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DANIELL TYPE EXTENSIONS OF $L_1$-GAUGES AND INTEGRALS

IVAN DOBRAKOV†* — JANA DOBRAKOVOVÁ**

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ABSTRACT. We show that a Daniell type extension procedure is possible if the subadditivity is replaced by a much weaker requirement of subadditive continuity. Particu larly, we prove that the Fatou and Daniell properties are preserved by this completion of the elementary $L_1$-gauge.

Introduction

An approach to vector integration based on application of the Daniell scheme, using outer $L_1$-pseudonorms induced by vector measures, named $L_1$-gauges, was elaborated by K. Bichteler in [3]. Its generalization to non locally convex spaces outlined in [4] led in [5] to substantial progress in stochastic integration, particularly in the $L^p$-theory for $p \in [0,1)$. A similar investigation of subadditive Daniell gauges and the integrals they dominate was done by M. Wilhelm in [20], [21] and [22] in an abstract setting. In [8], we will show that $L_1$-gauges of a non linear integration corresponding to Riesz type representations of non linear Hammerstein operators on $C(S, E)$ from [1] and [2] are only subadditively continuous, see (3) in Definition 1 below. Hence, it is of interest to investigate the Daniell scheme of integration in such a general setting. At the same time, we continue our program announced already in [9], see also [10]–[13].

Elementary $L_1$-gauges introduced by Definition 1 usually have an additional property connected with multiplication by scalars which determines the linear topological structure they induce, see Theorems 1 and 2. In Section 2, we show that an elementary $L_1$-gauge $J$ on a function lattice $\mathcal{F}$ always has an extension to a function lattice $\mathcal{L}$ such that $\mathcal{L}$ is complete in the pseudometric induced by this extension.

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Key words: vector lattice, subadditive continuity, elementary $L_1$-gauge, Fatou property, Daniell property.
In the main Section 3 of this paper, we prove that our extension preserves the Fatou and Daniell properties, see Theorems 8 and 9 respectively.

Finally, in Theorem 11, we show that, by applying our extension procedure again, we obtain no further extension. The proofs of these results are subtle and more involved then in the subadditive case treated in [3] and [20]. At the same time they show how far subadditivity can be replaced by the much weaker requirement of subadditive continuity.

We mentioned earlier that $L_1$-gauges of our extra generality occur in the non linear integration theory ([8]) corresponding to Riesz type representation of non linear Hammerstein operators on $C(S, E)$ from [1] and [2]. Namely, having a non linear measure $m$ on a $\delta$-ring $P$ of subsets of a set $T$, in [8], we show that

$$J(f) = J(f, T) = \sum_{k=1}^{\infty} k^{-2} \cdot \tilde{m}(kf, T) \cdot (1 + \hat{m}(kf, T))^{-1}$$

is in general only a subadditively continuous $L_1$-gauge with the Daniell property on $L_1(m)$.

Here $\tilde{m}(g, E)$ is defined as in part II of [7], and

$$\tilde{L}_1(m) = \{ g, \ g \text{ is } P\text{-measurable and } J(g, \cdot): \sigma(P) \to [0, \frac{\pi^2}{6}] \text{ is continuous}\}.$$

In other words, the analog $J: \tilde{L}_1(m) \to [0, \frac{\pi^2}{6}]$ of the $L_1$-pseudonorm $\int_T |f| \, d\mu$ of the classical $L_1(\mu)$ in non linear integration theory is in general only a subadditively continuous $L_1$-gauge. Let us note that here $J(f)$ must in general depend on the multiples $kf$, $k = 1, 2, \ldots$, since, otherwise, $\tilde{L}_1(m)$ will not be a linear space.

1. Elementary $L_1$-gauges

Suppose $T$ is a non-empty set. A collection $\mathcal{F}$ of functions on $T$ with values in $\mathbb{R} = (-\infty, +\infty)$ (in $\mathbb{R}^* = [-\infty, +\infty]$) is called an $\mathbb{R}$- ($\mathbb{R}^*$-)function lattice if $af + bg$, $f \lor g$, $f \land g \in \mathcal{F}$ provided $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$. $S(\mathbb{R})$ and $C_{00}(T)$ are the standard examples of $\mathbb{R}$-function lattices. For more information about function lattices, see [12].

**Definition 1.** By *elementary $L_1$-gauge* we mean a couple $(\mathcal{F}, J)$, where $\mathcal{F}$ is an $\mathbb{R}$-function lattice on $T$, $J: \mathcal{F} \to \mathbb{R}^+$ and has the following properties:

1. $J(0) = 0$ and $J(f) = J(|f|)$ for each $f \in \mathcal{F}$.
2. $0 \leq f \leq g \implies J(f) \leq J(g)$.

We say that $J$ is *monotone* on $\mathcal{F}^+ = \{ f, f \in \mathcal{F}, f \geq 0 \}$. 

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(3) $f_n \in \mathcal{F}$, $n = 1, 2, \ldots$, and $J(f_n - f_m) \to 0$ as $n, m \to \infty$ \implies $J(f_n) - J(f_m) \to 0$ as $n, m \to \infty$.

We say that $J$ is extendable subadditively continuous.

(4) $f, f_n \in \mathcal{F}^+$, $f_n \not\to f \implies J(f_n) \not\to J(f)$.

We say that $J$ has the Fatou property.

(5) $f_n, g_n \in \mathcal{F}$, $n = 1, 2, \ldots$ and $J(f_n) + J(g_n) \to 0$ as $n \to \infty$ \implies $J(f_n + g_n) \to 0$ as $n \to \infty$.

We say that $J$ has the pseudometric generating property.

Clearly, (3) implies the subadditive continuity, or autocontinuity of $J$ on $\mathcal{F}$, i.e.,

$f, f_n \in \mathcal{F}$, $n = 1, 2, \ldots$, and $J(f - f_n) \to 0 \implies J(f_n) \to J(f)$

\iff $\forall f \in \mathcal{F}$ $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

$g \in \mathcal{F}$ and $J(g) < \delta \implies J(f) - \varepsilon \leq J(f + g) \leq J(f) + \varepsilon$.

If $\mathcal{F}$ is complete in $\mathcal{G}$, $\mathcal{G}(f, g) = J(f - g)$, then the converse is also true.

We now list subsequent strengthenings of (3).

• $J$ is uniformly subadditively continuous, or uniformly autocontinuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$f, g \in \mathcal{F}$ $J(f - g) < \delta \implies |J(f) - J(g)| < \varepsilon$.

This property clearly implies (5).

• $J$ is $K$-subadditive if there is a $K \in [1, \infty)$ such that

$J(f + g) \leq J(f) + K \cdot J(g)$ for each $f, g \in \mathcal{F}$.

• $J$ is subadditive if $J(f + g) \leq J(f) + J(g)$ for each $f, g \in \mathcal{F}$.

• $J$ is additive on $\mathcal{F}^+$-clear.

Note that a subadditive (additive) $J$ is, in fact, countably subadditive (additive).

We say that $J$ is a Daniell elementary $L_1$-gauge if $f_n \in \mathcal{F}^+$, $n = 1, 2, \ldots$, and $f_n \searrow 0 \implies J(f_n) \searrow 0$. This Daniell property implies that $J$ is monotonically continuous on $\mathcal{F}^+$, and particularly the Fatou property of $J$.

As we will see in Theorems 1 and 2 below, the following additional properties of $J$ concerning multiplication by scalars from $\mathbb{R}$ determine the linear topological structure induced by $J$ on $\mathcal{F}$.

• $J$ is scalarly continuous if $J(a_n f) \to J(af)$ whenever $f \in \mathcal{F}$, $a_n, a \in \mathbb{R}$, $n = 1, 2, \ldots$, and $a_n \to a$, or equivalently, $\lim_{n \to \infty} J(f)^{\frac{1}{n}} = 0$ for each $f \in \mathcal{F}$. Obviously, each Daniell $J$ has this property.

• $J$ is locally bounded if there exists $a > 0$, and for each $n = 1, 2, \ldots$ there is a positive integer $N(n)$ such that $J\left(\frac{f}{N(n)}\right) < \frac{1}{n}$ whenever $f \in \mathcal{F}$ and $J(f) < a$.
• $J$ is $p$-homogeneous if there is a $p \in (0,1]$ such that $J(af) = |a|^p \cdot J(f)$ for each $a \in \mathbb{R}$ and each $f \in \mathcal{F}$.

• $J$ is subhomogeneous if $J(af) \leq |a| \cdot J(f)$ for each $a \in \mathbb{R}$, $f \in \mathcal{F}$.

• $J$ is homogeneous if $J(af) = |a| \cdot J(f)$ for each $a \in \mathbb{R}$, $f \in \mathcal{F}$.

For $E \subset T$ and $f : T \to \mathbb{R}$ put $\|f\|_E = \sup_{t \in E} |f(t)|$. Assertions (1) and (2) of the next lemma are immediate, while (3) follows from the well-known theorem of Dini.

**Lemma 1.**

1. Let $1 \in \mathcal{F}$, and let each $f \in \mathcal{F}$ be a bounded function. Then an elementary $L_1$-gauge $J : \mathcal{F} \to \mathbb{R}^+$ is scalarly continuous if and only if $J(f_n) \to 0$ whenever $f_n \in \mathcal{F}$, $n = 1,2,\ldots$, and $\|f_n\|_T \to 0$.

2. An elementary $L_1$-gauge $J : S(\mathbb{R}) \to \mathbb{R}^+$ is scalarly continuous if and only if $J(f_n \cdot \chi_E) \to 0$ whenever $E \in \mathbb{R}$, $f_n \in S(\mathbb{R})$, $n = 1,2,\ldots$, and $\|f_n\|_E \to 0$.

3. An elementary $L_1$-gauge $J : C_{00}(T) \to \mathbb{R}^+$ is scalarly continuous if and only if it is a Daniell $L_1$-gauge.

**Theorem 1.** Suppose $J : \mathcal{F} \to \mathbb{R}^+$ has the properties (1), (2) and (5) of Definition 1. Then:

1. There are $\delta_k, 0 < \delta_k \leq 2^{-k}, k = 1,2,\ldots$, such that $\delta_k \searrow 0$, $J(f + g) < \delta_k$ whenever $f,g \in \mathcal{F}$ and both $J(f), J(g) < \delta_{k+1}$, and $J\left(\sum_{i=k+1}^{k+p} f_i\right) < \delta_k$ for each $k,p = 1,2,\ldots$ whenever $f_k \in \mathcal{F}$ and $J(f_k) < \delta_k$ for each $k = 1,2,\ldots$.

2. There is an invariant pseudometric $\rho_c : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^+$ such that $\rho_c(f,g) \to 0$ if and only if $J(f - g) \to 0$, $f,g \in \mathcal{F}$. $(\mathcal{F}, \rho_c)$ is a pseudometric linear topological space if and only if $J$ is scalarly continuous.

**Proof.**

1. According to (5) of Definition 1, for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon)$ such that $J(f+g) < \epsilon$ whenever $f,g \in \mathcal{F}$ and both $J(f), J(g) < \delta(\epsilon)$. Take $\delta_1$ so that $0 < \delta_1 < 2^{-1}$, and put $\delta_2 = 2^{-1}\delta_1 \wedge \delta(2^{-1}\delta_1), \ldots, \delta_{n+1} = 2^{-1}\delta_n \wedge \delta(2^{-1}\delta_n), \ldots$.

2. Take $\delta_k$, $k = 1,2,\ldots$, from (1), and put $V_k = \{(f,g) \in \mathcal{F} \times \mathcal{F}, J(f - g) < \delta_k\}$. Clearly, $V_k$, $k = 1,2,\ldots$, is a countable base of an invariant uniformity. It remains for us to use the metrization Lemma 12 and Problem N in [14; Chapter VI].
**Theorem 2.** Suppose \( J : T \to \mathbb{R}^+ \) is a Daniell elementary \( L_1 \)-gauge. Then there is a subadditive Daniell \( L_1 \)-gauge \( J_c : T \to \mathbb{R}^+ \) such that \( J(f) \to 0 \) if and only if \( J_c(f) \to 0 \), \( f \in \mathcal{F} \).

If \( J \) is subhomogeneous (locally bounded), then \( J_c \) may be chosen so as to be homogeneous (\( p \)-homogeneous).

**Proof.** Take \( \rho_c \) as in Theorem 1, and put \( \|f\| = \rho_c(f,0) \). Then clearly \( f_n, g_n \in \mathcal{F}, \|g_n\| \leq |f_n|, \ n = 1,2, \ldots, \) and \( \|f_n\| \to 0 \) implies \( \|g_n\| \to 0 \). If \( \lambda_n \in \mathbb{R}, \ n = 1,2, \ldots, \) and \( \lambda_n \to 0 \), then \( \|\lambda_n f\| \to 0 \) for each \( f \in \mathcal{F} \) by scalar continuity of \( J \).

Hence, just as in the proof of Theorem VII.1.4 in [19], we obtain that \( J_c, J_c(f) = \sup\{\|g\|, \ 0 \leq g \leq |f|\} \), has the required property. \( J_c \) is clearly subhomogeneous (locally bounded) if and only if \( J \) is so. In that case, \( V = \{f, f \in \mathcal{F}, \ J(f) \leq 1\} \) is bounded convex neighbourhood of 0. Hence Kolmogorov’s theorem, see [16; Theorem III.2.1"], implies the existence of a homogeneous \( J_c \).

The locally bounded case follows from Theorem III.2.1 in [16].

Suppose \( J : \mathcal{F} \to \mathbb{R}^+ \) is uniformly autocontinuous and scalarly continuous. Then clearly, \( \sup_{f \in \mathcal{F}} |J(f + g) - J(f)| < \infty \) for each \( g \in \mathcal{F} \). Moreover, if \( J \) is locally bounded, then there is an \( a > 0 \) such that \( \sup\{|J(f + g) - J(f)|, f, g \in \mathcal{F}, \ J(g) < a\} < \infty \).

Less obvious is the next lemma:

**Lemma 2.** Suppose \( J : \mathcal{F} \to \mathbb{R}^+ \) is uniformly autocontinuous and \( p \)-homogeneous. Then there is a \( K \in [1, \infty) \) such that \( J(f + g) \leq J(f) + K \cdot J(g) \) whenever \( f, g \in \mathcal{F} \) and \( J(g) \leq 1 \).

**Proof.** Take \( g \in \mathcal{F} \) so that \( 0 < J(g) \leq 1 \), and let \( \varepsilon > 0 \). By the uniform autocontinuity of \( J \), there is a \( \delta, 0 < \delta < J(g) \), such that \( |J(f) - J(h)| < \varepsilon \) whenever \( f, h \in \mathcal{F} \) and \( J(f - h) < \delta \).

Next take \( \delta_1 > 0 \) so that \( J(h) < \delta \) whenever \( h \in \mathcal{F} \) and \( J(2^{-1}h) < \delta_1 \). Since, by assumption \( J \) is scalarly continuous, there is a positive integer \( N > 1 \) such that \( J((N - 1)^{-1}g) \geq \delta \) and \( J(N^{-1}g) < \delta \).

But then \( J(N^{-1}g) \geq \delta_1 \) since, otherwise, we have the contradiction: \( J((N - 1)^{-1}g) \leq J(2N^{-1}g) < \delta \). Hence \( \delta_1 \leq J(N^{-1}g) = N^{-p} \cdot J(g) \) for some \( p \in (0, 1] \) by the assumed \( p \)-homogeneity of \( J \).

But then \( N \leq (\delta_1^{-1} \cdot J(g))^{\frac{1}{p}} \leq \delta_1^{-\frac{1}{p}} \cdot J(g) \) since \( J(g) \leq 1 \). Hence \( J(f + g) = J(f + N \cdot N^{-1} \cdot g) \leq J(f) + N \cdot \varepsilon \leq J(f) + \varepsilon \cdot \delta_1^{-\frac{1}{p}} \cdot J(g) \) for each \( f \in \mathcal{F} \).
2. Extension of $L_1$-gauges

In what follows, $(F, J)$ will be a given elementary $L_1$-gauge. All functions considered are defined on $T$ and have values in $[-\infty, \infty]$. Unless otherwise specified, the arrows $\rightarrow (\nearrow$ or $\searrow$) denote pointwise (monotone) convergence of functions considered. Denote

$$F^0 = \{f, \ \exists f_n \in F, \ n = 1, 2, \ldots, \text{ such that } f_n \nearrow f \},$$
$$F_u = \{f, \ \exists f_n \in F, \ n = 1, 2, \ldots, \text{ such that } f_n \searrow f \}.$$  

For $f \in F^0^+ = \{f, \ f \in F^0 \text{ and } f \geq 0\}$ we put $J^0(f) = \lim_{n \to \infty} J(f_n)$, where $f_n \in F^+$, $n = 1, 2, \ldots$, and $f_n \nearrow f$. The monotonicity and the Fatou property of $J: F \to \mathbb{R}$ imply that $J^0: F^0^+ \to [0, \infty]$ is uniquely defined. Evidently, $J^0$ extends $J$, is monotone and has the pseudometric generating property. In a standard way, see [18; 6.2.III.d], it follows that $J^0$ has the Fatou property. Hence $J^0$ is countably subadditive provided $J$ is subadditive. Finally, $J^0$ has any of the properties: uniformly subadditively continuous, positively additive, locally bounded, $p$-homogeneous, subhomogeneous, homogeneous, provided $J$ has the corresponding property. Note that, except for the Fatou property and positive additivity, the analogs hold for the outer $L_1$-gauge $J^*$ which we now define.

Put

$$F^* = \{f, \ \exists g \in F^0^+ \text{ such that } |f| \leq g \},$$

and for $f \in F^*$ we define its outer $L_1$-gauge $J^*(f)$ by

$$J^*(f) = \inf \{J^0(g), \ g \in F^0^+, \ |f| \leq g \}.$$  

We define the null functions $N$ and the null sets by:

$$N = \{f, \ f \in F^* \text{ and } J^*(f) = 0\}, \quad \text{and} \quad N = \{N, \ N \subset T, \ \chi_N \in N\}.$$  

$F^*$ is clearly a hereditary $\mathbb{R}^*$-function lattice, $J^*$ extends $J^0$, and $J^*(F) = J^*(|f|)$ for each $f \in F^*$.

**THEOREM 3.**

1. Let $\delta_k$, $k = 1, 2, \ldots$, be as in Theorem 1. If $f, g \in F^*$ and $J^*(f), J^*(g) < \delta_{k+1}$, then $J^*(f + g) < \delta_k$, and if $f_i \in F^*$ and $J^*(f_i) < \delta_i$ for $i = 1, 2, \ldots$, then $J^*\left(\sum_{i=k+1}^{k+p} f_i\right) < \delta_k$ and $J^*\left(\sum_{i=k+1}^{\infty} f_i\right) \leq \delta_k$ for each $k, p = 1, 2, \ldots$.

2. $N$ is a hereditary $\sigma$ -sublattice of $F^*$, $N$ is a hereditary $\sigma$ -ring of subsets of $T$, and $f \in N$ if and only if the set $\{t, \ t \in T, \ f(T) \neq 0\} \in N$.

3. $F^*$ is complete in $\rho^*$, $\rho^*(f, g) = J^*(f - g)$.
If \( f, f_n \in F^* \), \( n = 1, 2, \ldots \), and \( J^*(f - f_n) \to 0 \) as \( n \to \infty \), then there is a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) such that \( f_{n_k} \to f \) almost everywhere \( N \) (a.e. \( N \)) as \( k \to \infty \).

**Proof.**

(1) If \( f, g, f_i \in F^* \), \( i = 1, 2, \ldots \), then the Fatou property of \( J^* \) implies the required assertions. From here, the general case easily follows from the definition of \( J^* \).

(2) The first assertion follows from (1), while the equivalence is a consequence of the inequalities:

\[
|f| \leq \lim_{n \to \infty} n \cdot |f| = +\infty \cdot \chi(t, t \in T, f(t) \neq 0) = \lim_{n \to \infty} n \cdot \chi(t, t \in T, f(t) \neq 0) \\
\geq \chi(t, t \in T, f(t) \neq 0).
\]

(3) Suppose \( f_n \in F^* \), \( n = 1, 2, \ldots \), and \( J^*(f_n - f_m) \to 0 \) as \( n, m \to 0 \). Take a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) such that \( J^*(f_{n_k+1} - f_{n_k}) < \delta_{k+1} \), where \( \delta_k, k = 1, 2, \ldots \), is from (1).

Then \( \sum_{i=k}^n |f_{n_i+1} - f_{n_i}| = h_k \backslash h \), and \( J^*(h) = 0 \) since \( J^*(h) \leq J^*(h_k) \leq \delta_k \)

for each \( k = 1, 2, \ldots \) by (1).

Define \( f(t) = \lim_{k \to \infty} f_{n_k}(t) \) if \( h(t) = 0 \), and \( f(t) = 0 \) if \( h(t) > 0 \). Obviously, \( f_{n_k} \to f \) a.e. \( N \) as \( k \to \infty \). Since \( |f - f_{n_k}| \leq \lim_{p \to \infty} \sum_{i=k}^p |f_{n_{i+1}} - f_{n_i}| \) for each \( k = 1, 2, \ldots \), \( J^*(f - f_{n_k}) \to 0 \) as \( k \to \infty \), hence also \( J^*(f - f_n) \to 0 \) as \( n \to \infty \).

(4) From the proof of (3) it is evident that a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) such that \( J^*(f - f_{n_k}) < \delta_k \) for \( k = 1, 2, \ldots \) has the required property.

**Definition 2.** We denote by \( \mathcal{L} \) the closure of \( F \) in \( (F^*, J^*) \), and for \( f \in \mathcal{L} \) we put \( J(f) = \lim_{n \to \infty} J(f_n) \), where \( f_n \in F \), \( n = 1, 2, \ldots \), is such that \( J^*(f - f_n) \to 0 \) as \( n \to \infty \).

The existence, finiteness and uniqueness of the above limit follow from the extendable subadditive continuity of \( J : F \to [0, \infty) \) and the pseudometric generating property of \( J^* : F^* \to [0, \infty] \). \( \mathcal{L} \) is clearly an \( \mathbb{R}^* \)-function lattice.

Using (4) of Theorem 3 we have the following equivalent:

**Definition 2'.** We say that \( f \in \mathcal{L} \) if \( f \in F^* \), and there are \( f_n \in F \), \( n = 1, 2, \ldots \), such that \( f_n \to f \) a.e. \( N \) and \( J(f_n - f_m) \to 0 \) as \( n, m \to \infty \). In that case, we put \( J(f) = \lim_{n \to \infty} J(f_n) \).

Using Definition 2 one easily checks the assertions of Theorem 4.
THEOREM 4. \( J : \mathcal{L} \to [0, \infty) \) extends \( J : \mathcal{F} \to \mathbb{R}^+ \), it shares the properties (1), (2), (3), and (5) of \( J : \mathcal{F} \to \mathbb{R}^+ \) from Definition 1, and, except for the Daniell property, it also shares any of the additional properties of \( J \) on \( \mathcal{F} \) listed in Section 1.

We note that the Fatou and Daniell properties of \( J \) on \( \mathcal{L} \) will be investigated in the Section 3 below.

LEMMA 3.

(1) \( \mathcal{N} = \{ f, \ f \in \mathcal{L} \text{ and } J(f) = 0 \} \).

(2) Let \( f \in \mathcal{L} \). Then \( \{ t, \ t \in T, \ |f(t)| = \infty \} \in \mathcal{N} \), and the factor space \( \mathcal{L} = \mathcal{L}/\mathcal{N} \) is a linear lattice.

Proof.

(1) Clearly, \( \mathcal{N} \subseteq \mathcal{L} \), and \( J(f) = 0 \) for \( f \in \mathcal{N} \). Suppose \( f \in \mathcal{L} \) and \( J(f) = 0 \). Take \( f_n \in \mathcal{F} \), \( n = 1, 2, \ldots \), so that \( J^*(f - f_n) \to 0 \). Thus \( J^*(f) = J^*(f - f_n + f_n) \to 0 \) by the pseudometric generating property of \( J^* \). Hence \( f \in \mathcal{N} \).

(2) Take \( f_n \in \mathcal{F} \), \( n = 1, 2, \ldots \), so that \( J^*(f - f_n) \to 0 \). Since each \( f_n \), \( n = 1, 2, \ldots \), is finite valued, we have \( J^* (+\infty \cdot \chi_{\{ t, \ t \in T, \ |f(t)| = +\infty \}}) = J^* ([|f - f_n + (-f + f_n - f + f_n)| - (|f - f_n - f + f_n| - f + f_n)] \leq J^* (6 \cdot |f - f_n|) \to 0 \).
(We are using the convention \((+\infty) + (-\infty) = (-\infty) + (+\infty) = 0\).) The last assertion is evident. □

Using Definition 2 one also easily checks the assertions of (1) of the next theorem. Assertion (2) of Theorem 5 follows in the same way as (2) in Theorem 1, and Theorem 2.

THEOREM 5.

(1) The analogs of assertions (1), (3), and (4) of Theorem 3 hold for \( (\mathcal{L}, J) \) and \( \rho, \ \rho(f,g) = J(f - g), \ f,g \in \mathcal{L} \).

(2) There is an invariant metric \( \rho_c \) on \( \mathcal{L} \times \mathcal{L} \) equivalent with \( \rho \). \( (\mathcal{L}, \rho_c) \) is a complete metric space for any such control metric \( \rho_c \).

It is a linear topological space if \( J \) is scalarly continuous. The analog of Theorem 2 holds.

3. Fatou and Daniell properties of the extension

Theorem 6 below is of fundamental importance for showing that our extension \( J : \mathcal{L} \to \mathbb{R}^+ \) shares various convergence properties of the elementary \( L_1 \)-gauge \( J : \mathcal{F} \to \mathbb{R}^+ \), particularly those mentioned above. In Theorem 11, we prove
that by applying the extension procedure to \( J: \mathcal{L} \to \mathbb{R}^+ \), we obtain no further enlargement of \( \mathcal{L} \).

We will need the following notions.

\[
\hat{F}^o = \{ f, \text{ there are } f_n \in \mathcal{F}, n = 1, 2, \ldots, \text{ such that } f_n \nearrow f \text{ and } J^o(f - f_n) \to 0 \},
\]

\[
\hat{F}_u = \{ f, \text{ there are } f_n \in \mathcal{F}, n = 1, 2, \ldots, \text{ such that } f_n \searrow f \text{ and } J^o(f_n - f) \to 0 \},
\]

\[
\hat{F} = \{ f, \text{ there exists } h \in \mathcal{F}^o \text{ such that } |f| \leq h \},
\]

\[
(\hat{F}^o)_u = \{ f, \text{ there are } f_n \in \hat{F}^o, n = 1, 2, \ldots, \text{ such that } f_n \searrow f \},
\]

\[
(\hat{F}_u)^o = \{ f, \text{ there are } f_n \in \hat{F}_u, n = 1, 2, \ldots, \text{ such that } f_n \nearrow f \}.
\]

For \( f \in \hat{F} \) put

\[
\hat{J}(f) = \inf\{ J^o(h), \ h \in \mathcal{F}^o, \ |f| \leq h \},
\]

and let \( \hat{\mathcal{L}} \) denote the closure of \( \mathcal{F} \) in \((\hat{F}, \hat{J})\), i.e., \( f \in \hat{\mathcal{L}} \) if \( f \in \hat{F} \), and there are \( f_n \in \mathcal{F}, n = 1, 2, \ldots, \) such that \( \hat{J}(f - f_n) \to 0 \).

Clearly, \( J^*(f) \leq \hat{J}(f) \) for each \( f \in \hat{F} \), hence

\[
\hat{\mathcal{N}} = \{ f, \ f \in \hat{\mathcal{F}}, \ \hat{J}(f) = 0 \} \subset \mathcal{N} \quad \text{and} \quad \hat{\mathcal{N}} = \{ E, \ E \subset T, \ \chi_E \in \hat{\mathcal{N}} \} \subset \mathcal{N}.
\]

Further, \( \hat{J}: \hat{\mathcal{F}} \to \mathbb{R}^+ = [0, \infty) \), and \( \hat{J} \) behaves similarly as \( J^*: \mathcal{F}^* \to \mathbb{R}^* \).

**Theorem 6.** Suppose \( f_n \in \mathcal{F}, n = 1, 2, \ldots, \) and \( J(f_n - f_m) \to 0 \) as \( n, m \to \infty \). Then there is a subsequence \( \{f_{nk}\} \subset \{f_n\} \), a sequence \( \varphi_k \in \mathcal{F}^o \), \( k = 1, 2, \ldots, \) and a sequence \( \psi_k \in \mathcal{F}_u \), \( k = 1, 2, \ldots, \) such that:

1. \( \varphi_k \searrow \varphi \in (\mathcal{F}^o)_u, \ \psi_k \nearrow \psi \in (\mathcal{F}_u)^o, \ \psi_k \leq f_{nk} \leq \varphi_k, \text{ for each } k = 1, 2, \ldots, \text{ and } J^o(\varphi_k - \psi_k) \to 0 \text{ as } k \to \infty, \)

2. \( f_{nk}(t) \to \varphi(t) \text{ a.e. } \hat{\mathcal{N}}, \text{ and } \varphi(t) = \psi(t) \text{ a.e. } \hat{\mathcal{N}}, \)

3. \( \hat{J}(f_n - \varphi) = \hat{J}(f_n - \psi) \to 0 \text{ as } n \to \infty. \)

**Proof.** Let \( \delta_k, k = 1, 2, \ldots, \) be the sequence of Theorem 1, and take a subsequence \( \{f_{nk}\} \subset \{f_n\} \) so that \( J(f_{nk+1} - f_{nk}) < \delta_k \) for each \( k = 1, 2, \ldots \). Then

\[
J^o\left( \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \right) \leq \delta_{k-1} < \delta_{k-2}
\]
by the Fatou property of $J^0$: $\mathcal{F}_0^+ \to [0, \infty]$. For $k = 1, 2, \ldots$ put
\[
\varphi_k = \left( f_{n_k} + \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \right) + \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|
\]
Clearly, $\varphi_k \in \mathcal{F}_0$ for each $k = 1, 2, \ldots$. If $|\varphi_k(t)| < \infty$, then $|f_{n_i}(t)| < \infty$ and $|\varphi_i(t)| < \infty$ for $i \geq k$, hence
\[
\varphi_k(t) - \varphi_{k+1}(t) = f_{n_k}(t) - f_{n_{k+1}}(t) + 2|f_{n_{k+1}}(t) - f_{n_k}| \leq 0.
\]
If $\varphi_k(t) = -\infty$, then $f_{n_i}(t) = -\infty$ for $i \geq k$, hence $\varphi_{k+1}(t) = -\infty$. Thus $\varphi_k \sim \varphi \in (\mathcal{F}_0)^\circ$.

For $k = 1, 2, \ldots$ put
\[
\psi_k = \left( f_{n_k} - \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \right) - \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|
\]
Then $\psi_k \in \mathcal{F}_0$ for each $k = 1, 2, \ldots$, and similarly as above, $\psi_k \sim \psi \in (\mathcal{F}_0)^\circ$.

Obviously, $\psi_k \leq f_{n_k} \leq \varphi_k$ for each $k = 1, 2, \ldots$. Since
\[
\varphi_k - \psi_k \leq 4 \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|, \quad J^0(\varphi_k - \psi_k) < \delta_{k-6}
\]
for $k = 7, 8, \ldots$. Hence $\varphi(t) = \psi(t)$ a.e. $\hat{\mathcal{N}}$.

Since $\psi = \liminf_k \psi_k \leq \liminf_k f_{n_k} \leq \limsup_k f_{n_k} \leq \limsup_k \varphi_k = \varphi$,
$f_{n_k}(t) \to \varphi(t)$ a.e. $\hat{\mathcal{N}}$.

Clearly,
\[
|f_{n_k} - \varphi| \leq |f_{n_k} - \varphi_k| + |\varphi_k - \varphi|
\]
\[
|f_{n_k} - \varphi_k| \leq 2 \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|
\]
and
\[
\varphi_k - \varphi = \sum_{i=k}^{\infty} (\varphi_i - \varphi_{i+1}) \leq 3 \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}|
\]
Hence $\hat{J}(f_{n_k} - \varphi) < \delta_{k-7}$ for $k = 8, 9, \ldots$.

Thus $\hat{J}(f_n - \varphi) \to 0$ as $n \to \infty$ by the pseudometric generating property of $\hat{J}$ on $\hat{\mathcal{F}}$, see (5) in Definition 1.

\[ \square \]

**Corollary 1.** $\hat{\mathcal{L}}$ is complete in $\hat{\rho}$, $\hat{\rho}(f, g) = \hat{J}(f - g)$, and $f \in \hat{\mathcal{L}}$ if and only if $f \in \hat{\mathcal{F}}$, and there are $f_n \in \mathcal{F}$, $n = 1, 2, \ldots$, such that $f_n(t) \to f(t)$ a.e. $\hat{\mathcal{N}}$, and $\hat{J}(f_n - f_m) \to 0$ as $n, m \to \infty$. 

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COROLLARY 2. \( f \in \hat{\mathcal{L}}(\mathcal{L}) \) if and only if \( f \in \hat{\mathcal{F}}(\mathcal{F}^*) \), and for each \( \varepsilon > 0 \) there are \( g \in \hat{\mathcal{F}}_u \) and \( h \in \hat{\mathcal{F}}^o \) such that \( g(t) \leq f(t) \leq h(t) \) a.e. \( N \) and \( J(h - g) < \varepsilon \).

COROLLARY 3. \( \mathcal{F} \subset \hat{\mathcal{F}}^o \oplus \hat{\mathcal{F}}_u \oplus \mathcal{N} \subset \hat{\mathcal{L}} \subset ((\hat{\mathcal{F}}^o)_u \oplus \mathcal{N}) \cap ((\hat{\mathcal{F}}^o)_u \oplus \mathcal{N}) \), \( \hat{\mathcal{F}}^o \oplus \hat{\mathcal{F}}_u \oplus \mathcal{N} \subset \mathcal{L} \subset ((\hat{\mathcal{F}}^o)_u \oplus \mathcal{N}) \cap ((\hat{\mathcal{F}}^o)_u \oplus \mathcal{N}) \), and \( \mathcal{L} = \hat{\mathcal{L}} \oplus \mathcal{N} \), where \( \oplus \) means addition of elements.

THEOREM 7. \( \hat{J}(f) = J(f) \) for \( f \in \hat{\mathcal{L}} \).

Proof. Let \( f \in \hat{\mathcal{L}} \), \( f_n \in \mathcal{F} \), \( n = 1, 2, \ldots \), and \( \hat{J}(f - f_n) \to 0 \). Then \( J^*(f - f_n) \to 0 \), hence \( J(f_n) \to J(f) \). Thus, by Theorem 6, \( \hat{J}(f) = \inf \{ J^o(h), \ h \in \hat{\mathcal{F}}^o \} \), \( |f| \leq h \} \leq \inf_{k} J^o(\varphi_k) = \lim_{k \to \infty} J(\varphi_k) = \lim_{k \to \infty} J(f_{nk}) = J(f) \).

Conversely, for \( \varepsilon > 0 \) take \( h_\varepsilon \in \hat{\mathcal{F}}^o \subset \mathcal{L} \) so that \( \varphi \leq h_\varepsilon \) and \( J(h) = J^o(h) \leq \hat{J}(\varphi) + \varepsilon \), where \( \varphi \) is from Theorem 6. Then \( J(f) = J(\varphi) \leq J(h) = J^o(h) \leq \hat{J}(\varphi) + \varepsilon = \hat{J}(f) + \varepsilon \).

THEOREM 8. Suppose \( f, f_n \in \mathcal{L}^+ \), \( n = 1, 2, \ldots \), and \( f_n(t) \not\to f(t) \) a.e. \( N \). Then \( J(f_n) \not\to J(f) \). In particular, \( J: \mathcal{L} \to \mathbb{R}^+ \) has the Fatou property.

Proof. First we show that \( J: \hat{\mathcal{L}} \to \mathbb{R}^+ \) has the Fatou property. Let \( f, f_n \in \hat{\mathcal{L}}^+ \), \( n = 1, 2, \ldots \), and \( f_n \not\to f \). Take \( h \in \hat{\mathcal{F}}^o \) so that \( h \geq f \), and let \( \varepsilon > 0 \). Let finally \( \delta_k, k = 1, 2, \ldots \), be a sequence according to assertions (1) of Theorems 1 and 5.

Owing to autocontinuity of \( J: \hat{\mathcal{L}} \to \mathbb{R}^+ \), see assertion (1) of Theorem 5, for each \( n = 1, 2, \ldots \) there is a \( k_n > k_{n-1} \), \( k_0 = 1 \), such that \( g \in \hat{\mathcal{L}} \) and \( \hat{J}(g) < \delta_{k_n} \) implies \( J(f_n + g) \leq J(f_n) + \varepsilon \). According to Theorem 6, for each \( n = 1, 2, \ldots \) there is an \( h_n \in \hat{\mathcal{F}}^o \) such that \( h_n \geq f_n \) and \( J(h_n - f_n) < \delta_{k_n+1} \).

Obviously, \( \left( \bigvee_{i=1}^{n} h_i \right) \land h \not\to h' \geq f \), hence

\[
J \left( \left( \bigvee_{i=1}^{n} h_i \right) \land h \right) = J^o \left( \left( \bigvee_{i=1}^{n} h_i \right) \land h \right) \not\to J^o(h') \geq \hat{J}(f) = J(f)
\]
by the Fatou property of \( J^o: \mathcal{F}^o \to [0, \infty] \) and by Theorem 7. Clearly,

\[
g_n = \left( \bigvee_{i=1}^{n} h_i \right) \land h - f_n \leq \sum_{i=1}^{n} (h_i - f_i),
\]

hence, \( J(g_n) < \delta_{k_n} \). Thus

\[
J \left( \left( \bigvee_{i=1}^{n} h_i \right) \land h \right) = J(f_n + g_n) \leq J(f_n) + \varepsilon,
\]

\[
\begin{array}{l}
\text{THEOREM 7}. \ \ \hat{J}(f) = J(f) \text{ for } f \in \hat{\mathcal{L}}.
\end{array}
\]
hence, 

$$J(f) = \hat{J}(f) \leq J^0(h') = \lim_{n \to \infty} J\left(\left(\bigvee_{i=1}^{n} h_i\right) \land h\right) \leq \lim_{n \to \infty} J(f_n) + \varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary, $J(f) \leq \lim_{n \to \infty} J(f_n)$. The converse inequality follows from monotonicity of $J$ on $L^+$. 

Suppose now $f, f_n \in L^+, n = 1, 2, \ldots$, and $f_n \not\succ f$ a.e. $\mathcal{N}$. Take $N \in \mathcal{N}$ so that $f_n \not\succ f$ on $T - N$, and put $f'_n = f_n \cdot \chi_{T-N} + f \cdot \chi_N$ for $n = 1, 2, \ldots$. 

Then $f'_n \in L^+$ and $J(f'_n) = J(f_n)$ for $n = 1, 2, \ldots$, and $f'_n \not\succ f$ on $T$. Now it is easy to verify that $L = \hat{L}_N$, where $\hat{L}_N$ corresponds to $J_N : \mathcal{F} \odot \mathcal{N} \to \mathbb{R}^+$, $J_N(f + u) = J(f)$ for $f \in \mathcal{F}$ and $u \in \mathcal{N}$. 

Hence $J(f_n) = J(f'_n) \not\succ J(f)$ by the first part of the proof. 

**THEOREM 9.** Suppose $J : \mathcal{F} \to \mathbb{R}^+$ is a Daniell elementary $L_1$-gauge. Then $J(f_n) \Downarrow 0$ whenever $f_n \in L^+, n = 1, 2, \ldots$, and $f_n(t) \Downarrow 0$ a.e. $\mathcal{N}$. Hence $J : L \to \mathbb{R}^+$ has the Daniell property, $\widehat{\mathcal{F}^o} = \mathcal{F}^o \cap L$ and $\widehat{\mathcal{F}}_u = \mathcal{F}_u \cap L$. 

**Proof.** Without loss of generality, suppose $f_n \in L^+, n = 1, 2, \ldots, f_n \Downarrow 0$, and let $\varepsilon > 0$. By the pseudometric generating property of $J : L \to \mathbb{R}^+$, see Definition 1 and Theorem 4, take $\delta > 0$ so that $J(f + g) < \varepsilon$ whenever $f, g \in L$ and both $J(f), J(g) < \delta$. Take a sequence $\delta_k, k = 1, 2, \ldots$, as in assertions (1) of Theorems 1 and 5, and let $k_0$ be such that $\delta_{k_0} < \delta$. By Corollary 2 of Theorem 6, there are $h_n \in \widehat{\mathcal{F}^o}^+, n = 1, 2, \ldots$, such that $h_n(t) \geq f_n(t)$ a.e. $\mathcal{N}$, and $J(h_n - f_n) \leq \delta_{k_0+n}$ for each $n = 1, 2, \ldots$, and $h_n \Downarrow h$. 

Since $h = \lim_{n \to \infty} (h_n - f_n) \leq \sum_{n=1}^{\infty} (h_n - f_n)$, $\hat{J}(h) \leq \delta_{k_0} < \delta$. Take $h' \in \widehat{\mathcal{F}^o}$, $n = 1, 2, \ldots$, such that $h' \geq h$ and $\hat{J}(h') < \delta$, and let $u_n \in \mathcal{F}^+$, $n = 1, 2, \ldots$, be such that $u_n \not\succ h'$. Then $h_n \leq (h_n - u_n) + h'$ for each $n = 1, 2, \ldots$, and $h_n - u_n \Downarrow 0$. 

Since $h_n - u_n \in \widehat{\mathcal{F}^o}$, $n = 1, 2, \ldots$, there are $v_n \in \mathcal{F}^+$, $n = 1, 2, \ldots$, such that $v_n \leq h_n - u_n$ and $J(h_n - u_n - v_n) < \delta_{k_0+1+n}$ for each $n = 1, 2, \ldots$. 

Put $w_n = \bigwedge_{i=1}^{n} v_i$, $n = 1, 2, \ldots$. Then $w_n \in \mathcal{F}^+$ for each $n = 1, 2, \ldots$, and $w_n \Downarrow 0$. Hence, by the Daniel property of $J : \mathcal{F} \to \mathbb{R}^+$, there is an $n_0$ such that $J(w_n) < \delta_{k_0+1}$ for $n \geq n_0$. 

Since $h_n - u_n - w_n = \bigwedge_{i=1}^{n} (h_i - u_i) - \bigwedge_{i=1}^{n} v_i \leq \sum_{i=1}^{n} (h_i - u_i - v_i)$, $J(h_n - u_n - w_n) < \delta_{k_0+1}$ for each $n = 1, 2, \ldots$. Hence 

$$J(h_n - u_n) = J(h_n - u_n - w_n + w_n) < \delta_{k_0} < \delta \quad \text{for} \quad n \geq n_0.$$ 

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Thus $J(f_n) \leq J(h_n) < \varepsilon$ for $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, the theorem is proved.

We now show that by applying our extension procedure to $J: \mathcal{L} \to \mathbb{R}^+$ we obtain no further enlargement of $\mathcal{L}$. This is of basic importance in connection with the Stone type representation of $J: \mathcal{L} \to \mathbb{R}^+$.

Denote by $J^*_1$ the extension of $J$ to $\mathcal{L}^+ \supset \mathcal{F}$. Clearly, $J^*_1(f) \leq J^*(f)$ for each $f \in \mathcal{F}^* = \mathcal{L}^*$, where

$$J^*_1(f) = \inf \{ J^*_1(h), \ h \in \mathcal{L}^+, \ |f| \leq h \} \quad \text{for} \quad f \in \mathcal{F}^*.$$

**Theorem 10.** Suppose $J: \mathcal{F} \to \mathbb{R}^+$ is uniformly autocontinuous. Then $J^*_1(f) = J^*(f)$ for each $f \in \mathcal{F}^* = \mathcal{L}^*$. Hence $J(f) = J^*(f)$ for $f \in \mathcal{F}$ in this case.

**Proof.** Let $f \in \mathcal{F}^*$, $\varepsilon > 0$, and $u \in \mathcal{L}^+$ be such that $u \geq f$ and $J^*(u) \leq J^*_1(f) + 3^{-1} \cdot \varepsilon$. Take $u_n \in \mathcal{L}^+$, $n = 1, 2, \ldots$, so that $u_n \nearrow u$ and $J(u_n) / J^*_1(u)$.

Let $\delta_k$, $k = 1, 2, \ldots$, be a sequence as in assertion (1) of Theorems 1 and 5, and take $k_0$ so that $\delta_{k_0} < \delta(3^{-1} \cdot \varepsilon)$, where $\delta(3^{-1} \cdot \varepsilon)$ is such that $|J^*(v) - J^*(w)| < 3^{-1} \cdot \varepsilon$ whenever $v, w \in \mathcal{F}^*$ and $J^*(v - w) < \delta(3^{-1} \cdot \varepsilon)$. By Corollary 2 of Theorem 6, for each $n = 1, 2, \ldots$ there exists $h_n \in \mathcal{F}^+$ such that $h_n(t) \geq u_n(t)$ a.e. $\mathcal{N}$ and $J^*(h_n - u_n) < \delta_{k_0 + n}$. For $n = 1, 2, \ldots$ put $h'_n = \bigvee_{i=1}^n h_i$. Then

$h'_n(t) \nearrow h'(t) \geq f(t)$ a.e. $\mathcal{N}$, and $h'_n - u_n \leq \sum_{i=1}^n (h_i - u_i)$.

Hence $J^*(h') = \lim_{n \to \infty} J^*(h'_n)$ and $J^*(h'_n - u_n) < \delta_{k_0} < \delta(3^{-1} \cdot \varepsilon)$ for each $n = 1, 2, \ldots$. Since $h'(t) \geq f(t)$ a.e. $\mathcal{N}$, there is a $v \in \mathcal{F}^+$ such that $h' + v \geq f$ and $J^*(v) < \delta(3^{-1} \cdot \varepsilon)$.

Thus

$$J^*(f) \leq J^*(h' + v) \leq J^*(h') + 3^{-1} \cdot \varepsilon$$

$$= 3^{-1} \cdot \varepsilon + \lim_{n \to \infty} J^*(h'_n) = 3^{-1} \cdot \varepsilon + \lim_{n \to \infty} J^*(h'_n - u_n + u_n)$$

$$\leq 2 \cdot 3^{-1} \cdot \varepsilon + \lim_{n \to \infty} J^*(u_n) = 2 \cdot 3^{-1} \cdot \varepsilon + J^*_1(u)$$

$$\leq J^*_1(f) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $J^*_1(f) = J^*(f)$. \hfill \Box

**Theorem 11.** Suppose $f_k \in \mathcal{F}^* = \mathcal{L}^*$, $k = 1, 2, \ldots$. Then $J^*_1(f_k) \to 0$ if and only if $J^*(f_k) \to 0$. Hence, by applying the extension procedure to $J: \mathcal{L} \to \mathbb{R}^+$, we obtain no further enlargement of $\mathcal{L}$.

**Proof.** Since $J^*_1 \leq J^*$ on $\mathcal{F}^* = \mathcal{L}^*$, consider $f_k \in \mathcal{F}^*$, $k = 1, 2, \ldots$, such that $J^*_1(f_k) < \delta_k$, where $\delta_k$, $k = 1, 2, \ldots$, is a sequence as in assertion (1) of Theorems 1 and 5.
By Theorem 8, for each $k = 1, 2, \ldots$ there exists $f'_k \in L^0^+$ and a sequence $f''_{k,n} \in L^+$, $n = 1, 2, \ldots$, such that $f'_k \geq f_k$ and $f''_{k,n} \triangleright f'_k$ for each $k$, and $J(f''_{k,n}) > J(f'_k) < \delta_k$.

By virtue of Corollary 2 of Theorem 6, for each $k, n = 1, 2, \ldots$ there exists $h'_{k,n} \in \mathcal{F}^o^+$ such that $h'_{k,n}(t) \geq f''_{k,n}(t)$ a.e. $\mathcal{N}$ and $J(h'_{k,n} - f''_{k,n}) < \delta_{k+n}$.

For $k, n = 1, 2, \ldots$ put $h_{k,n} = \bigvee_{i=1}^n h'_{k,i}$. Then $h_{k,n} - f'_{k,n} \leq \sum_{i=1}^n (h'_{k,i} - f'_{k,i})$, hence $J(h_{k,n} - f'_{k,n}) < \delta_k$ and $J(f''_{k,n}) < \delta_k$ for each $k, n = 1, 2, \ldots$.

Thus $h_{k,n} \triangleright h_k \in \mathcal{F}^o^+$, $h_k(t) \geq f'_{k}(t)$ a.e. $\mathcal{N}$, and $J(h_k) = \lim_{n \to \infty} J(h_{k,n}) \leq \delta_{k-1} < \delta_{k-2}$ for each $k = 3, 4, \ldots$.

Take $u_k \in \mathcal{F}^o^+$ so that $h_k + u_k \geq f'_k$ and $J(u_k) < \delta_{k-2}$, $k = 3, 4, \ldots$. Then $J^*(f'_k) \leq J^*(f'_{k,n}) \leq J(h_k + u_k) < \delta_{k-3}$ for each $k = 4, 5, \ldots$, hence $J^*(f'_k) \to 0$ as $k \to \infty$.

The theorem is proved. □

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DANIELL TYPE EXTENSIONS OF $L_1$-GAUGES AND INTEGRALS


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