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SEMILINEARLY AND SEMILATTICE RIGHT ORDERED GROUPS

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ABSTRACT. A right partially ordered group (rpo-group) $G = (G, \cdot, \leq)$ is a group endowed with a partial order relation $\leq$ which is compatible to the right with the group multiplication. If an rpo-group G is up-directed and the set $\{x \in G;\ a \leq x\}$ is linearly ordered for each $a \in G$, then G is called a semilinearly ordered group. If $(G, \leq)$ is moreover an upper semilattice then G is said to be a semilinear $\vee$-group. It is shown that the partial order of any semilinear $\vee$-group is isolated, that each normal convex directed subgroup of a semilinearly ordered group is isolated and each convex $\vee$-subgroup of a semilinear $\vee$-group is 2-isolated.

A *right partially ordered group* (*rpo-group*) is a system $G = (G, \cdot, \leq)$ such that $(G, \cdot)$ is a group, $(G, \leq)$ is a partially ordered set and $a \leq b$ implies $ac \leq bc$ for all $a, b, c \in G$. Let $P = P(G) = \{x \in G;\ e \leq x\}$, where e is the unit element of G. Then P is called the *positive cone* of G.

It is obvious that if G is a group and $\emptyset \neq P \subseteq G$, then P is the positive cone of a right partial ordering of the group G if and only if

a) $P \cdot P \subseteq P$,
b) $P \cap P^{-1} = \{e\}$.

(For this fact, and also for all necessary properties of right partially ordered groups, see the book [3].)

A right partially ordered group G is called *directed* (or more precisely *up-directed*) if for each $a, b \in G$ there is $c \in G$ such that $a, b \leq c$. The directedness of a right partially ordered group is equivalent to the condition

c) $G = P^{-1} \cdot P$.

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Note that if G is partially (two-sided) ordered, that means G is right partially ordered and moreover $a \leq b$ implies $ca \leq cb$ for all $a, b, c \in G$, and if it is up-directed, then it is also down-directed, i.e. for each $a, b \in G$ there exists also $d \in G$ with $d \leq a, b$. But for right partially ordered groups which are directed, this second condition is not true in general.

If G is a right partially ordered group such that $a \leq b$ or $b \leq a$ for each elements a, b in G, then G is called a (*linearly*) *right ordered group* (in short: *ro-group*). The theory of right ordered groups is developed in the books [2], [3] and [4].

In the present paper we will deal with the class of semilinearly ordered groups. This class of right partially ordered groups is close to the class of ro-groups and has no non-linear analogy for two-sided ordered groups.

DEFINITION. A right partially ordered group G is called a *semilinearly ordered group* if it is directed and satisfies the following condition:

$$a \leq x \ \& \ a \leq y \implies x \leq y \text{ or } y \leq x .$$

It is obvious that every ro-group is semilinearly ordered. In 1903, Frege asked a question whether there exists a properly semiordered group, i.e. a semilinearly ordered group not being an ro-group. In 1987, Adeleke, Dummett and Neumann (see [1]) found such a semilinearly ordered group and so answered this question in the affirmative. Another example was constructed by Varaksin in [6]. Moreover, he proved that every semilinearly ordered nilpotent group is an ro-group. This result can be generalized for any locally nilpotent group. (See [3; Corollary 8.3.4].) Kopytov (see [3; Example 8.1.1, Theorem 8.4.1]) found a large class of properly semiordered groups and gave a general method for constructions of such groups.

Structure properties of semilinearly ordered semigroups were studied in [5] and [3]. A survey of the results and methods of the theory of semilinearly ordered groups is contained in [3; Chapter 8].

In the paper we study some questions of isolatedness in semilinear $\vee$-groups. It is proved that the order of any semilinear $\vee$-group is isolated and this is used to show that the semilinearly ordered groups constructed by Kopytov are not $\vee$-groups. Further it is shown that the normal convex directed subgroups of semilinearly ordered groups are isolated and that the convex $\vee$-subgroups of semilinear $\vee$-groups are at least 2-isolated. Finally, the existence of proper right-relatively convex subgroups of properly semilinearly ordered groups is proved.

PROPOSITION 1. *Let G be an rpo-group and $a, b, c \in G$. If the join $a \vee b$ of a and b exists, then the join $ac \vee bc$ exists too and $(a \vee b)c = ac \vee bc$.*

Proof. Let $a \vee b$ exist. Then $(a \vee b)c \geq ac, bc$. If $u \in G$ is such that $u \geq ac, bc$, then $uc^{-1} \geq a, b$, thus $uc^{-1} \geq a \vee b$, and hence $u \geq (a \vee b)c$. $\square$

DEFINITION. A right partially ordered group $G = (G, \cdot, \leq)$ is called a *right* $\vee$*-group* if $(G, \leq)$ is an upper semilattice. If G is moreover semilinearly ordered then it is called a *semilinear* $\vee$*-group.*

DEFINITION. The (right) partial order of a (right) partial ordered group G is called *isolated* if for each $n \in \mathbb{N}$ and $a \in G$, $a^n \geq e$ implies $a \geq e$.

It is obvious that the order of any ro-group is isolated. But there are semilinearly ordered groups with orders that are not isolated. (See [3; Section 8.1.2], or a note in this paper below.) Now we will examine the situation for semilinear $\vee$-groups.

THEOREM 2. *If G is a semilinear $\vee$-group then the partial order on G is isolated.*

Proof. Let G be a semilinear $\vee$-group.

1. Let $a \in G$, $a^2 > e$, $a \not\geq e$.

a) Suppose that $a < e$. Then $a^2 > a$, hence $a > e$, a contradiction.

b) Let $a^2 > e$, $a \parallel e$. Then $a \vee a^2 \geq a \vee e > e$. Let us suppose that $a \vee a^2 = a \vee e$. Since $a \vee a^2 = (e \vee a)a$, we get $(a \vee e)a = a \vee e$, thus $a = e$, a contradiction.

Hence, let $a \vee a^2 > a \vee e$. Since $a^2 > e$ and $a \vee e > e$, we have that a^2 and $a \vee e$ are comparable.

If $a^2 \geq a \vee e$ then $a \geq e \vee a^{-1} > e$, a contradiction.

If $a^2 < a \vee e$ then $a \vee a^2 \leq a \vee e$, a contradiction.

Therefore $a^2 > e$ implies $a > e$.

2. Let $n \in \mathbb{N}$, $n > 2$. Let us suppose that for each $m \in \mathbb{N}$, $m \leq n - 1$:

$$b^m \geq e \implies b \geq e \qquad \text{for each} \quad b \in G\,.$$

Let $a \in G$, $a^n > e$, $a \not\geq e$. Then $a^m \not\geq e$ for each $m \leq n - 1$.

a) Let $a \parallel e$. We know that a^n and $a \vee e$ are comparable.

α) Let $a^n \leq a \vee e$. Since $e < a^n$ implies $a \vee e \leq a \vee a^n$, we have $a \vee e = a \vee a^n$. Moreover $a > a^{1-n}$, $a \parallel e$ and $a^{1-n} \parallel e$, hence $a \vee e = a^{1-n} \vee e$. Thus $a^{1-n} \vee e = a^n \vee a$, therefore $(a^n \vee a)a^{-n} = a^n \vee a$, that means $a^{-n} = e$, a contradiction.

β) Let $a^n > a \vee e$. Then $a^n > a$, and so $a^{n-1} > e$, a contradiction.

b) Let $a < e$. Then $a^n > e > a$, hence $a^{n-1} > e$, a contradiction.

Therefore, for all $n \in \mathbb{N}$ we have:

$$a^n \geq e \implies a \geq e \qquad \text{for each} \quad a \in G\,.$$

□

Note. In the book [3; Section 8.1.2], a set of properly semilinearly ordered groups have been constructed. Namely, let $p \in \mathbb{Z}$, $p \neq 0$, $p \neq \pm 1$ and G_p be the multiplicative group of matrices

$$\begin{pmatrix} p^k & q \\ 0 & 1 \end{pmatrix}$$

where q is any rational number in form $|p|^s m$, where k, s and m are integers and p is not a divisor of m. If P_p is the subset of G_p of such matrices that $k \geq 1$, $q \in \mathbb{Z}$, or $k = 0$, $q \in \mathbb{Z}$, $q \geq 0$, then in [3] it is proved that P_p is the positive cone of a properly semilinear right order on G_p.

In [3] it is proved that in (G_p, P_p), the order is not isolated. So we have as an immediate consequence of Theorem 2:

COROLLARY 3. *For any $p \in \mathbb{Z}$, $p \neq 0$, $p \neq \pm 1$, the properly semilinearly ordered group (G_p, P_p) is not a semilinear $\vee$-group.*

So, it remains as an open question:

PROBLEM. Find a semilinear $\vee$-group that is not (linearly) right ordered.

Now, recall that if G is a group then its subgroup H is called *isolated* if for any $a \in G$ and $n \in \mathbb{N}$, $a^n \in H$ implies $a \in H$. It is obvious (see also [3; Proposition 2.1.1]) that any convex subgroup of a right ordered group is isolated. We will deal with isolated subgroups in semilinarly ordered groups.

THEOREM 4.

a) If G is a semilinearly ordered group then each of its proper normal convex directed subgroups is isolated.

b) If P is the positive cone of a semilinearly ordered group and $P^* = \{x \in P \cup P^{-1};\ xy^{-1} \in P \cup P^{-1}$ if $y \in P \cup P^{-1}\}$, then $G_r = P^* \cap P^{*-1}$ is an isolated right ordered subgroup of G.

Proof.

a) If H is a normal convex directed subgroup of G then by [5; Theorem 6] (see also [3; Theorem 8.3.2]), H is a right ordered subgroup of G. Hence, if $a \in G$, $n \in \mathbb{N}$, $a^n \in H$ and $a \geq e$, then from the convexity of H we get $a \in H$. Similarly for $a < e$.

b) By [3; Theorem 8.3.3], G_r is a convex right ordered subgroup of G. Hence, analogously as in a), G_r is isolated. $\square$

If G is a group and H is a subgroup of G, then H is called *2-isolated* if $a^2 \in H$ implies $a \in H$ for any $a \in G$.

THEOREM 5. *If G is a semilinear $\vee$-group and H is a convex $\vee$-subgroup of G then H is 2-isolated.*

Proof. Let $a \in G$ and $a^2 \in H$.

1. If $a^2 \geq e$ then by Theorem 2, $a \geq e$, hence $a^2 \geq a \geq e$, and from the convexity of H it follows that $a \in H$.

2. Similarly for $a^2 < e$.

3. Let $a^2 \in H$, $a^2 \parallel e$. Then also $a \parallel e$, $a^{-1} \parallel e$, $a^{-2} \parallel e$, and moreover, $a^2 \vee e$ and $a \vee e$ are comparable.

a) Let $a^2 \vee e \geq a \vee e$. Since $a^2 \vee e \in H$, the convexity of H implies $a \vee e \in H$, and thus $(a \vee e)a^{-2} \in H$. Hence $(e \vee a^{-1})a^{-1} = a^{-1} \vee a^{-2} \in H$.

If $a^{-1} \vee e \notin H$, then $a^{-1} \vee e > a^2 \vee e$. Hence $a \vee e \leq a^2 \vee e < a^{-1} \vee e$, and thus $e \vee a^{-1} \leq a \vee a^{-1} < a^{-2} \vee a^{-1}$. Therefore $e < e \vee a^{-1} < a^{-2} \vee a^{-1} \in H$, so $e \vee a^{-1} \in H$, a contradiction. Hence $a^{-1} \vee e \in H$. From the fact that $(e \vee a^{-1})a^{-1} \in H$ we get $a^{-1} \in H$, and hence also $a \in H$.

b) Let $a^2 \vee e < a \vee e$. Then $a^2 \not\geq a$.

i) If $a^2 < a$, then $a^2 \vee e = a \vee e$, a contradiction.

ii) Let $a^2 \parallel a$. Suppose that $a^{-1} \vee e > a^2 \vee e$.

α) Let $a^2 \vee e < a \vee e < a^{-1} \vee e$. Then $a \vee a^{-1} < e \vee a^{-1}$, but by the assumption $a \vee a^{-1} > e$, hence $a \vee a^{-1} \geq a^{-1} \vee e$, a contradiction.

β) Let $a^2 \vee e < a^{-1} \vee e < a \vee e$. Then analogously as in α) we get a contradiction.

γ) Let $a^2 \vee e < a^{-1} \vee e = a \vee e$. Then $a \vee e = a^2 \vee a = (a \vee e)a$, thus $a = e$, a contradiction.

Therefore, if $a^2 \vee e < a \vee e$ and $a^2 \parallel a$, then $a^{-1} \vee e \leq a^2 \vee e$, and hence $a^{-1} \vee e \in H$. Thus $(a^{-1} \vee e)a^2 \in H$, i.e. $(e \vee a)a \in H$. Analogously as in part a) we get $e \vee a \in H$, and also $a \in H$. □

Recall that if G is a group and H is a subgroup of G, then H is called *right-relatively convex* if H is convex with respect to some right order of the group G. For semilinearly ordered groups we have:

PROPOSITION 6. *If G is a properly semilinearly ordered group, then G contains proper right-relatively convex subgroups.*

Proof. Every semilinearly ordered group is by [3; Theorem 8.2.1], right orderable and by [3; Proposition 5.1 9], every right orderable group without proper right-relatively convex subgroups is an abelian group. By the assumption, G is not abelian, hence we get the assertion. □

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