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SEMILINEARLY AND SEMILATTICE
RIGHT ORDERED GROUPS

JIŘÍ RACHŮNEK
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ABSTRACT. A right partially ordered group (rpo-group) $G = (G, \cdot, \leq)$ is a
group endowed with a partial order relation $\leq$ which is compatible to the right
with the group multiplication. If an rpo-group $G$ is up-directed and the set
$\{x \in G; \ a \leq x\}$ is linearly ordered for each $a \in G$, then $G$ is called a semilin-
erly ordered group. If $(G, \leq)$ is moreover an upper semilattice then $G$ is said
to be a semilinear $\vee$-group. It is shown that the partial order of any semilinear
$\vee$-group is isolated, that each normal convex directed subgroup of a semilinearly
ordered group is isolated and each convex $\vee$-subgroup of a semilinear $\vee$-group is
2-isolated.

A right partially ordered group (rpo-group) is a system $G = (G, \cdot, \leq)$ such
that $(G, \cdot)$ is a group, $(G, \leq)$ is a partially ordered set and $a \leq b$ implies $ac \leq bc$
for all $a, b, c \in G$. Let $P = P(G) = \{x \in G; \ e \leq x\}$, where $e$ is the unit element
of $G$. Then $P$ is called the positive cone of $G$.

It is obvious that if $G$ is a group and $\emptyset \neq P \subseteq G$, then $P$ is the positive
cone of a right partial ordering of the group $G$ if and only if

a) $P \cdot P \subseteq P$,

b) $P \cap P^{-1} = \{e\}$.

(For this fact, and also for all necessary properties of right partially ordered
groups, see the book [3].)

A right partially ordered group $G$ is called directed (or more precisely up-
directed) if for each $a, b \in G$ there is $c \in G$ such that $a, b \leq c$. The directedness
of a right partially ordered group is equivalent to the condition

c) $G = P \cdot P$.

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subgroup.
Note that if \( G \) is partially (two-sided) ordered, that means \( G \) is right partially ordered and moreover \( a \leq b \) implies \( ca \leq cb \) for all \( a, b, c \in G \), and if it is up-directed, then it is also down-directed, i.e. for each \( a, b \in G \) there exists also \( d \in G \) with \( d \leq a, b \). But for right partially ordered groups which are directed, this second condition is not true in general.

If \( G \) is a right partially ordered group such that \( a \leq b \) or \( b \leq a \) for each elements \( a, b \) in \( G \), then \( G \) is called a (linearly) right ordered group (in short: \( \text{ro-group} \)). The theory of right ordered groups is developed in the books [2], [3] and [4].

In the present paper we will deal with the class of semilinearly ordered groups. This class of right partially ordered groups is close to the class of ro-groups and has no non-linear analogy for two-sided ordered groups.

**Definition.** A right partially ordered group \( G \) is called a semilinearly ordered group if it is directed and satisfies the following condition:

\[ a \leq x \ & \ a \leq y \implies x \leq y \ or \ y \leq x. \]

It is obvious that every ro-group is semilinearly ordered. In 1903, Frege asked a question whether there exists a properly semiordered group, i.e. a semilinearly ordered group not being an ro-group. In 1987, Adeleke, Dummett and Neumann (see [1]) found such a semilinearly ordered group and so answered this question in the affirmative. Another example was constructed by Varaksin in [6]. Moreover, he proved that every semilinearly ordered nilpotent group is an ro-group. This result can be generalized for any locally nilpotent group. (See [3; Corollary 8.3.4].) Kopytov (see [3; Example 8.1.1, Theorem 8.4.1]) found a large class of properly semiordered groups and gave a general method for constructions of such groups.

Structure properties of semilinearly ordered semigroups were studied in [5] and [3]. A survey of the results and methods of the theory of semilinearly ordered groups is contained in [3; Chapter 8].

In the paper we study some questions of isolatedness in semilinear \( \vee \)-groups. It is proved that the order of any semilinear \( \vee \)-group is isolated and this is used to show that the semilinearly ordered groups constructed by Kopytov are not \( \vee \)-groups. Further it is shown that the normal convex directed subgroups of semilinearly ordered groups are isolated and that the convex \( \vee \)-subgroups of semilineral \( \vee \)-groups are at least 2-isolated. Finally, the existence of proper right-relatively convex subgroups of properly semilinearly ordered groups is proved.

**Proposition 1.** Let \( G \) be an \( \text{rpo-group} \) and \( a, b, c \in G \). If the join \( a \vee b \) of \( a \) and \( b \) exists, then the join \( ac \vee bc \) exists too and \( (a \vee b)c = ac \vee bc \).

**Proof.** Let \( a \vee b \) exist. Then \( (a \vee b)c \geq ac, bc \). If \( u \in G \) i such ha \( u \geq ac, bc \), then \( uc^{-1} \geq a, b \), thus \( uc^{-1} \geq a \vee b \), and hence \( u > (a \vee b)c \). \( \square \)
**Definition.** A right partially ordered group \( G = (G, \cdot, \leq) \) is called a right \( \lor \)-group if \( (G, \leq) \) is an upper semilattice. If \( G \) is moreover semilinearly ordered then it is called a semilinear \( \lor \)-group.

**Definition.** The (right) partial order of a (right) partial ordered group \( G \) is called isolated if for each \( n \in \mathbb{N} \) and \( a \in G \), \( a^n \geq e \) implies \( a \geq e \).

It is obvious that the order of any ro-group is isolated. But there are semilinearly ordered groups with orders that are not isolated. (See [3; Section 8.1.2], or a note in this paper below.) Now we will examine the situation for semilinear \( \lor \)-groups.

**Theorem 2.** If \( G \) is a semilinear \( \lor \)-group then the partial order on \( G \) is isolated.

**Proof.** Let \( G \) be a semilinear \( \lor \)-group.

1. Let \( a \in G \), \( a^2 > e \), \( a \not\geq e \).
   a) Suppose that \( a < e \). Then \( a^2 > a \), hence \( a > e \), a contradiction.
   b) Let \( a^2 > e \), \( a \parallel e \). Then \( a \lor a^2 \geq a \lor e > e \). Let us suppose that \( a \lor a^2 = a \lor e \). Since \( a \lor a^2 = (e \lor a)a \), we get \( (a \lor e)a = a \lor e \), thus \( a = e \), a contradiction.
   Hence, let \( a \lor a^2 > a \lor e \). Since \( a^2 > e \) and \( a \lor e > e \), we have that \( a^2 \) and \( a \lor e \) are comparable.
   If \( a^2 \geq a \lor e \) then \( a \geq e \lor a^{-1} > e \), a contradiction.
   If \( a^2 < a \lor e \) then \( a \lor a^2 \leq a \lor e \), a contradiction.
   Therefore \( a^2 > e \) implies \( a > e \).
2. Let \( n \in \mathbb{N} \), \( n > 2 \). Let us suppose that for each \( m \in \mathbb{N} \), \( m \leq n - 1 \):
   \[
   b^m \geq e \implies b \geq e \quad \text{for each } b \in G.
   \]
   Let \( a \in G \), \( a^n > e \), \( a \not\geq e \). Then \( a^m \not\geq e \) for each \( m \leq n - 1 \).
   a) Let \( a \parallel e \). We know that \( a^n \) and \( a \lor e \) are comparable.
   \( a^n \leq a \lor e \). Since \( e < a^n \) implies \( a \lor e \leq a \lor a^n \), we have \( a \lor e = a \lor a^n \).
   Moreover \( a > a^{1-n} \), \( a \parallel e \) and \( a^{1-n} \parallel e \), hence \( a \lor e = a^{1-n} \lor e \).
   Thus \( a^n \lor e = a^n \lor a \), therefore \( (a^n \lor a)a^{-n} = a^n \lor a \), that means \( a^{-n} = e \), a contradiction.
   b) Let \( a^n > a \lor e \). Then \( a^n > a \), and so \( a^{n-1} > e \), a contradiction.
   b) Let \( a < e \). Then \( a^n > e \lor a \), hence \( a^{n-1} > e \), a contradiction.
   Therefore, for all \( n \in \mathbb{N} \) we have:
   \[
   a^n \geq e \implies a \geq e \quad \text{for each } a \in G.
   \]
\[\square\]
Note. In the book [3; Section 8.1.2], a set of properly semilinearly ordered groups have been constructed. Namely, let \( p \in \mathbb{Z} \), \( p \neq 0 \), \( p \neq \pm 1 \) and \( G_p \) be the multiplicative group of matrices

\[
\begin{pmatrix}
p^k & q \\
0 & 1
\end{pmatrix}
\]

where \( q \) is any rational number in form \( \frac{|p|^s m}{p} \), where \( k \), \( s \) and \( m \) are integers and \( p \) is not a divisor of \( m \). If \( P_p \) is the subset of \( G_p \) of such matrices that \( k \geq 1 \), \( q \in \mathbb{Z} \), or \( k = 0 \), \( q \in \mathbb{Z} \), \( q \geq 0 \), then in [3] it is proved that \( P_p \) is the positive cone of a properly semilinear right order on \( G_p \).

In [3] it is proved that in \((G_p, P_p)\), the order is not isolated. So we have as an immediate consequence of Theorem 2:

**Corollary 3.** For any \( p \in \mathbb{Z} \), \( p \neq 0 \), \( p \neq \pm 1 \), the properly semilinearly ordered group \((G_p, P_p)\) is not a semilinear \( \vee \)-group.

So, it remains as an open question:

**Problem.** Find a semilinear \( \vee \)-group that is not (linearly) right ordered.

Now, recall that if \( G \) is a group then its subgroup \( H \) is called isolated if for any \( a \in G \) and \( n \in \mathbb{N} \), \( a^n \in H \) implies \( a \in H \). It is obvious (see also [3; Proposition 2.1.1]) that any convex subgroup of a right ordered group is isolated. We will deal with isolated subgroups in semilinearly ordered groups.

**Theorem 4.**

a) If \( G \) is a semilinearly ordered group then each of its proper normal convex directed subgroups is isolated.

b) If \( P \) is the positive cone of a semilinearly ordered group and \( P^* = \{ x \in P \cup P^{-1} ; \ xy^{-1} \in P \cup P^{-1} \text{ if } y \in P \cup P^{-1} \} \), then \( G_r = P^* \cap P^*^{-1} \) is an isolated right ordered subgroup of \( G \).

**Proof.**

a) If \( H \) is a normal convex directed subgroup of \( G \) then by [5; Theorem 6] (see also [3; Theorem 8.3.2]), \( H \) is a right ordered subgroup of \( G \). Hence, if \( a \in G \), \( n \in \mathbb{N} \), \( a^n \in H \) and \( a \geq e \), then from the convexity of \( H \) we get \( a \in H \). Similarly for \( a < e \).

b) By [3; Theorem 8.3.3], \( G_r \) is a convex right ordered subgroup of \( G \). Hence, analogously as in a), \( G_r \) is isolated.

If \( G \) is a group and \( H \) is a subgroup of \( G \), then \( H \) is called 2-isolated if \( a^2 \in H \) implies \( a \in H \) for any \( a \in G \).

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**THEOREM 5.** If $G$ is a semilinear $\vee$-group and $H$ is a convex $\vee$-subgroup of $G$ then $H$ is 2-isolated.

**Proof.** Let $a \in G$ and $a^2 \in H$.

1. If $a^2 \geq e$ then by Theorem 2, $a \geq e$, hence $a^2 \geq a \geq e$, and from the convexity of $H$ it follows that $a \in H$.

2. Similarly for $a^2 < e$.

3. Let $a^2 \in H$, $a^2 \parallel e$. Then also $a \parallel e$, $a^{-1} \parallel e$, $a^{-2} \parallel e$, and moreover, $a^2 \vee e$ and $a \vee e$ are comparable.

   a) Let $a^2 \vee e \geq a \vee e$. Since $a^2 \vee e \in H$, the convexity of $H$ implies $a \vee e \in H$, and thus $(a \vee e)a^{-1} = a^{-1} \vee a^{-2} \in H$. Hence $(e \vee a^{-1})a^{-1} = a^{-1} \vee a^{-2} \in H$.

   If $a^{-1} \vee e \notin H$, then $a^{-1} \vee e > a^2 \vee e$. Hence $a \vee e \leq a^2 \vee e < a^{-1} \vee e$, and thus $e \vee a^{-1} \leq a \vee a^{-1} < a^2 \vee a^{-1}$. Therefore $e < e \vee a^{-1} < a^2 \vee a^{-1} \in H$, so $e \vee a^{-1} \in H$, a contradiction. Hence $a^{-1} \vee e \in H$. From the fact that $(e \vee a^{-1})a^{-1} \in H$ we get $a^{-1} \in H$, and hence also $a \in H$.

   b) Let $a^2 \vee e < a \vee e$. Then $a^2 \not\geq a$.

   i) If $a^2 < a$, then $a^2 \vee e = a \vee e$, a contradiction.

   ii) Let $a^2 \parallel a$. Suppose that $a^{-1} \vee e > a^2 \vee e$.

   a) Let $a^2 \vee e < a \vee e < a^{-1} \vee e$. Then $a \vee a^{-1} < e \vee a^{-1}$, but by the assumption $a \vee a^{-1} > e$, hence $a \vee a^{-1} \geq e \vee a^{-1}$, a contradiction.

   b) Let $a^2 \vee e < a^{-1} \vee e < a \vee e$. Then analogously as in $\alpha$) we get a contradiction.

   γ) Let $a^2 \vee e < a^{-1} \vee e = a \vee e$. Then $a \vee e = a^2 \vee a = (a \vee e)a$, thus $a = e$, a contradiction.

   Therefore, if $a^2 \vee e < a \vee e$ and $a^2 \parallel a$, then $a^{-1} \vee e \leq a^2 \vee e$, and hence $a^{-1} \vee e \in H$. Thus $(a^{-1} \vee e)a^2 \in H$, i.e. $(e \vee a)a \in H$. Analogously as in part a) we get $e \vee a \in H$, and also $a \in H$. □

Recall that if $G$ is a group and $H$ is a subgroup of $G$, then $H$ is called **right-relatively convex** if $H$ is convex with respect to some right order of the group $G$. For semilinearly ordered groups we have:

**PROPOSITION 6.** If $G$ is a properly semilinearly ordered group, then $G$ contains proper right-relatively convex subgroups.

**Proof.** Every semilinearly ordered group is by [3; Theorem 8.2.1], right orderable, and by [3; Proposition 5.1.9], every right orderable group without proper right-relatively convex subgroups is an abelian group. By the assumption, $G$ is not abelian, hence we get the assertion. □
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