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ON MAGNIFYING ELEMENTS IN GRUPOIDS

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Let S be a grupoid. We say that the element $t \in S$ is a *left magnifier* iff there exists a subset $M \subset S$, $M \neq S$ such that $t \cdot M = S$. The definition of a right magnifying element is similar. This notion was introduced for semigroups by Ljapin [1] and studied by Schwarz [3].

There is a vague conviction that almost everything which is valid for finite algebraic structures (e. g. for finite groups or finite semigroups) is valid also in the compact case. As a matter of fact the structure of a compact semigroup (see e. g. [2]) is very similar to the structure of a finite semigroup. Another example: Ljapin proved that in a finite semigroup there exist no magnifying elements. Schwarz in [3] showed that a compact semigroup contains no magnifying elements.

It is easy to see that if a grupoid S is finite then no magnifying element can exist. The aim of this paper is to show that in some convergence grupoids the existence of a magnifying element is imposible. Thus we generalize the result of [3] in two directions. We assume no topology on S (only convergence structure) and weaken the associativity (which is replaced by the alternativity).

To show that the above statement is only a very vague one, we give an example of a compact Hausdorff grupoid which is associative in a rather weak sense and contains a left magnifying element.

The following notions will be used. A grupoid S is said to be *alternative* iff every its subgrupoid generated by two elements is a semigroup. A grupoid S is said to be *with associative powers* iff every its subgrupoid generated by one element is a semigroup. In such grupoids the power a^n of an element $a \in S$ is unambiguously defined.

A *convergence space* F is a set F with a distinguished class of sequences $\{a_n\}$ ($a_n \in F$) which are called convergent. We assume that to each convergent sequence there correspodens a unique element a of F called the *limit* of the sequence and denoted by $a = \lim_n a_n$ (or simply $a_n \to a$) and the following properties hold:

(i) $\lim_n a_n = a$ if $a_n = a$ for $n = 1, 2, \ldots$.
(ii) If $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and $a_n \to a$, then $a_{n_k} \to a$.

We do not assume that this convergence is determined by a topology. However, it is clear that a topological space is also a convergence space.

The *sequential closure* vA of a set $A \subset F$ is a set of all limits of all convergent sequences $\{a_n\}$ taking their values in A (i. e. $a_n \in A$). If $vA = A$, we say that A is *sequentially closed.*

A *convergence grupoid* S is a grupoid provided with a convergence structure in which multiplication is continuous, i. e., if $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$ (the elements a_n, b_n, a and b being in S).

A convergence grupoid S is called *sequentially compact* iff every sequence $\{a_n\}$ of elements from S contains a convergent subsequence.

Examples for illustration of the above definitions are given in [4].

In the following S denotes always a sequentially compact alternative grupoid. We recall the following results proved in [4]. Such a grupoid contains always at least one idempotent. Let $t \in S$. Consider the sequence $T = \{t, t^2, t^3, \ldots\}$. This sequence contains always a subsequence $\{t^{n_k}\}$ such that $t^{n_k} \to e$, where e is an idempotent. If any subsequence of T converges to an idempotent, then this idempotent is necessarily equal to e. Hence we may say that t belongs to the idempotent e. The sequential closure vT contains a unique idempotent e and $(vT)e$ is a subgroup of S.

1. Proposition. *Let M be a sequentially closed subset of S and let $M \cdot t = S$. Let t belong to the idempotent e. Then $M \cdot e = S$.*

Proof. From $M \cdot t = S$ it follows that $S \cdot t = S$ and so

$$S \cdot t^2 = (S \cdot t) \cdot t = S \cdot t = S.$$

Hence

$$M \cdot t^2 = M \cdot (t \cdot t) = (M \cdot t) \cdot t = S \cdot t = S$$

(where in the second step we used the alternativity of S) and so $M \cdot t^n = S$ by induction. Let $t^{n_k} \to e$ and let s be in S. Since $M \cdot t^{n_k} = S$, there exists a sequence $a_{n_k} \in M$ such that $a_{n_k} \cdot t^{n_k} = s$. We may assume that $a_{n_k} \to a \in M$, since S is sequentially compact and M is sequentially closed. Hence $a \cdot e = s$ and $a \in M$ and so $M \cdot e = S$.

2. Proposition. *If $M \subset S$ and $M \cdot t = S$, then the idempotent $e \in vT$ is a right unit of S.*

Proof. $M \cdot t = S$ implies $S \cdot t = S$. Since S is sequentially closed by the preceding proposition $S \cdot e = S$. And so for every $a \in S$ we have $a = u \cdot e$ for some $u \in S$. Hence for every a in S (using the alternativity) we have

$$a \cdot e = (u \cdot e) \cdot e = u \cdot (e \cdot e) = u \cdot e = a,$$

which concludes the proof.

3. Proposition. *Let $M \cdot t = S$, where $M \subset S$. Then M is sequentially closed.*

Proof. Let $m \in M$ and let $t^{n_k} \to e$. Then by the alternativity of S

$$m \cdot (t \cdot e) = \lim_k m \cdot (t \cdot t^{n_k}) = \lim_k (m \cdot t) \cdot t^{n_k} = (m \cdot t) \cdot e$$

and so

$$M \cdot (t \cdot e) = (M \cdot t) \cdot e = S \cdot e = S.$$

Hence $M \cdot g = S$, where $g = t \cdot e \in vT \cdot e$. Then $M = S \cdot g^{-1}$, where $g \cdot g^{-1} = e$ and $g^{-1} \in vT \cdot e$. Let now $a_n \in M$ and $a_n \to a$, then $a_n = b_n \cdot g^{-1}$, where $b_n \in S$. We may assume by the sequential compactness of S that $b_n \to b \in S$. And so

$$a = b \cdot g^{-1} \in S \cdot g^{-1} = M.$$

4. Theorem. *Let S be an alternative, sequentially compact convergence grupoid. Then S contains no left and no right magnifying elements.*

Proof. Let $M \cdot t = S$, $M \subset S$ and $t \in S$. Then M is a closed subset of S by the last proposition. By proposition 1 $M \cdot e = S$. Since e is a right unit we get that $M \cdot e = M$ and so $M = S$. Hence t cannot be a magnifying element of S.

We give now the example mentioned above. This example shows that the result of the last theorem is the best possible in some sense, namely the alternativity cannot be replaced by the associativity of powers.

5. Example. Let $\{1, 2, 3, \ldots\}$ be the topological space with discrete topology. And let

$$P = \{0, 1, 2, 3, \ldots\}$$

be its one point compactification. Let us define the grupoid structure on P as follows: $1 \odot i = i - 1$ for $i = 2, 3, 4, \ldots$, $1 \odot 0 = 1 \odot 1 = 0$ and $x \odot y = 0$ if $x \neq 1$.

Then P is a compact Hausdorff grupoid with associative powers. It is easy to see that the element 1 is left magnifying since

$$\{0, 2, 3, \ldots\} \subsetneqq 1 \odot \{0, 2, 3, \ldots\} = P.$$

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ОБ УВЕЛИЧИТЕЛЬНЫХ ЭЛЕМЕНТАХ В ГРУПОИДАХ

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Резюме

Целью этой заметки является доказательство утверждения, что в некоторых групоидах сходимости не может существовать никакой увеличительный элемент.