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ON INTRINSIC QUASIMETRICS PRESERVING MAPS ON NON-ABELIAN PARTIALLY ORDERED GROUPS

MILAN JASEM

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ABSTRACT. In [JASEM, M.: Intrinsic metric preserving maps on partially ordered groups, Algebra Universalis 36 (1996), 135–140], it was proved that a stable surjective map $f$ from an abelian directed group $G_1$ onto a directed group $G_2$ is a homomorphism if it satisfies the following condition:

(C) If $|x-y|=|z-t|$, then $|f(x)-f(y)|=|f(z)-f(t)|$ for each $x, y, z, t \in G_1$.

In this paper a stable map $f: G_1 \to G_2$ satisfying (C) is studied, where $G_1$ and $G_2$ are non-abelian directed groups. It is shown that a stable injective map $f: G_1 \to G_2$ satisfying (C) is a homomorphism in the case that $G_1$ is a 2-isolated directed group and $G_2$ is a linearly ordered group. The question whether $f$ is a homomorphism also in the case of non-linearly ordered group $G_2$ remains open.

In [5], Swamy defined an intrinsic metric on a lattice ordered group (l-group) $G$ as a map $d: G \times G \to G$ satisfying the following conditions for each $a, b, c \in G$:

(M$_1$) $d(a, b) \geq 0$ and $d(a, b) = 0$ if and only if $a = b$,
(M$_2$) $d(a, b) = d(b, a)$,
(M$_3$) $d(a, c) \leq d(a, b) + d(b, c)$,

and showed that any abelian l-group is autometrized by $d(x, y) = |x - y|$.

Holland [1] considered whether other metrics might be naturally defined on an l-group.

Rachůnek [4] generalized the notion of an intrinsic metric to any partially ordered group (po-group). He defined an intrinsic metric on a po-group $G$ as a map $d: G \times G \to \exp G$ satisfying (M$_2$) and the following conditions for each $a, b, c \in G$:

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Let $G_1$ and $G_2$ be po-groups and let $d_1$ and $d_2$ be intrinsic quasimetrics on $G_1$ and $G_2$, respectively. A map $f : G_1 \to G_2$ preserves intrinsic quasimetrics $d_1$ and $d_2$ if and only if $d_1(x, y) = d_1(z, t)$ implies $d_2(f(x), f(y)) = d_2(f(z), f(t))$ for each $x, y, z, t \in G_1$. A map $f : G_1 \to G_2$ is called stable if $f(0) = 0$.

We recall some notations and notions concerning po-groups used in the paper. Let $G$ be a po-group. The group operation will be written additively. We denote by $U(a_1, \ldots, a_n)$ the set of all upper bounds of the set $\{a_1, \ldots, a_n\}$ in $G$. If for $a, b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a, b\}$ in $G$, then it will be denoted by $a \lor b$ ($a \land b$). For $a \in G$, $|a| = U(-a, a)$. In the case that $G$ is an l-group, $|a| = -a \lor a$ for $a \in G$ as usual. We denote by $\exp G$ the set of all subsets of $G$. The set of all positive integers will be denoted by $\mathbb{N}$. A po-group $G$ is called 2-isolated if $2a > 0$ implies $a > 0$ for each $a \in G$.

**Theorem 1.** Let $G$ be a 2-isolated directed group, $n \in \mathbb{N}$. Let $d(a, b) = n|a - b|$ for each $a, b \in G$. Then $d$ is an quasimetric on $G$.

**Proof.** Let $a, b \in G$, $n \in \mathbb{N}$. If $x \in d(a, b)$, then $x = x_1 + \cdots + x_n$, where $x_i \geq a - b$, $x_i \geq b - a$, $i = 1, \ldots, n$. Thus $2x_i \geq 0$ and hence $x_i \geq 0$ for $i = 1, \ldots, n$. Then $x \geq 0$. Therefore $d(a, b) \subseteq U(0)$.

If $d(a, b) = U(0)$, then $0 = z_1 + \cdots + z_n$, where $z_i \geq a - b$, $z_i \geq b - a$ for $i = 1, \ldots, n$. Then $2z_i \geq 0$ and hence $z_i \geq 0$ for $i = 1, \ldots, n$. Then $0 = z_1 + \cdots + z_n \geq z_i$ and thus $z_i = 0$ for $i = 1, \ldots, n$. Then we get $0 \geq a - b$, $0 \geq b - a$. This yields $a = b$. \hfill \Box

The following example shows that a stable map $f : G_1 \to G_2$ satisfying the condition (C), where $G_1$ and $G_2$ are even linearly ordered groups, need not be a homomorphism.
EXAMPLE. Let $\mathbb{Z}$ be the additive group of all integers with the natural order. Let $G_1 = G_2 = \mathbb{Z}$. For even integer $x \in G_1$ we put $f(x) = 0$, for odd integer $x \in G_1$ we put $f(x) = 1$. Then the stable map $f : G_1 \to G_2$ satisfies the condition (C), but $f$ is not a homomorphism.

So it is needed to put an additional condition on a stable map satisfying (C) to be a homomorphism, for example surjectivity, injectivity.

Remark. Let $G$ be a po-group, $a, b \in G$. If $a \geq 0$, then $|a| = |b|$ implies $a \geq b$, $a \geq -b$. If $a \geq 0$, $b \geq 0$, then $|a| = |b|$ implies $a = b$. We shall often need these assertions and we shall apply them without special references.

THEOREM 2. Let $G_1$ be a 2-isolated directed group and let $G_2$ be a linearly ordered group. Let $f : G_1 \to G_2$ be a stable injective map satisfying the following condition for each $a, b, c, d \in G_1$:

If $|a - b| = |c - d|$, then $|f(a) - f(b)| = |f(c) - f(d)|$.

Then $f$ is a homomorphism.

Proof. First we prove that $f(-z) = -f(z)$, $f(2z) = 2f(z)$ for each $z \in G_1$. Let $z \in G_1$, $f(z) \geq 0$. Assume that $f(-z) > 0$. Since $|z - 0| = |-z - 0|$, we have $|f(z) - f(0)| = |f(-z) - f(0)|$. Then $f(z) = f(-z)$. This implies $z = -z$. Since $G_1$ is 2-isolated, we have $-z = 0$. Thus $f(-z) = 0$, a contradiction. Therefore $f(-z) \leq 0$. Then from $|f(z) - f(0)| = |f(-z) - f(0)|$ we obtain $f(-z) = -f(z)$. Since $|2z - z| = |z - 0|$, we have $|f(2z) - f(z)| = |f(z) - f(0)|$. This yields $f(z) \geq f(z) - f(2z)$. Therefore $f(2z) \geq 0$. From $|2z - 0| = |z - (-z)|$ we obtain $|f(2z) - f(0)| = |f(z) - f(-z)| = |2f(z)|$. Hence $f(2z) = 2f(z)$.

Let $z \in G_1$, $f(z) < 0$. Assume that $f(-z) < 0$. Since $|z - 0| = |-z - 0|$, we have $|f(z) - f(0)| = |f(-z) - f(0)|$. Thus $f(z) = f(-z)$ and hence $z = -z$. This yields $z = 0$. Then $f(-z) = 0$, a contradiction. Therefore $f(-z) \geq 0$. Then from $|f(z) - f(0)| = |f(-z) - f(0)|$ it follows that $f(-z) = -f(z)$. Since $|2z - z| = |z - 0|$, we have $|f(2z) - f(z)| = |f(z) - f(0)|$. This implies $-f(z) \geq f(2z) - f(z)$. Thus $0 \geq f(2z)$. From $|2z - 0| = |z - (-z)|$ it follows that $|f(2z) - f(0)| = |f(z) + f(-z)| = 2|f(z)|$. Hence $f(2z) = 2f(z)$.

Let $x, y \in G_1$. Now we prove that $f(x + y) = f(x) + f(y)$.

a) Let $f(x) \geq 0$, $f(y) \geq 0$. Assume that $f(x + y) < 0$. From $|(x + y) - y| = |x - 0|$ we obtain $|f(x + y) - f(y)| = |f(x) - f(0)|$. Since $f(x + y) - f(y) \leq 0$, we have $f(y) - f(x + y) = f(x)$. Hence $f(x + y) = -f(y) + f(x)$. From $|(x + y) - 0| = |x - (-y)|$ it follows that $|f(x + y) - f(0)| = |f(x) - f(0)| = |f(x) + f(y)|$. Hence $f(x + y) = f(x) + f(y)$. Then $-f(y) + f(x) = f(x) + f(y) \geq f(x)$. This yields $0 \geq f(y)$. Thus $f(y) = 0$ and hence $y = 0$. Then $f(x + y) = f(x) \geq 0$, a contradiction. Therefore $f(x + y) \geq 0$. From $|(x + y) - 0| = |x - (-y)|$ it follows that $|f(x + y) - f(0)| = |f(x) - f(0)| = |f(x) + f(y)|$. This implies $f(x + y) = f(x) + f(y)$.
b) Let $f(x) \geq 0$, $f(y) < 0$. From $|(x + y) - y| = |x - 0|$ we obtain $|f(x + y) - f(y)| = |f(x) - f(0)|$. Then $f(x) \geq f(x + y) - f(y)$. This implies $f(x) \geq f(x + y)$. Further we have $f(-y) = -f(y) \geq 0$. In view of a) we have $f(x - y) = f(x + (-y)) = f(x) + f(-y) = f(x) - f(y)$. Since $|x - (x + y)| = |(x - y) - x|$, we have $|f(x) - f(x + y)| = |f(x - y) - f(x)| = |f(x) - f(y) - f(x)|$. Clearly $f(x) - f(y) - f(x) \geq 0$. Hence $f(x) - f(x + y) = f(x) - f(y) - f(x)$. Therefore $f(x + y) = f(x) + f(y)$.

c) Let $f(x) < 0$, $f(y) \leq 0$. Since $f(-x) = -f(x) \geq 0$ and $f(-y) = -f(y) \geq 0$, in view of a) we have $f(-y - x) = f((-y) + (-x)) = f(-y) + f(-x) = -f(y) - f(x) \geq 0$. Then $f(x + y) = f(-(y - x)) = -f(-y - x) = f(x) + f(y)$.

d) Let $f(x) < 0$, $f(y) > 0$. Then $f(-y) = -f(y) \leq 0$. In view of c) we get $f(x - y) = f(x + (-y)) = f(x) + f(-y) = f(x) - f(y)$. From $|(x + y) - y| = |x - 0|$ it follows that $|f(x + y) - f(y)| = |f(x) - f(0)|$. Thus $-f(x) \geq f(y) - f(x + y)$. This implies $f(x + y) \geq f(x)$. Since $|x - (x + y)| = |(x - y) - x|$, we have $|f(x) - f(x + y)| = |f(x - y) - f(x)| = |f(x) - f(y) - f(x)|$. Clearly $f(x) - f(y) - f(x) \leq 0$. Then we get $f(x + y) - f(x) = f(x) + f(y) - f(x)$. Therefore $f(x + y) = f(x) + f(y)$.

Remark. It is clear from the proof of Theorem 2 that a stable injective map $f : G_1 \to G_2$ ($G_1, G_2$ as in Theorem 2) preserving intrinsic quasimetrics $d_1(a, b) = n|a - b|$ ($n \in \mathbb{N}$) and $d_2(a, b) = |a - b|$ on $G_1$ and $G_2$ is also a homomorphism.

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