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*Mathematica Slovaca*, Vol. 30 (1980), No. 4, 401--404

Persistent URL: http://dml.cz/dmlcz/132712

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ON THE \((m, n)\)-BASIS OF A DIGRAPH

MATUŠ HARMINC

In the presented paper there is introduced the notion of an \((m, n)\)-basis of a digraph (where \(m\) and \(n\) are positive integers). There is investigated the existence of an \((m, n)\)-basis for digraphs of certain types. Some results of Richardson [6] and von Neumann and Morgenstern [5] are generalized.

Let us recall some fundamental notions. A finite directed graph \(D = (V, A)\) with the set of points \(V\) and with the set of lines \(A \subseteq V \times V\) with no loops or multiple lines is called a digraph. The concepts of a path, a cycle, an indegree of a point \(v\) (denoted \(\text{id}(v)\)) are used like in [4]. A transmitter is a point whose indegree is 0. The \(n\)-th power of a given digraph \(D\) is the digraph \(D^{(n)}\), which has the same point set as \(D\) and a line \(uv\) is in \(D^{(n)}\) if and only if there is a path in \(D\) from \(u\) to \(v\) of length \(d \leq n\) (see [4]). Throughout the paper the symbols \(c, d, k, m, n\) denote positive integers. For each set \(M \subseteq V\) of points of \(D\) we denote by \(H(M)\) the set consisting of terminal points of those lines that have initial points in \(M\).

A set \(S \subseteq V\) is \(m\)-independent if for no two distinct points \(u, v \in S\) there exists a path of length \(d \leq m\) from \(u\) to \(v\). A set \(S \subseteq V\) is an \(n\)-cover in \(D\) if for each \(v \in V - S\) there exists at least one \(u \in S\) such that there exists a path of length \(d \leq n\) from \(u\) to \(v\). A set \(S\) is an \(n\)-basis for \(D\) if it is \(n\)-independent and an \(n\)-cover for \(D\) (see Harary, Norman, Cartwright [4]). This concept is a generalization of the concept of a 1-basis (a solution) of a digraph ([1], [6]). (Some authors study the dual concept — the kernel of a digraph [2].)

**Definition.** A subset \(S\) of \(V\) in a digraph \(D = (V, A)\) is called an \((m, n)\)-basis of \(D\) if

(i) \(S\) is \(m\)-independent, and
(ii) \(S\) is an \(n\)-cover in \(D\).

By definition of the \((m, n)\)-basis it is clear that an \((m, n)\)-basis is a \((k, c)\)-basis of the same digraph for all \(k \leq m\) and \(c \geq n\).

For each positive integer \(n\) the notion of an \((n, n)\)-basis coincides with the notion of an \(n\)-basis. In [4] it is established that, instead of studying the existence of an \((n, n)\)-basis of a digraph \(D\) it suffices to study the existence of a 1-basis of \(D^{(n)}\). We note that the situation with an \((m, n)\)-basis in the case \(m \neq n\) is rather different. Further we remark that the problem of the existence of a 1-basis for an arbitrary digraph is not solved in general (see [1]).
Theorem 1. a) Every digraph has an \((m, n)\)-basis for \(n \geq 2m\).
b) For each pair \((m, n)\), \(n < 2m\), there exists a digraph without an \((m, n)\)-basis.

Proof. The proof of a) will be established in two steps.

1) Using mathematical induction with respect to the number of points of a digraph we shall prove the theorem for \(m = 1\). The digraph with one point and digraphs with two points have a \((1, 2)\)-basis. Let each digraph with \(k\) points have a \((1, 2)\)-basis for each \(k < c\) and let a digraph \(D\) have \(c\) points. Let us take a point \(v\) and construct a digraph \(G\) generated in \(D\) by a set of points \(V - \{v\} - H(\{v\})\). Let \(S\) be a \((1, 2)\)-basis for \(G\). If there exists \(u \in S\) such that \(v \in H(\{u\})\), then \(S\) is a \((1, 2)\)-basis for \(D\) too. In the opposite case we can easily verify that \(S \cup \{v\}\) is a \((1, 2)\)-basis for \(D\).

2) Now let us construct the digraph \(D^{(m)}\) and denote by \(S^{(m)}\) a \((1, 2)\)-basis for \(D^{(m)}\). We have \(S^{(m)} \subseteq V\); the 1-independence in \(D^{(m)}\) is equivalent to the \(m\)-independence in \(D\) and similarly the 2-cover for \(D^{(m)}\) is the \(2m\)-cover for \(D\). The set \(S^{(m)}\) is an \((m, 2m)\)-basis for \(D\), i.e. an \((m, n)\)-basis for \(D\) for each \(n \geq 2m\).

b) If \(n < 2m\), then a digraph that consists of two cycles of length \(2m + 1\) having a unique common line, has no \((m, 2m - 1)\)-basis and therefore no \((m, n)\)-basis for \(n < 2m\).

Corollary 1 (Landau [4]). In every tournament there exists a point \(v\) such that every point different from \(v\) is reachable from \(v\) by a path of length one or two.

Corollary 2. If in a digraph \(D\) there is no path of length \(n + 1\), then \(D\) has an \((m, n)\)-basis for each \(m\).

To prove this it is sufficient to take an \((m, 2m)\)-basis \(S\) for \(D\) (such a basis exists according to Theorem 1). Since every path is of length at most \(n\), the set \(S\) is an \((m, n)\)-basis for \(D\), too.

It is possible to establish stronger results than Theorem 1 for some special classes of digraphs. A digraph \(D = (V, A)\) is called:

- transitive, if \(uv \in A, vw \in A\) implies \(uw \in A\) for each triple of distinct points \(u, v, w\);
- acyclic, if \(D\) has no cycle;
- symmetric, if for each pair of distinct point \(u, v\) the condition \(uv \in A\) is equivalent to \(vu \in A\);
- asymmetric, if \(uv \in A\) implies \(vu \not\in A\) for each pair of distinct points \(u, v\).

Theorem 2. a) Every transitive digraph has an \((m, n)\)-basis for each pair \((m, n)\).
b) Every acyclic digraph has an \((m, n)\)-basis for \(m \leq n\). Let \(m > n\); there exists an acyclic digraph having no \((m, n)\)-basis.
c) Every symmetric digraph has an \((m, n)\)-basis for each \(m \leq n\). Let \(m > n\); there exists a symmetric digraph having no \((m, n)\)-basis.
Proof. a) In a transitive digraph the existence of a path from \( u \) to \( v \) is equivalent to the existence of the line \( uv \). According to this fact and to the definition of an \((m, n)\)-basis it is evident that the following assertion holds: A set \( S \) is an \((m, n)\)-basis for a transitive digraph if and only if it is a \( 1 \)-basis for this digraph. As a transitive digraph has a \( 1 \)-basis (see [2], [4]), part a) is proved.

b) It is known (cf. [4]) that an acyclic digraph has an \( m \)-basis for every \( m \), therefore it has an \((m, n)\)-basis for each \( n < m \). Moreover, the following holds: Assume that \( D \) is an acyclic digraph, \( n < m \) and let \( W \) be the set of transmitters of \( D \). Then \( D \) has an \((m, n)\)-basis iff \( V = W \cup H(W) \cup H(H(W)) \cup \ldots \cup H^n(W) \). (And in this case \( W \) is the \((m, n)\)-basis of \( D \).)

c) If a digraph \( D \) is symmetric, the digraph \( D^{(m)} \) is symmetric, too. We denote by \( S \) a \( 1 \)-basis of \( D^{(m)} \) (in a symmetric digraph such a basis exists, cf. Berge [2]). The set \( S \) is an \( m \)-basis for \( D \); it is also an \((m, n)\)-basis for \( D \) for \( n \geq m \). The digraph which consists of the points \( u_0, u_1, \ldots, u_m \) and of the lines \( u_0u_1, u_1u_2, \ldots, u_{m-1}u_m \) has no \((m, m-1)\)-basis and therefore no \((m, n)\)-basis for \( n < m \).

**Theorem 3.** For any asymmetric digraph \( D \) the following statements are equivalent.

(i) \( V = W \cup H(W) \), where \( W \) is the set of transmitters.

(ii) For any pair \((m, n)\) the digraph \( D \) has an \((m, n)\)-basis.

(iii) The digraph \( D \) has a \((2, 1)\)-basis.

Proof. (i) implies (ii): \( W \) is an \( m \)-independent set for any \( m \) and it is a \( 1 \)-cover. (ii) implies (iii) immediately. Let \( S \) be a \((2, 1)\)-basis for the digraph \( D \), \( s \in V - W - H(W) \). If \( s \in S \), there exists \( w \in S \) such that \( ws \in A \). If \( s \in S \), we take \( w = s \). Then there exists \( v \in V - S \) such that \( uv \in A \). Because the set \( S \) is a \( 1 \)-cover, there is \( u \in V \) in \( S \) such that \( uw \in A \); since \( D \) is asymmetric we have \( u \neq w \). This is a contradiction, since \( S \) is a \( 2 \)-independent set.

**Corollary.** Every asymmetric digraph with no transmitter has no \((m, 1)\)-basis for each \( m > 1 \).

Proof. If \( m > 1 \), then every \((m, 1)\)-basis is a \((2, 1)\)-basis. In an asymmetric digraph we have a contradiction between (i) of Theorem 3 and the assumption of the corollary.

**Theorem 4.** Let \( D \) have an \((m, n)\)-basis. Let \( C \) be a cycle such that \( \text{id}(v) = 1 \) for each \( v \in C \). Let \( d \) be the length of \( C \). Then

\[
d \in \langle 2; n+1 \rangle \cup \bigcup_{c=2} \langle c(m+1); c(n+1) \rangle.
\]

Proof. The cycle \( C \) as described in the theorem must have the property that each point of \( C \) is covered only by points from \( C \). We denote by \( c \) the number of
those points of the cycle, which are contained in an \((m, n)\)-basis \(S\). In the case \(c = 1\) we obtain that for the length \(d\) of the cycle \(C\) we have \(d \in \langle 2; n + 1 \rangle\). Let \(c \equiv 2\). If \(d < c(m + 1)\), the set \(S\) is not \(m\)-independent. If \(d > c(n + 1)\), it is not an \(n\)-cover. Thus \(d \in \langle c(m + 1); c(n + 1) \rangle\).

The opposite assertion is not valid in general: A digraph consisting of the points \(u_1, u_2, u_3, v_1, v_2, v_3\) and of the lines \(u_1u_2, u_2u_3, u_3u_1, u_1v_1, u_2v_2, u_3v_3\) has \(d \in \langle 2; n + 1 \rangle\) for \(m = 3, n = 2\), but has no \((3, 2)\)-basis.

After this paper has been submitted I have found that the step 1 (in the part a) of Theorem 1 was proved already by V. Chvatal and L. Lovasz (Hypergraphs Seminar, Lecture Notes in Mathematics, 411, Springer-Verlag, Berlin 1974).

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Received December 12, 1978