Ekrem Savaş; Fatih Nuray

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ON σ-STATISTICALLY CONVERGENCE AND LACUNARY σ-STATISTICALLY CONVERGENCE

EKREM SAVAŞ — FATIH NURAY

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ABSTRACT. In this note we introduce the concepts of σ-statistically convergence and lacunary σ-statistically convergence and give some inclusion relations.

1. Introduction and background

A complex number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n 1/n \left|\{k \le n : \ |x_k - L| \ge \varepsilon\}\right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $S\text{-}\lim x = L$ or $x_k \to L\,(S)$.

The idea of the statistical convergence of sequence of real numbers was introduced by Fast [2]. Schonberg [12] studied statistical convergence as a summability method and listed some of the elementary properties of statistical convergence. Both of these authors noted that if a bounded sequence is statistically convergent to L, then it is Cesáro summable to L.

Subsequently, statistical convergent sequences have been discussed in Šalát [9], Fridy [4], Maddox [6] and others independently. Most recently, in [1] it is shown that if a sequence is strongly p-Cesáro summable or w_p-convergent to L, $0 < p < \infty$, then the sequence must be statistically convergent to L and that a bounded statistically convergent sequence must be w_p-convergent. It is also shown that the statistically convergent sequences do not form a locally convex FK-space.

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Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞, the space of real bounded sequences, is said to be an invariant mean or σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, \ldots)$, and
(3) $\phi\big((x_{\sigma(n)})\big) = \phi(x)$ for all $x \in l_\infty$.

The mappings σ are one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is the translation mapping $n \to n+1$, a σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [5].

If $x = (x_n)$, the set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [10] that

$$V_\sigma = \big\{x = (x_n) : \lim_m t_{mn}(x) = Le \text{ uniformly in } n\,,\ L = \sigma\text{-}\lim x\big\}\,,$$

where $t_{mn}(x) = (x_n + Tx_n + \cdots + T^m x_n)/(m+1)$.

Several authors including Schaefer [12], Mursaleen [7], Savaş [11] and others have studied invariant convergent sequences. Recently, Mursaleen [8] defined strongly σ-convergent sequences by saying that $x_k \to L\,[V_\sigma]$ if and only if

$$\lim_n 1/n \sum_{k=0}^{n-1} |x_{\sigma^k(m)} - L| \to 0 \qquad \text{uniformly in} \quad m\,.$$

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences. It is known ([8]) that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.

For $\sigma(m) = m+1$ the space $[V_\sigma]$ is the space of strongly almost convergent sequences.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. Freedman, Sember and Raphael [3] defined the space N_θ in the following way: For any lacunary sequence $\theta = (k_r)$,

$$N_\theta = \Big\{x = (x_k) : \text{ for some } L\,,\ \lim_r 1/h_r \sum_{k \in I_r} |x_k - L| = 0\Big\}.$$

Quite recently, lacunary strong σ-convergent sequences were introduced by Savaş [10] as below:

$$L_\theta = \Big\{x = (x_k) : \lim_r 1/h_r \sum_{k \in I_r} |x_{\sigma^k(m)} - L| = 0\,,\ \text{uniformly in } m\Big\}.$$

In [10], it is also shown that there is a strong connection between $[V_\sigma]$ and L_θ as in Lemma 1.

Lemma 1. ([10]) *$L_\theta \iff [V_\sigma]$ for every lacunary sequence θ.*

The purpose of this paper is to introduce two concepts of convergence, S_θ and $S_{\sigma\theta}$, and to give some inclusion relations between S_σ- and $S_{\sigma\theta}$-convergence and also between L_θ and $S_{\sigma\theta}$-convergence in the same way as L_θ is related to $[V_\sigma]$.

Now we are ready to begin.

2. Definitions and theorems

Before giving the promised inclusion relations we will give two new definitions.

Definition 1. *A complex number sequence $x = (x_k)$ is said to be σ-statistically convergent to the number L if for every $\varepsilon > 0$*

$$\lim_n 1/n \left|\left\{0 \le k \le n : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| = 0 \qquad \textit{uniformly in} \quad m = 1, 2, \ldots .$$

In this case we write S_σ-$\lim x = L$ or $x_k \to L\,(S_\sigma)$ and we define

$$S_\sigma = \left\{x = (x_k) : \ \textit{for some } L, \ S_\sigma\text{-}\lim x = L\right\}.$$

Definition 2. *Let $\theta = (k_r)$ be a lacunary sequence; the number sequence $x = (x_k)$ is $S_{\sigma\theta}$-convergent to L provided that for every $\varepsilon > 0$,*

$$\lim_r 1/h_r \left|\left\{k \in I_r : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| = 0 \qquad \textit{uniformly in} \quad m = 1, 2, \ldots .$$

In this case we write $S_{\sigma\theta}$-$\lim x = L$ or $x_k \to L\,(S_{\sigma\theta})$ and we define

$$S_{\sigma\theta} = \left\{x = (x_k) : \ \textit{for some } L, \ S_{\sigma\theta}\text{-}\lim x = L\right\}.$$

We now give some inclusion relations between L_θ-convergence and $S_{\sigma\theta}$-convergence and show that these are equivalent for bounded sequences. We also study relation between S_θ-convergence and $S_{\sigma\theta}$-convergence.

Theorem 1. *Let $\theta = (k_r)$ be a lacunary sequence; then*

(i) *$x_k \to L\,(L_\theta)$ implies $x_k \to L\,(S_{\sigma\theta})$,*
(ii) *$x \in l_\infty$ and $x_k \to L\,(S_{\sigma\theta})$ imply $x_k \to L\,(L_\theta)$,*
(iii) *$S_{\sigma\theta} \cap l_\infty = L_\theta$.*

Proof.

(i) If $\varepsilon > 0$ and $x_k \to L\,(L_\theta)$, we can write

$$\sum_{k\in I_r} |x_{\sigma^k(m)} - L| \ge \sum_{\substack{k\in I_r \\ |x_{\sigma^k(m)}-L|\ge\varepsilon}} |x_{\sigma^k(m)} - L| \ge \varepsilon \left|\left\{k \in I_r :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right|$$

which yields the result.

(ii) Suppose that $x_k \to L\,(S_{\sigma\theta})$ and $x \in l_\infty$, say $|x_{\sigma^k(m)} - L| \le M$ for all k and m. Given $\varepsilon > 0$, we get

$$\begin{aligned} &1/h_r \sum_{k\in I_r} |x_{\sigma^k(m)} - L| \\ &= 1/h_r \sum_{\substack{k\in I_r \\ |x_{\sigma^k(m)}-L|\ge\varepsilon}} |x_{\sigma^k(m)} - L| + 1/h_r \sum_{\substack{k\in I_r \\ |x_{\sigma^k(m)}-L|<\varepsilon}} |x_{\sigma^k(m)} - L| \\ &\le M/h_r \left|\left\{k \in I_r :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| + \varepsilon \end{aligned}$$

from which the result follows.

Let θ be given and define x_k to be $1, 2, \ldots, \left[\sqrt{h_r}\right]$ for $k = \sigma^n(m)$, $n = k_{r-1}+1, k_{r-1}+2, \ldots, k_{r-1} + \left[\sqrt{h_r}\right]$; $m \ge 1$, and $x_k = 0$ otherwise (where [ ] denotes the greatest integer function). Note that x is not bounded.

Further, for $0 < \varepsilon < 1$ we have

$$1/h_r \left|\left\{k \in I_r :\ |x_{\sigma^k(m)} - 0| \ge \varepsilon\right\}\right| = \left[\sqrt{h_r}\right]/h_r \to 0 \qquad \text{as} \quad r \to \infty,$$

i.e. $x_k \to 0\,(S_{\sigma\theta})$. But,

$$1/h_r \sum_{k\in I_r} |x_{\sigma^k(m)} - 0| = 1/h_r\Big(\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right]+1\right)/2\Big) \to 1/2 \ne 0 \qquad \text{as} \quad r \to \infty,$$

hence $x_k \not\to 0\,(L_\theta)$. Thus, inclusion (i) is proper and this example shows that the boundedness condition cannot be omitted from the hypothesis (ii).

(iii) This is an immediate consequence of (i), (ii), Lemma 1 and $[V_\sigma] \subset l_\infty$. This completes the proof.

We now give a lemma which will be used in the proof of Theorem 2.

LEMMA 2. *Suppose for given* $\varepsilon_1 > 0$ *and every* $\varepsilon > 0$, *there exists* n_0 *and* m_0 *such that*

$$1/n \left|\left\{0 \le k \le n-1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| < \varepsilon_1$$

for all $n \ge n_0$ *and* $m \ge m_0$, *then* $x = (x_k) \in S_\sigma$.

Proof. Let $\varepsilon_1 > 0$ be given. For every $\varepsilon > 0$, choose n_0', m_0 such that

$$1/n \left|\left\{0 \le k \le n-1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| < \varepsilon_1/2 \tag{1}$$

for all $n \ge n_0'$ and $m \ge m_0$. It is enough to prove that there exists n_0'' such that for $n \ge n_0''$, $0 \le m \le m_0$,

$$1/n \left|\left\{0 \le k \le n-1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| < \varepsilon_1 . \tag{2}$$

Since taking $n_0 = \max(n_0', n_0'')$, (2) will hold for $n \ge n_0$ and for all m, which gives the result.

Once m_0 has been chosen, $0 \le m \le m_0$, m_0 is fixed. So put

$$K = \left|\left\{0 \le k \le m_0 - 1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| .$$

Now taking $0 \le m \le m_0$ and $n \ge m_0$, by (1) we have

$$\begin{aligned} &1/n \left|\left\{0 \le k \le n-1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \\ &\le 1/n \left|\left\{0 \le k \le m_0-1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \\ &\quad + 1/n \left|\left\{m_0 \le k \le n-1 :\ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \\ &\le K/n + 1/n \left|\left\{m_0 \le k \le n-1 :\ |x_{\sigma^k(m_0)} - L| \ge \varepsilon\right\}\right| \\ &\le K/n + \varepsilon_1/2 , \end{aligned}$$

and taking n, sufficiently large, we can write

$$\le K/n + \varepsilon_1/2 < \varepsilon_1 ,$$

which gives (2), and hence the result follows.

THEOREM 2. *$S_{\sigma\theta} = S_\sigma$ for every lacunary sequence θ.*

Proof. Let $x \in S_{\sigma\theta}$. Then, from Definition 2, given $\varepsilon_1 > 0$, there exist r_0 and L such that

$$1/h_r \left|\left\{0 \le k \le h_r - 1 : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| < \varepsilon_1$$

for $r \ge r_0$ and $m = k_{r-1} + 1 + u$, $u \ge 0$.

Let $n \ge h_r$, write $n = ih_r + t$, where $0 \le t \le h_r$, i is an integer. Since $n \ge h_r$, $i \ge 1$. Now

$$\begin{aligned}
&1/n \left|\left\{0 \le k \le n - 1 : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \\
&\le 1/n \left|\left\{0 \le k \le (i+1)h_r - 1 : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \\
&= 1/n \sum_{j=0}^{i} \left|\left\{jh_r \le k \le (j+1)h_r - 1 : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \\
&\le 1/n(i+1)h_r\,\varepsilon_1 \le 2ih_r\,\varepsilon_1/n \qquad (i \ge 1)
\end{aligned}$$

for $h_r/n \le 1$, and since $ih_r/n \le 1$,

$$1/n \left|\left\{0 \le k \le n - 1 : \ |x_{\sigma^k(m)} - L| \ge \varepsilon\right\}\right| \le 2\varepsilon_1 .$$

Then by Lemma 2, $S_{\sigma\theta} \subset S_\sigma$. It is easy to see that $S_\sigma \subset S_{\sigma\theta}$.

This completes the proof.

When $\sigma(m) = m + 1$, from Definitions 1 and 2 we have the definitions of almost statistically convergence and lacunary almost statistically convergence of a sequence. So, similar inclusions to Theorems 1 and 2 hold between strongly almost convergent sequences and almost statistical convergent sequences, which have not appeared anywhere by this time.

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Firat University
Department of Mathematics
Elaziğ
Turkey