

Lemma 2. *Let p.d. matrix* $\mathbf{V} \in T$. *Then:*

a) *A BIQUE exists for all* $\tau \in \underline{\Gamma}$ *with* $\underline{\Gamma} = \left\{\boldsymbol{f}'\boldsymbol{\phi} :\ \boldsymbol{f} \in \mathcal{M}(\underline{\mathbf{Q}}),\ \underline{\mathbf{Q}} = \left[\operatorname{tr}(\underline{\mathbf{M}}\underline{\mathbf{V}}_i\underline{\mathbf{M}}\underline{\mathbf{V}}_j)\right]\right\}$ *if an only if*

$$\underline{Z} = \left\{\underline{\mathbf{M}}\underline{\mathbf{V}}(\boldsymbol{\phi})\underline{\mathbf{M}} :\ \underline{\mathbf{V}}(\boldsymbol{\phi}) = \sum_{i=1}^{p} \phi_i \mathbf{V}^{-1/2}\mathbf{V}_i\mathbf{V}^{-1/2} \in \underline{T}\right\},$$

where $\underline{T} = \{\underline{\mathbf{V}}(\phi) : \phi \in \mathbb{R}^p\}$, *forms a quadratic subspace.*

b) *If a BIQUE of* τ *exists, it is uniquely determined by* $\hat{\tau} = \underline{\mathbf{y}}'\underline{\mathbf{A}}\underline{\mathbf{y}}$, $\underline{\mathbf{A}} \in \underline{Z}$, $\operatorname{tr}(\underline{\mathbf{A}}\underline{\mathbf{V}}_k) = f_k$, $k = 1, \dots, p$.

Let us apply Lemma 2 to the case $p = 1$, $\phi_1 = \sigma^2$ and $\mathbf{V}_1 = \mathbf{V}$ is p.d. matrix. Now we have $T = \{\sigma^2\mathbf{V} : \sigma \in \mathbb{R}\}$, $\underline{T} = \{\sigma^2\mathbf{I} : \sigma \in \mathbb{R}\}$ and $\underline{Z} = \{\sigma^2\underline{\mathbf{M}} : \sigma \in \mathbb{R}\}$. If $\underline{\mathbf{M}}\underline{\mathbf{A}}\underline{\mathbf{M}} \in \underline{Z}$, then $\underline{\mathbf{M}}\underline{\mathbf{A}}\underline{\mathbf{M}} = \alpha\mathbf{M}$, $\alpha \in \mathbb{R}$. Hence $\underline{\mathbf{M}}\underline{\mathbf{A}}\underline{\mathbf{M}}\underline{\mathbf{M}}\underline{\mathbf{A}}\underline{\mathbf{M}} = \alpha^2\underline{\mathbf{M}}$, i.e. $\underline{\mathbf{M}}\underline{\mathbf{A}}\underline{\mathbf{M}}\underline{\mathbf{M}}\underline{\mathbf{A}}\underline{\mathbf{M}} \in \underline{Z}$ and $\underline{Z}$ is quadratic subspace. By Lemma 2, there exists a BIQUE $\hat{\sigma}^2 = \underline{\mathbf{y}'\mathbf{A}\mathbf{y}}$ of σ^2, where $\underline{\mathbf{A}} = \alpha\underline{\mathbf{M}}$, $\operatorname{tr}(\underline{\mathbf{A}}) = 1$. Hence $\alpha \operatorname{tr}(\underline{\mathbf{M}}) = 1$, i.e. $\alpha = 1/\operatorname{tr}(\underline{\mathbf{M}}) = 1/(n-m)$, $\underline{\mathbf{A}} = \underline{\mathbf{M}}/(n-m)$, $\hat{\sigma}^2 = [1/(n-m)]\underline{\mathbf{y}}'\underline{\mathbf{M}}\underline{\mathbf{y}} = [1/(n-m)]\underline{\mathbf{y}}'[\mathbf{I} - \underline{\mathbf{X}}(\underline{\mathbf{X}}'\underline{\mathbf{X}})^{-1}\underline{\mathbf{X}}']\underline{\mathbf{y}}$. Rewriting it for the variables of the previous model (1) we get $\hat{\sigma}^2$ in the form (2). Hence we see that (2) is BIQUE for the variance component σ^2.

We can get the same result from the BAIQUE, i.e. if we solve the equations

$$c_{11}\underline{\mathbf{M}}\mathbf{A}\underline{\mathbf{M}} = \mu_1\underline{\mathbf{M}}, \qquad \operatorname{tr}(\mathbf{A}\underline{\mathbf{M}}) = 1,$$

which solve the problem analogously to that given in [3] for $p = 1$; a solution is obviously $\mathbf{A} = \underline{\mathbf{M}}/(n-m)$.

Therefore the BAIQUE for one variance component σ^2 in the case of normally distributed $\boldsymbol{y}$ and $\mathbf{V}$ p.d. is a BIQUE.

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