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CONVERGENCE WITH A FIXED REGULATOR IN ARCHIMEDEAN LATTICE ORDERED GROUPS

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ABSTRACT. A convergence with the same regulator u for all sequences in an Archimedean lattice ordered group G is dealt with in this paper. It is shown that a u-Cauchy completion (C-completion) G^* of G is an l-subgroup of the Dedekind completion of G. Some results on the relations between G and G^* are proved. The question of the existence of a greatest C-complete l-ideal of G is investigated.

This paper can be considered as a continuation of the article [3]. In [3] we were dealing with a convergence in a lattice ordered group which is determined by a fixed regulator. J. Martinez [10] examined a convergence with regulators depending on sequences in Archimedean lattice ordered groups. Related notions for vector lattices were studied by Vulikh [11], and Luxemburg and Zaanen [9].

In the present paper we restrict ourselves to the case when the lattice ordered group G under consideration is Archimedean. Let $0 < u \in G$ be a convergence regulator in G. The main results of the paper are as follows.

The Dedekind completion $G^\wedge$ of G is u-Cauchy complete and a u-Cauchy completion G^* of G is an l-subgroup of $G^\wedge$. This implies that G is a dense l-subgroup of G^*. Using this fact, some further results on the relationships between G and G^* are established.

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We are interested in the existence of a greatest u-Cauchy complete l-ideal of G.

A u-Cauchy completion of the direct product of Archimedean lattice ordered groups is constructed.

1. Preliminaries and auxiliary results

The standard terminology for lattice ordered groups will be used (cf. [1], [5], [6]). We recall the basic relevant notions. The group operation in a lattice ordered group will be written additively.

Let G be a lattice ordered group, $\mathbb{N}$ the set of all positive integers and $\mathbb{Q}$ ($\mathbb{R}$) the additive group of all rationals (reals) with the natural linear order. G is called *Archimedean* if for any $x, y \in G$, $nx \leq y$ for all $n \in \mathbb{N}$ implies $x \leq 0$. It is well known that Archimedean lattice ordered groups are Abelian.

A *strong unit* of G is an element $e \in G$, $0 < e$, such that to each $x \in G$ there exists $n \in \mathbb{N}$ satisfying $ne > x$.

G is *torsion free*, i.e., $x \neq 0$ implies $nx \neq 0$ for each $n \in \mathbb{N}$.

Setting $|x| = x \vee (-x)$, we define the *absolute value* of $x \in G$. The relations

$$|x \vee z - y \vee z| \leq |x - y|, \qquad |x \wedge z - y \wedge z| \leq |x - y| \tag{1}$$

are fulfilled for all $x, y, z \in G$.

Let $x, x_i \in G$ for every $i \in I$. If $\bigvee_{i \in I} x_i$ exists in G, then so do $\bigwedge_{i \in I} (-x_i)$, $\bigvee_{i \in I} (x + x_i)$. Moreover, $\bigwedge_{i \in I} (-x_i) = -\bigvee_{i \in I} x_i$, $\bigvee_{i \in I} (x + x_i) = x + \bigvee_{i \in I} x_i$ and dually.

If G is Abelian and $x, y \in G$, then $|x + y| \leq |x| + |y|$; if $n \in \mathbb{N}$ and $nx < ny$, then $x < y$; $n(x \vee y) = nx \vee ny$ for each $n \in \mathbb{N}$ and dually.

Define an l-subgroup H of G to be *dense* in G if for each $0 < g \in G$ there exists $h \in H$ with $0 < h \leq g$.

If every nonempty upper bounded subset of G possesses a least upper bound (or equivalently if each nonempty lower bounded subset of G has a greatest lower bound) in G, then G is called *complete*. Note that a complete lattice ordered group is Archimedean.

Definition 1.1. (cf. [1; p. 71]) Let G, $G^\wedge$ be lattice ordered groups with the following properties:

(i) G is an l-subgroup of $G^\wedge$.
(ii) $G^\wedge$ is complete.
(iii) Every element of $G^\wedge$ is the least upper bound of a subset of G.

Then $G^\wedge$ is said to be a *Dedekind completion* of G.

THEOREM 1.2. (cf. [1; Theorem 8.2.2]) *If G is Archimedean lattice ordered group, then it admits a unique Dedekind completion.*

Remark that G is dense in $G^{\wedge}$.

Luxemburg and Zaanen in their monograph [9] studied the notion of a *u-uniform convergence* in a vector lattice V.

DEFINITION 1.3. (cf. [9]) Let V be a vector lattice and $0 \leq u \in V$. A sequence $(x_n)_{n \in \mathbb{N}}$ (briefly (x_n)) in V is said to *converge u-uniformly* to an element $x \in V$ whenever for every $\varepsilon \in \mathbb{R}$, $0 < \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that

$$|x_n - x| \leq \varepsilon u \qquad \text{for each} \quad n \in \mathbb{N}, \quad n \geq n_0 .$$

This definition was adapted for using in lattice ordered groups as follows (cf. [3]):

DEFINITION 1.4. Let G be a lattice ordered group and $0 < u \in G$. We say that a sequence (x_n) in G *u-converges* to an element $x \in G$, written $x_n \xrightarrow{u} x$ (or x is a *u-limit* of (x_n)), if for every $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$p|x_n - x| \leq u \qquad \text{for each} \quad n \in \mathbb{N}, \quad n \geq n_0 ;$$

u is called a *convergence regulator*.

If $G = \mathbb{Q}$, the u-convergence coincides with the usual convergence for every $u \in \mathbb{Q}$, $0 < u$.

Let us recall some notions from [3] concerning the convergence determined by a fixed convergence regulator in lattice ordered groups. Unless otherwise specified, all results of this section have proofs which may be found in [3].

DEFINITION 1.5. Let G be a lattice ordered group and $0 < u \in G$. A sequence (x_n) in G is called *u-fundamental* whenever for every $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with

$$p|x_n - x_m| \leq u \qquad \text{for all} \quad m, n \in \mathbb{N}, \quad m \geq n \geq n_0 .$$

THEOREM 1.6. *Let G be an Archimedean lattice ordered group and $0 < u \in G$. Then u-limits are uniquely determined.*

In what follows, G is assumed to be an Archimedean lattice ordered group and $0 < u \in G$ the convergence regulator in G. By a convergent (fundamental) sequence and a limit, a u-convergent (u-fundamental) sequence and a u-limit will be meant respectively. The notation $x_n \to x$ (or $x_n \to x$ in G) will be applied instead of $x_n \xrightarrow{u} x$.

By a *zero sequence* in G a sequence (x_n) with $x_n \to 0$ is understood. F (E) stands for the set of all fundamental (zero) sequences in G.

In 1.7–1.9, (x_n), (y_n) are sequences in G and $x, y \in G$.

LEMMA 1.7. *Let* $\Box \in \{+, \wedge, \vee\}$.

(i) *If* $x_n \to x$ *and* $y_n \to y$, *then* $x_n \Box y_n \to x \Box y$.
(ii) *If* $(x_n) \in F$ *and* $(y_n) \in F$, *then* $(x_n \Box y_n) \in F$.
(iii) *If* $x_n \to x$, *then* $kx_n \to kx$ *for each integer* k.
(iv) *If* $(x_n) \in F$, *then* (x_n) *is a bounded sequence.*

LEMMA 1.8. *Let* $x_n \to x$ *and* $x_n \geq 0$ *for every* $n \in \mathbb{N}$. *Then* $x \geq 0$.

Proof. According to 1.7(i), $x_n = x_n \vee 0 \to x \vee 0$. The hypothesis and Theorem 1.6 imply $x = x \vee 0$, which entails $x \geq 0$. □

From Lemmas 1.7 and 1.8 we obtain:

COROLLARY 1.9. *Let* $x_n \to x$, $y_n \to y$ *and* $x_n \leq y_n$ *for each* $n \in \mathbb{N}$. *Then* $x \leq y$.

Every convergent sequence in G is fundamental in G. If also the converse holds, then G is called *u-Cauchy complete* (briefly, *C-complete*).

DEFINITION 1.10. Let G, H be Archimedean lattice ordered groups with the following properties:

(i) G is an l-subgroup of H.
(ii) H is C-complete.
(iii) Every element of H is a limit of some sequence in G.

Then H is said to be a *u-Cauchy completion* (briefly *C-completion*) of G.

Let $(x_n), (y_n) \in F$. If we put $(x_n) + (y_n) = (x_n + y_n)$ and $(x_n) \leq (y_n)$ if and only if $x_n \leq y_n$ for every $n \in \mathbb{N}$, then $(F, +, \leq)$ becomes an Archimedean lattice ordered group. E is an l-ideal of F. Let us form the factor group $G^* = F/E$. We use $(x_n)^*$ to denote the coset of G^* containing the sequence (x_n). G^* is a lattice ordered group. We have $(x_n)^* + (y_n)^* = (x_n + y_n)^*$ and $(x_n)^* \leq (y_n)^*$ if and only if there exist sequences $(x'_n) \in (x_n)^*$ and $(y'_n) \in (y_n)^*$ with $(x'_n) \leq (y'_n)$, or equivalently, for each $(x^1_n) \in (x_n)^*$ there is $(y^1_n) \in (y_n)^*$ with $(x^1_n) \leq (y^1_n)$; $(x_n)^* \vee (y_n)^* = (x_n \vee y_n)^*$ and dually. It is easy to verify that $(x_n)^* \leq (y_n)^*$ if and only if $(x_n) \leq (y_n) + (t_n)$ for some sequence $(t_n) \in E^+$.

The element $(u, u, \dots)^*$ is considered as a convergence regulator in G^*. The mapping $\phi \colon G \to G^*$, defined by $\phi(x) = (x, x, \dots)^*$ for every $x \in G$, is an embedding of the lattice ordered group G into G^*. Under this embedding, G is an l-subgroup of G^*, u is a convergence regulator in G^* and we have:

Remark 1.11. Every element $(x_n)^* \in G^*$ is a limit of some sequence in G, namely $x_n \to (x_n)^*$.

2. u-Cauchy completion and the Dedekind completion of an Archimedean lattice ordered group

Remind that G is assumed to be an Archimedean lattice ordered group and $0 < u \in G$ is a convergence regulator in G and G^*; u will be taken as a convergence regulator in $G^\wedge$.

This section deals with a relation between G^* and the Dedekind completion $G^\wedge$ of G. $G^\wedge$ is an Archimedean lattice ordered group. This is a consequence of the following statement.

THEOREM 2.1. ([5; Proposition 5.4.2]) *A complete lattice ordered group is Archimedean.*

Let (x_n) be an upper bounded sequence in a complete lattice ordered group G and $p \in \mathbb{N}$. Then $\bigvee\limits_{n\in\mathbb{N}} x_n$ and $\bigvee\limits_{n\in\mathbb{N}} px_n \in \mathbb{N}$ do exist in G.

The following result is well known.

LEMMA 2.2. *Let G be a complete lattice ordered group, (x_n) an upper bounded sequence in G and $x_n \geq 0$ for every $n \in \mathbb{N}$. Then $p\bigvee\limits_{n\in\mathbb{N}} x_n = \bigvee\limits_{n\in\mathbb{N}} px_n$ for each $p \in \mathbb{N}$.*

Let (A_n) be a fundamental sequence in $G^\wedge$. By Lemma 1.7(iv) the sequence (A_n) is bounded in $G^\wedge$. Hence there exists $B_n = A_n \wedge A_{n+1} \wedge \dots$ in $G^\wedge$ for each $n \in \mathbb{N}$.

LEMMA 2.3. *If (A_n) is a fundamental sequence in $G^\wedge$, then so is (B_n).*

Proof. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$p|A_n - A_m| \leq u \qquad \text{for each} \quad m,n \in \mathbb{N}, \quad m \geq n \geq n_0 .$$

Let $m,n \in \mathbb{N}$, $m \geq n \geq n_0$. By using (1) we get

$$\begin{aligned}
&p|B_n - B_m| \\
=&p|(A_n \wedge A_{n+1} \wedge \dots \wedge A_{m-1}) \wedge (A_m \wedge A_{m+1} \wedge \dots) - (A_m \wedge A_{m+1} \wedge \dots) \wedge A_m| \\
\leq&p|A_n \wedge A_{n+1} \wedge \dots \wedge A_{m-1} - A_m| \\
=&p|A_m - (A_n \wedge A_{n+1} \wedge \dots \wedge A_{m-1})| \\
=&p\big|A_m + \big((-A_n) \vee (-A_{n+1}) \vee \dots \vee (-A_{m-1})\big)\big| \\
=&p|(A_m - A_n) \vee (A_m - A_{n+1}) \vee \dots \vee (A_m - A_{m-1})| \\
\leq&p|A_m - A_n| \vee p|A_m - A_{n+1}| \vee \dots \vee p|A_m - A_{m-1}| \leq u .
\end{aligned}$$

Thus (B_n) is a fundamental sequence in $G^\wedge$. □

THEOREM 2.4. *Let G be an Archimedean lattice ordered group. Then $G^{\wedge}$ is C-complete.*

P r o o f . Let (A_n) be a fundamental sequence in $G^{\wedge}$ and let (B_n) be as above. Then Lemmas 2.3 and 1.7(iv) imply that the sequence (B_n) is bounded (this follows also from the definition of B_n). Hence there exists $B = \bigvee_{m\in\mathbb{N}} B_m = \bigvee_{\substack{m\in\mathbb{N}\\ m\geq n+1}} B_m$. We intend to show that $A_n \to B$.

Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that

$$p|A_n - A_m| \leq u \qquad \text{for all} \quad m, n \in \mathbb{N}, \quad m \geq n \geq n_0 .$$

Suppose that $n \in \mathbb{N}$, $n \geq n_0$. Applying Lemma 2.2 we get

$$\begin{aligned}
p|A_n - B| &= p|B - A_n| = p\Big| \bigvee_{\substack{m\in\mathbb{N}\\ m\geq n+1}} B_m - A_n \Big| \\
&= p|B_{n+1} \vee B_{n+2} \vee \dots - A_n| \\
&= p|(B_{n+1} - A_n) \vee (B_{n+2} - A_n) \vee \dots| \\
&= p|(A_{n+1} \wedge A_{n+2} \wedge \dots - A_n) \vee (A_{n+2} \wedge A_{n+3} \wedge \dots - A_n) \vee \dots| \\
&\leq p\left(|A_{n+1} \wedge A_{n+2} \wedge \dots - A_n| \vee |A_{n+2} \wedge A_{n+3} \wedge \dots - A_n| \vee \dots\right) \\
&= p|(A_{n+1} - A_n) \wedge (A_{n+2} - A_n) \wedge \dots| \\
&\qquad\qquad \vee p|(A_{n+2} - A_n) \wedge (A_{n+3} - A_n) \wedge \dots| \vee \dots \\
&= p|(A_n - A_{n+1}) \vee (A_n - A_{n+2}) \vee \dots| \\
&\qquad\qquad \vee p|(A_n - A_{n+2}) \vee (A_n - A_{n+3}) \vee \dots| \vee \dots \\
&\leq p\left(|A_n - A_{n+1}| \vee |A_n - A_{n+2}| \vee \dots\right) \\
&\qquad\qquad \vee p\left(|A_n - A_{n+2}| \vee |A_n - A_{n+3}| \vee \dots\right) \vee \dots \\
&= p|A_n - A_{n+1}| \vee p|A_n - A_{n+2}| \vee \dots \leq u.
\end{aligned}$$

as desired. □

Let (x_n) be a sequence in G and $x \in G$. It is easily seen that $x_n \to x$ in G if and only if $x_n \to x$ in $G^{\wedge}$ and that (x_n) is fundamental in G if and only if (x_n) is fundamental in $G^{\wedge}$.

Suppose that $x \in G^*$. With respect to Theorem 1.11, there exists a sequence (x_n) in G such that $x_n \to x$ in G^*. Since (x_n) is fundamental in G, it is also fundamental in $G^{\wedge}$. By Theorem 2.4, there exists $A \in G^{\wedge}$ with $x_n \to A$ in $G^{\wedge}$. Define the mapping $\varphi\colon G^* \to G^{\wedge}$ by the rule $\varphi(x) = A$.

Assume that also for a sequence (x'_n) in G, $x'_n \to x$ in G^* holds. Then there exists $A' \in G^{\wedge}$ with $x'_n \to A'$ in $G^{\wedge}$. By Lemma 1.7(i) we have $x_n - x'_n \to 0$,

$x_n - x'_n \to A - A'$ in $G^\wedge$. Applying Theorem 1.6 we get $A = A'$. Therefore φ is correctly defined. An analogous argument proves that φ is injective.

Let $x = (x_n)^*$, $y = (y_n)^*$ and $\varphi(x) = A$, $\varphi(y) = B$. Theorem 1.11 implies that $x_n \to x$, $y_n \to y$ in G^*. Then $x_n \to A$, $y_n \to B$ in $G^\wedge$. By Lemma 1.7(i), $x_n \wedge y_n \to x \wedge y$ in G^* and $x_n \wedge y_n \to A \wedge B$ in $G^\wedge$. Consequently, $\varphi(x \wedge y) = A \wedge B = \varphi(x) \wedge \varphi(y)$. Dually, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$.

It is easy to verify that φ preserves the group operation.

At the end we identify x and $\varphi(x)$ for every $x \in G$. We have proved the validity of the following Theorem.

THEOREM 2.5. *Let G be an Archimedean lattice ordered group. Then G^* is an l-subgroup of $G^\wedge$.*

Since G is dense in $G^\wedge$, we get:

COROLLARY 2.6. *G is a dense l-subgroup in G^*.*

The following question remained open in [3]: Is G^* Archimedean lattice ordered group for each Archimedean lattice ordered group G? The following Corollary of Theorems 2.1 and 2.5 gives the positive answer to this question.

COROLLARY 2.7. *G^* is an Archimedean lattice ordered group.*

From Corollary 2.7 and [3; Theorems 3.16, 3.17] we obtain:

THEOREM 2.8. *G^* is a C-completion of G. It is uniquely determined up to isomorphisms over G.*

In general, G^* does not coincide with $G^\wedge$.

EXAMPLE 2.9. Let G be the set of all eventually constant sequences of real numbers. G is an Archimedean lattice ordered group under the addition and the ordering performed componentwise. The Dedekind completion $G^\wedge$ of G is the lattice ordered group of all bounded sequences of real numbers. This is a consequence of [4; Theorem 2.5]. We choose the constant sequence $u = (1, 1, \dots)$ as a convergence regulator in G and also in $G^\wedge$. With respect to Theorem 2.5, G^* is an l-subgroup of $G^\wedge$. There is no sequence in G that converges to the element $(1, 0, 1, 0, \dots) \in G^\wedge$. Thus $G^\wedge$ fails to have the property (iii) of C-completion of G from the Definition 1.10. We conclude that $G^* \neq G^\wedge$.

Let G be a lattice ordered group and $x \in G$. The set

$$x^\perp = \{y \in G : \ |y| \wedge |x| = 0\}$$

is said to be a *polar of* x. For $X \subseteq G$, we set $X^\perp = \bigcap\{x^\perp : x \in X\}$. $X^\perp$ is called a *polar of* X; $x^\perp$ and $X^\perp$ are convex l-subgroups of G.

A lattice ordered group G is called *projectable* if $G = g^{\perp\perp} \times g^{\perp}$ for each $g \in G$ (cf. [5]). It is well known that each complete lattice ordered group is projectable.

Hence we have

$$G^{\wedge} = u^{\perp\perp} \times u^{\perp}. \tag{2}$$

Every element $z \in G^{\wedge}$ can be uniquely expressed in the form $z = z^1 + z^2$, $z^1 \in u^{\perp\perp}$, $z_2 \in u^{\perp}$. Let (x_n) be a sequence in $G^{\wedge}$, $x \in G^{\wedge}$. Under the above notation, $x_n = x_n^1 + x_n^2$ for each $n \in \mathbb{N}$, $x = x^1 + x^2$; $u^1 = u$, $u^2 = 0$, as $u \in u^{\perp\perp}$.

Lemma 2.10. *Let* (2) *be valid. If* (x_n) *is a sequence in* $G^{\wedge}$ *and* $x \in G^{\wedge}$ *such that* $x_n \to x$, *then*

(i) $x_n^1 \to x^1$,

(ii) *there exists* $n_0 \in \mathbb{N}$ *with* $x_n^2 = x^2$ *for each* $n \in \mathbb{N}$, $n \geq n_0$.

P r o o f. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$|x_n - x| \leq u \qquad \text{for every} \quad n \in \mathbb{N}, \quad n \geq n_0.$$

Consequently, for each $n \in \mathbb{N}$, $n \geq n_0$,

$$|x_n^1 - x^1| = |x_n - x|^1 \leq u \qquad \text{and} \qquad |x_n^2 - x^2| = |x_n - x|^2 \leq 0$$

is satisfied as desired. □

From Lemma 2.10, there immediately follows:

Lemma 2.11. *Let* (2) *be valid. If* (x_n) *is a sequence in* $u^{\perp}$ *and* $x \in G^{\wedge}$ *such that* $x_n \to x$ *in* $G^{\wedge}$, *then* $x \in u^{\perp}$ *and there exists* $n_0 \in \mathbb{N}$ *such that* $x_n = x$ *for each* $n \in \mathbb{N}$, $n \geq n_0$.

3. u-Cauchy completion of the direct product of Archimedean lattice ordered groups

Let G be the direct product of lattice ordered groups G_i, $i \in I$. This fact is expressed by writing

$$G = \prod_{i \in I} G_i. \tag{3}$$

The ith component of an element $x \in G$ is denoted by $x(i)$. Since G is Archimedean, all G_i are Archimedean as well.

In Lemmas 1–3 it is supposed that G fulfils (3) and that $u(i)$ is a convergence regulator in G_i for each $i \in I$. For a sequence (x_n^i) in G_i and $x^i \in G_i$, we will write $x_n^i \to x^i$ instead of $x_n^i \xrightarrow{u(i)} x^i$. F_i (E_i) will denote the set of all fundamental (zero) sequences in G_i for each $i \in I$.

Lemma 3.1. *Let (x_n) be a sequence in G and $x \in G$. Then $x_n \to x$ if and only if $x_n(i) \to x(i)$ for each $i \in I$.*

Proof. Let $p \in \mathbb{N}$. Then $p|x_n - x| \le u$ holds if and only if $p|x_n - x|(i) \le u(i)$ for each $i \in I$. As $|x_n - x|(i) = |x_n(i) - x(i)|$, the proof is finished. □

The proof of the following Lemma is analogous.

Lemma 3.2. *Let (x_n) be a sequence in G. Then $(x_n) \in F$ if and only if $\big(x_n(i)\big) \in F_i$ for each $i \in I$.*

Let $i \in I$. E_i is an l-ideal of F_i. We can form the factor group $G_i^* = F_i/G_i$; G_i^* is a lattice ordered group under the natural group and lattice operations.

Lemma 3.3. *G_i^* is Archimedean for each $i \in I$.*

Proof. Let $i \in I$. Assume that $(x_n^i)^*$, $(y_n^i)^* \in G_i^*$ and $k(x_n^i)^* \le (y_n^i)^*$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ there exists a sequence $(t_n^{ik}) \in E_i$ with $k(x_n^i) \le (y_n^i) + (t_n^{ik})$. Let (x_n) be a sequence in G with $x_n(i) = x_n^i$ and $x_n(j) = 0$ for all $j \in I$, $j \ne i$ and all $n \in \mathbb{N}$. Sequences (y_n) and (t_n^k) in G are defined similarly. By Lemmas 3.1 and 3.2, $(t_n^k) \in E$ for each $k \in \mathbb{N}$ and $(x_n) \in F$. We have $k(x_n) \le (y_n) + (t_n^k)$, which entails $k(x_n)^* \le (y_n)^*$ for each $k \in \mathbb{N}$. The assumption implies $(x_n)^* \le E$. Then $(x_n) \le (v_n)$ for some $(v_n) \in E$. Applying Lemma 3.1, $v_n(i) \in E_i$. The relation $x_n^i = x_n(i) \le v_n(i)$ yields $(x_n^i)^* \le E_i$, as desired. □

Theorem 3.4. *We have*

$$G^* \simeq \prod_{i \in I} G_i^* .$$

Proof. Let $(x_n)^* \in G^*$. Using Lemma 3.2, from $(x_n) \in F$ it follows that $\big(x_n(i)\big) \in F_i$ for each $i \in I$, so $\big(x_n(i)\big)^* \in G_i^*$ for each $i \in I$. Let X be an element of $\prod_{i \in I} G_i^*$ with $X_{(i)} = \big(x_n(i)\big)^*$ for each $i \in I$. Define the mapping $\psi\colon G^* \to \prod_{i \in I} G_i^*$ by the rule $\psi\big((x_n)^*\big) = X$.

Let $(x_n)^*$, $(y_n)^* \in G^*$. We have $(x_n)^* = (y_n)^*$ if and only if $(x_n - y_n) \in E$, i.e., $(x_n - y_n)(i) = \big(x_n(i) - y_n(i)\big) \in E_i$ for each $i \in I$ by Lemma 3.1. That means $\big(x_n(i)\big)^* = \big(y_n(i)\big)^*$ for each $i \in I$. We conclude that ψ is correctly defined and one-to-one.

To show that ψ is a mapping from G^* onto $\prod_{i \in I} G_i^*$, suppose that $Y \in \prod_{i \in I} G_i^*$. For each $i \in I$ there is a sequence $(y_n^i) \in F_i$ with $Y_{(i)} = (y_n^i)^*$. For each $n \in \mathbb{N}$ denote by y_n the element of G with $y_n(i) = y_n^i$ for each $i \in I$. Lemma 3.2 implies $(y_n) \in F$. Consequently, $(y_n)^* \in G^*$ is the origin of Y under the mapping ψ.

One readily sees that ψ preserves the group and lattice operations. □

4. Some further results on G and G^*

In this section we investigate the question which properties of G remain valid for G^*. Further, it is shown that the system of all $\mathrm{C_b}$-complete l-ideals of G has a greatest element. An analogous problem is dealt with for C-completeness.

Evidently, a chain in G is a chain in G^*.

A subset A of G is called an *antichain* in G if $x \parallel y$ for each $x, y \in A$, $x \neq y$.

LEMMA 4.1. *Let A be an antichain in G. Then A is an antichain in G^*.*

Proof. Let $x, y \in A$, $x \neq y$. Then $x \parallel y$. Evidently, $(x, x, \dots)^* \neq (y, y, \dots)^*$. We have to show that $(x, x, \dots)^* \parallel (y, y, \dots)^*$. Let $(x, x, \dots)^* < (y, y, \dots)^*$. Then there are sequences $(x_n) \in (x, x, \dots)^*$ and $(y_n) \in (y, y, \dots)^*$ with $(x_n) \leq (y_n)$. By Theorem 1.11, $x_n \to x$ and $y_n \to y$. Corollary 1.9 implies $x \leq y$, a contradiction. □

A chain K in G is called *maximal* if for each chain H in G with $K \subseteq H$ the relation $K = H$ is valid. The notion of a *maximal antichain* in G is defined similarly.

A maximal chain (antichain) in G need not be a maximal chain (antichain) in G^*.

EXAMPLE 4.2. Let G be the direct product of lattice ordered groups G_1, G_2, written $G = G_1 \times G_2$ with $G_1 = G_2 = \mathbb{Q}$. The set $K = \{(q, q) \in G : \ q \in \mathbb{Q}\}$ is a maximal chain in G and $A = \{(-q, q) \in G : \ q \in \mathbb{Q}\}$ is a maximal antichain in G. On the other hand, $K(A)$ fails to be a maximal chain (antichain) in G^*. Indeed, by Theorem 3.4 we get $G^* \simeq G_1^* \times G_2^* \simeq \mathbb{R} \times \mathbb{R}$.

It is well known that every Archimedean linearly ordered group is a subgroup of $\mathbb{R}$. Therefore we get:

THEOREM 4.3. *G^* is a linearly ordered group if and only if G is a linearly ordered group.*

A nonempty system S of strictly positive elements from G is called *disjoint* if $x \wedge y = 0$ for each $x, y \in S$, $x \neq y$. We say that S is a *maximal disjoint system* in G if $0 \leq g \in G$ and $g \wedge x = 0$ for each $x \in S$ imply $g = 0$.

LEMMA 4.4. *Let S be a maximal disjoint system in G. Then S is a maximal disjoint system in G^*.*

Proof. Let $x, y \in S$, $x \neq y$. Then $x \wedge y = 0$. Therefore $(x, x, \dots)^* \wedge (y, y, \dots)^* = (x \wedge y, x \wedge y, \dots)^* = (0, 0, \dots)^* = E$. Hence S is a disjoint system in G^*. Assume that $E \leq (x_n)^* \in G^*$ and $(x_n)^* \wedge x = E$ is fulfilled for every

$x \in S$. By way of contradiction, suppose that $(x_n)^* > E$. In view of Corollary 2.6, there exists $g \in G$ with $0 < g \le (x_n)^*$. Therefore $g \wedge x = 0$ for every $x \in S$, which contradicts to the maximality of S. □

A group H is called *divisible* if for each $x \in H$ and each $k \in \mathbb{N}$ there exists $y \in H$ such that $ky = x$.

Lemma 4.5. *If G is divisible, then so does G^*.*

P r o o f. We have to prove that for each $k \in \mathbb{N}$ and each $x \in G^*$ there exists $y \in G^*$ with $ky = x$.

Let $k \in \mathbb{N}$ and $x \in G^*$. With respect to Theorem 1.11, there exists a sequence (x_n) in G with $x_n \to x$ in G^*. Then (x_n) is fundamental in G^* and also in G. For any $n \in \mathbb{N}$ there is $y_n \in \mathbb{N}$ such that $ky_n = x_n$. Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$p|y_n - y_m| \le p|ky_n - ky_m| = p|x_n - x_m| \le u$$

for each $m, n \in \mathbb{N}$, $m \ge n \ge n_0$. Therefore $(y_n) \in F$. Again, according to Theorem 1.11, there exists $y \in G^*$ with $y_n \to y$ in G^*. By 1.7(iii), $ky_n \to ky$. Then from $ky_n \to x$ in G^* and Theorem 1.6 we conclude $ky = x$. □

A subset S of G will be called *Cauchy complete* (briefly *C-complete*) if for each sequence (x_n) in S such that $(x_n) \in F$ there exists $x \in S$ with $x_n \to x$ in G.

If for each sequence (x_n) in S, bounded in S, $(x_n) \in F$ there exists $x \in S$ with $x_n \to x$ in G, then we say that S is a *Cauchy b-complete subset* of G (briefly *C_b-complete*).

J. J a k u b í k [8] studied o-convergence in lattice ordered groups. By the same method as used in [8], the following three results can be proved.

Lemma 4.6. *Let $a, b, c \in G$, $a \le b \le c$. If intervals $[a, b]$ and $[b, c]$ are C-complete subsets of G, then $[a, c]$ is also a C-complete subset of G.*

Lemma 4.7. *Let $a, b, c \in G$, $a \le b$. If $[a, b]$ is a C-complete subset of G, then $[a+c, b+c]$ is also a C-complete subset of G.*

Lemma 4.8. *Let $a, b \in G$, $0 \le a, b$. If $[0, a]$ and $[0, b]$ are C-complete subsets of G, then $[0, a+b]$ is also a C-complete subset of G.*

Let us form the set

$$M = \left\{x \in G : \ [0, |x|] \text{ is a C-complete subset of } G\right\}.$$

LEMMA 4.9. *M is an l-ideal of G.*

Proof. Let $x, y \in M$. Then $[0,|x|]$ and $[0,|y|]$ are C-complete subsets of G. Using Lemma 4.8, $[0,|x|+|y|]$ is a C-complete subset of G as well. From $|x+y| \leq |x|+|y|$ and Corollary 1.9 we infer that $[0,|x+y|]$ is a C-complete subset of G, so $x+y \in M$. If $x \in M$, then also $-x \in M$ because of the relation $|x| = |-x|$. We have shown that M is a subgroup of G. Since $|x \vee y| \leq |x| \vee |y| \leq |x|+|y|$, the same argument as above proves that $x \vee y \in M$ and thus M is a sublattice of G. It is apparent that M is a convex subset of G. □

LEMMA 4.10. *M is the greatest C_b-complete l-ideal of G.*

Proof. We start by proving that M is a C_b-complete subset of G. Let (x_n) be a sequence in M, bounded in M, and $(x_n) \in F$. There are $a, b \in M$ such that $x_n \in [a,b]$ for every $n \in \mathbb{N}$. It suffices to show that $[a,b]$ is a C-complete subset of G. By Lemma 4.9, $b-a \in M$ and so $[0, b-a]$ is a C-complete subset of G. Applying Lemma 4.7, $[a,b]$ is a C-complete subset of G. Suppose that M' is an l-ideal of G that is a C_b-complete subset of G. Choose any $g \in M'$. From $[0,|g|] \subseteq M'$ it follows that $[0,|g|]$ is a C-complete subset of G, implying that $g \in M$. We conclude $M' \subseteq M$. □

The idea of proofs of Lemma 4.9 and Theorem 4.10 are similar to those used in [2] examining a system of intervals in lattice ordered groups.

The question whether there exists a greatest C-complete l-ideal of G remains open.

Nevertheless, the following two results are valid.

An l-ideal of G generated by C-complete l-ideals of G need not be C-complete for some convergence regulator in G.

EXAMPLE 4.11. Let $G = \prod_{i \in \mathbb{N}} G_i$, $G_i = \mathbb{R}$ for every $i \in \mathbb{N}$. Then $G'_i = \{g \in G : g(j) = 0 \text{ for each } j \in \mathbb{N},\ j \neq i\}$ is an l-ideal of G and it is a C-complete subset of G for every $i \in I$. Denote by H the l-ideal of G generated by the set $\bigcup_{i \in I} G'_i$. H consists of all elements of G having a finite support. Let us form the sequence (x_n) in H by setting $x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right)$ for every $n \in \mathbb{N}$. If $u \in G$, $u = (1,1,1,\dots)$ is considered as a convergence regulator in G and $x = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$, then $x_n \to x$ in G. Whence (x_n) is a fundamental sequence in G. Therefore H fails to be C-complete subset of G, as $x \notin H$.

THEOREM 4.12. *Let $S = \{G_i\}_{i \in I}$ be the system of all C-complete l-ideals of G such that $u \in G_i$ for each $i \in I$. Then the system S has a greatest element.*

Proof. We claim that the l-ideal H of G generated by the set $\bigcup_{i \in I} G_i$ is the greatest element of S. It is enough to show that H is a C-complete subset of G. Let (x_n) be a sequence in H such that $(x_n) \in F$. Because $u \in H$, 1.7(iv) yields that (x_n) is bounded in H. There are $a, b \in H$, $a < b$, with $x_n \in [a, b]$ for every $n \in \mathbb{N}$. We have $0 < b - a \in H$. There are $i_1, i_2, \dots, i_n$ from I such that there exist $0 < c_1 \in G_{i_1}$, $0 < c_2 \in G_{i_2}$, ..., $0 < c_n \in G_{i_n}$ with $b - a \leq c_1 + c_2 + \dots + c_n$. Since G_i is a C-complete subset of G, $[0, c_i] \subseteq G_i$ and Corollary 1.9 yield that $[0, c_i]$ is a C-complete subset of G for each $i \in I$. By Lemma 4.8 and induction we get that $[0, c_1 + c_2 + \dots + c_n]$ is a C-complete subset of G. From $[0, b-a] \subseteq [0, c_1 + c_2 + \dots + c_n]$ and Corollary 1.9 we infer $[0, b-a]$ is a C-complete subset of G. Applying Lemma 4.7, $[a, b]$ is a C-complete subset of G and the proof is complete. □

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