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THE MINIMAL RIGHT A-IDEAL OF THE FREE SEMIGROUP ON A COUNTABLE SET

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In [1] O. Grošek and L. Satko have studied some properties of left, right and two-sided A-ideals of a semigroup S. A left A-ideal G of a semigroup S is a subset $G \subset S$ such that for all $s \in S$ there exists a $g \in G$ satisfying the relation $sg \in G$. The right A-ideal is defined dually and the two-sided A-ideal is a subset of S that is a left A-ideal and a right a right A-ideal of S.

In the presented paper we prove the

Theorem. *The free semigroup on a countable set has a minimal right A-ideal.*

The proof of this Theorem is based on Lemma 1 and Lemma 2.

Lemma 1. *Let $S=(\sigma_n)_{n=1}^{\infty}$ be a free semigroup on a countable set X, where $(\sigma_n)_{n=1}^{\infty}$ is a simple sequence.*

Then there exist a simple sequence $(s_r)_{r=1}^{\infty}$ of elements of S and a sequence $G=(g_n)_{n=1}^{\infty}$ of elements of S such that the following conditions are satisfied:

$$(*)\qquad \begin{aligned} &1)\quad g_{k+1}=g_k s_k, \quad \text{for all} \quad k \in N=\{1, 2, 3, \dots\}\\ &2)\quad d\colon H=\{(g_p, g_q) \in G \times G \mid p<q\} \to S, \end{aligned}$$

$d(g_p, g_q)=s_p \dots s_{q-1}$ is a bijection.

We shall give the proof of this Lemma in several steps. Here $l(\sigma_n)$ will denote the length of the word $\sigma_n \in S$.

First we construct a finite sequence $(s_r)_{r=1}^{k-1}$ of elements of S and a finite sequence $G_k=(g_r)_{r=1}^{k}$ of elements of S for all $k \in N$, $k \geq 2$, satisfying the following conditions:

$$(**)\qquad \begin{aligned} &1)\quad g_{r+1}=g_r s_r \quad \text{for all} \quad r<k\\ &2)\quad d\colon H_k=\{(g_p, g_q) \in G_k \times G_k \mid p<q\} \to S, \end{aligned}$$

$d(g_p, g_q)=s_p \dots s_q$ is an injection.

I) For $k=2$ let $(s_1)=(\sigma_1)$ and $G_2=(g_1, g_2)$, where g_1 is an arbitrary element of S and $g_2=g_1 s_1$. Evidently (s_1) and G_2 satisfy conditions $(**)$.

II) Suppose the sequences $(s_r)_{r=1}^{2^i-1}$ and G_{2^i} having the properties (**) to be defined for $i \in N$. We want to define $(s_r)_{r=1}^{2^i}$ and G_{2^i+1} having the properties (**).

Let $m_{2^i} = \max\{l(d(g_p, g_q)) | (g_p, g_q) \in H_{2^i}\}$. Further let $(s_r)_{r=1}^{2^i}$, where $l(s_{2^i}) \geq 2m_{2^i}$, $s_{2^i} = x^l$ if $X = \{x\}$ and $s_{2^i} = xyx^2y^2 \ldots x^l y^l$ if X contains at least two distinct elements x and y. Let $G_{2^i+1} = (g_r)_{r=1}^{2^i+1}$, where $g_{2^i+1} = g_{2^i}s_{2^i}$.

Evidently the property 1) of (**) is satisfied.

The property 2) of (**) follows from comparing lengths of words.

a) The relation $l(d(g_p, g_{2^i+1})) > l(d(g_q, g_{2^i+1}))$ holds for every $(g_p, g_q) \in H_{2^i}$, hence $d(g_p, g_{2^i+1}) \neq d(g_q, g_{2^i+1})$.

b) We have $l(d(g_r, g_{2^i+1})) \geq l(s_{2^i}) \geq 2m_{2^i} > m_{2^i} \geq l(d(g_p, g_q))$ for every $(g_p, g_q) \in H_{2^i}$ and for every $g_r \in G_{2^i}$, therefore $d(g_r, g_{2^i+1}) \neq d(g_p, g_q)$. This means that $d\colon H_{2^i+1} \to S$ is an ijection.

Hence 2) of (**) holds.

III) Let us suppose that the sequences $(s_r)_{r=1}^{2^i}$ and G_{2^i+1} have the properties (**). We define the sequences $(s_r)_{r=1}^{2^i+1}$ and G_{2^i+2} satisfying the properties (**).

Let S_{2^i+1} be the subsequence of S that can be obtained from S by omitting all elements of $d(H_{2^i+1})$.

Let us now consider the sequence $(s_r)_{r=1}^{2^i+1}$, where s_{2^i+1} is the first term of S_{2^i+1}. Further let $G_{2^i+2} = (g_r)_{r=1}^{2^i+2}$, where $g_{2^i+2} = g_{2^i+1}s_{2^i+1}$.

We shall prove that $(s_r)_{r=1}^{2^i+1}$ and G_{2^i+2} have the properties (**).

Evidently the property 1) of (**) is satisfied.

The property 2) of (**) will be proved in several steps.

a) For every $(g_p, g_q) \in H_{2^i+1}$ we have $l(d(g_p, g_{2^i+2})) > l(d(g_q, g_{2^i+2}))$, hence $d(g_p, g_{2^i+2}) \neq d(g_q, g_{2^i+2})$.

b) The relation $d(g_p, g_q) \neq s_{2^i+1} = d(g_{2^i+1}, g_{2^i+2})$ holds for every $(g_p, g_q) \in H_{2^i+1}$ by the choice of s_{2^i+1}.

c) For every $(g_p, g_q) \in H_{2^i}$ and every $g_t \in G_{2^i}$ we have $l(d(g_p, g_q)) \leq m_{2^i} < 2m_{2^i} \leq l(d(g_t, g_{2^i+1})) < l(d(g_t, g_{2^i+2}))$, hence $d(g_p, g_q) \neq d(g_t, g_{2^i+2})$.

d) For every $(g_p, g_q) \in H_{2^i}$ and also if $g_p = g_q \in G_{2^i}$ we have $l(d(g_q, g_{2^i+1})) < l(d(g_p, g_{2^i+2}))$, therefore $d(g_q, g_{2^i+1}) \neq d(g_p, g_{2^i+2})$.

e) It remains to be shown that $d(g_p, g_{2^i+1}) \neq d(g_q, g_{2^i+2})$ for all $(g_p, g_q) \in H_{2^i}$.

We shall prove it indirectly.

Let

$$d(g_p, g_{2^i+1}) = d(g_q, g_{2^i+2}) \qquad \text{(i)}.$$

From (i) we get

$$s_p \ldots s_{q-1} s_q \ldots s_{2^i} = s_q \ldots s_{2^i} s_{2^i+1} \qquad \text{(ii)}.$$

Hence

$$l(s_p \ldots s_{q-1} s_q \ldots s_{2^i}) = l(s_q \ldots s_{2^i} s_{2^i+1}) \qquad \text{(iii)}.$$

Since both words in (ii) contain the word $s_q \ldots s_{2^i}$, by compairing lengths of these words we get

$$l(s_p \ldots s_{q-1}) = l(s_{2i+1}) = l^* \qquad \text{(iv)}.$$

α) We have $l(d(g_p, g_q)) = l(s_{2i+1})$, $(g_p, g_q) \in H_{2i}$. If $X = \{x\}$, we get $d(g_p, g_q) = s_{2i+1}$, $(g_p, g_q) \in H_{2i}$ which contradicts the choice of s_{2i+1}. Hence in the case of $X = \{x\}$ we have $d(g_p, g_{2i+1}) \neq d(g_q, g_{2i+2})$ for every $(g_p, g_q) \in H_{2i}$.

β) Now we shall deal with the case of X containing at least two distinct elements x and y. By definition $s_{2i} = xyx^2y^2 \ldots x^iy^i$ and $l(xyx^2y^2 \ldots x^iy^i) = l(s_{2i}) \geqslant 2m_{2i} \geqslant 2l(d(g_p, g_q)) = 2l(s_p \ldots s_{q-1}) = 2l^*$. Therefore we may write

$$s_{2i} = uv, \quad \text{where} \quad l(u) = l^* \quad \text{and} \quad l(v) \geqslant l^*.$$

By (ii) we have

$$s_p \ldots s_{q-1} s_q \ldots s_{2i-1} uv = s_q \ldots s_{2i-1} uv s_{2i+1}.$$

The equality of these two words implies the equality of the initial sections of these words having the same length. Since

$$l(s_p \ldots s_{q-1} s_q \ldots s_{2i-1}) = l^* + l(s_q \ldots s_{2i-1}) = l(s_q \ldots s_{2i-1} u) \qquad \text{(v)},$$

we have

$$s_p \ldots s_q \ldots s_{2i-1} = s_q \ldots s_{2i-1} u \qquad \text{(vi)}.$$

From (ii) and (vi) we get

$$s_q \ldots s_{2i} s_{2i+1} = s_p \ldots s_q \ldots s_{2i} = s_p \ldots s_q \ldots s_{2i-1} uv = s_q \ldots s_{2i-1} uuv.$$

This implies

$$s_q \ldots s_{2i-1} s_{2i} s_{2i+1} = s_q \ldots s_{2i-1} uuv.$$

Hence $s_{2i} s_{2i+1} = uuv$.

Since $2l(u) \leqslant l(s_{2i})$, we have $s_{2i} = uuw$, where w may be the empty word. But this is a contradiction with the form of the word $s_{2i} = xyx^2y^2 \ldots x^iy^i$.

This means that for all $(g_p, g_q) \in H_{2i}$ we have $d(g_p, g_{2i+1}) \neq d(g_q, g_{2i+2})$.

From a)—e) we get that the property 2) of (∗∗) is satisfied for the sequences $(s_r)_{r=1}^{2i+1}$ and G_{2i+2}.

Therefore the sequences $(s_r)_{r=1}^{2i+1}$ and G_{2i+2} have the properties (∗∗).

From I)—III) using induction it follows that the sequences $(s_r)_{r=1}^{k-1}$ and G_k have the properties (∗∗) for all $k \in N$, $k \geqslant 2$.

IV) Let us consider $(s_r)_{r=1}^{\infty}$ and $G = (g_n)_{n=1}^{\infty}$. We want to show that these sequences have the properties (∗).

The property 1) of (∗) follows immediatelly from the property 1) of (∗∗).

The injectivity of the function $d\colon H \to S$ follows from the injectivity of the functions $d\colon H_k \to S$.

The surjectivity of the function $d\colon H \to S$ is a consequence of the fact that during the construction all elements of $S = (\sigma_n)_{n=1}^{\infty}$ are used as values of the function $d\colon H \to S$. The element σ_i will be used last in the construction of G_{2i}. Therefore the function $d\colon H \to S$ is a bijection. Hence the property 2) of (∗) is also satisfied.

Lemma 2. *Let $S=(\sigma_n)_{n=1}^{\infty}$ be a free semigroup on a countable set X, where $(\sigma_n)_{n=1}^{\infty}$ is a simple sequence. Let $(s_r)_{r=1}^{\infty}$ be a simple sequence of elements of S and $G=(g_n)_{n=1}^{\infty}$ be a sequence of elements of S satisfying (*). Then $G=(g_n)_{n=1}^{\infty}$ is a minimal right A-ideal of S.*

Proof. a) The mapping d: $H \to S$ is a surjection, hence for every $\sigma_n \in S$ there exists an element $(g_p, g_q) \in H$ such that $g_p\sigma_n = g_q$ i.e. for every $\sigma_n \in S$ there exists an element $g_p \in G$ such that $g_p\sigma_n = g_q \in G$. This means that G is a right A-ideal.

b) If we omit an arbitrary element $g_k \in G$ from G, then $G \backslash \{g_k\}$ is not a right A-ideal.

By the assumption we have $g_{k+1} = g_k s_k$, hence $d(g_k, g_{k+1}) = s_k$. Since $d \cdot H \to S$ is injective, for the element $s_k \in S$ there exists only one pair $(g_p, g_q) \in H \subset G \times G$ such that $d(g_p, g_q) = s_k$, namely $(g_p, g_q) = (g_k, g_{k+1})$.

Let us suppose that $G \backslash \{g_k\}$ is a right A-ideal. Then for the element $s_k \in S$ there exists an element $g_p \in (G \backslash \{g_k\}) \subset G$ such that $g_p s_k = g_q \in (G \backslash \{g_k\}) \subset G$. Clearly $(g_p, g_q) \in H \subset G \times G$ and $d(g_p, g_q) = s_k$ hold. From this it follows that $(g_p, g_q) = (g_k, g_{k+1})$, therefore we have $g_k = g_p \in (G \backslash \{g_k\})$, which is impossible. Hence $G \backslash \{g_k\}$ is not a right A-ideal.

We have proved that G is a minimal right A-ideal. Lemma 1 and Lemma 2 imply directly our Theorem.

Remark. The proof of the Theorem involves the construction of a minimal A-ideal of the infinite cyclic semigroup. O. Grošek and L. Satko have constructed minimal A-ideals of the infinite cyclic semigroup, distinct from minimal A-ideals used in the proof of our Theorem.

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МИНИМАЛЬНЫЙ ПРАВЫЙ А-ИДЕАЛ СВОБОДНОЙ ПОЛУГРУППЫ НА СЧЕТНОМ МНОЖЕСТВЕ

Роберт Шулка

Резюме

Приводится конструкция некоторых минимальных А-идеалов свободной полугруппы на счетном множестве, чем доказывается их существование.