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ON MODULAR SET FUNCTIONS AND MEASURES

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ABSTRACT. The paper deals with real-valued signed modular set functions *MSF*'s with finite variation and range of definition on some set lattice $\mathscr{K}$. The class $Mo(\mathscr{K})$ of all such *MSF*'s can be viewed as a Riesz space. The classes of tight, σ-additive and τ-smooth *MSF*'s defined in the paper are proved to be complete normal subspaces of $Mo(\mathscr{K})$. The conditions, under which these classes are isomorphic with spaces of tight, regular, σ-additive and τ-smooth Baire (Borel) measures, are studied.

Introduction

Let X, $\mathscr{G}(X)$ be a topological space and let $\mathscr{G}_0(X)$, $\mathscr{F}_0(X)$ and $\mathscr{K}_0(X)$ be the classes of all open, closed and closed compact Baire sets defined by the topology $\mathscr{G}(X)$. If m is a Baire measure on $\mathscr{B}_0(X)$, then the basic "topological" properties of m like the tightness and smoothness are determined by the behaving of m on the set lattices $\mathscr{F}_0(X)$ and $\mathscr{K}_0(X)$, respectively (see [17]). The question studied in this paper is, under which additional conditions is a modular, not necessarily non-negative set function defined on $\mathscr{F}_0(X)$ or $\mathscr{K}_0(X)$ able to determine a measure on $\mathscr{B}_0(X)$ with suitable topological properties.

The problem is solved in an abstract level for signed modular set functions. Such functions often agree with regular contents, see [16], lemma 2.4. The first part of the paper deals with real-valued modular set functions (shortly *MSF*'s) with finite variation (see latter definitions) and with range of definition on a set lattice $\mathscr{K}$. The set $Mo(\mathscr{K})$ of all such *MSF*'s is proved to be a Riesz space containing the sets $Mo(\mathscr{K}, t)$, $Mo(\mathscr{K}, \sigma)$ and $Mo(\mathscr{K}, \tau)$ as complete normal subspaces. Here $Mo(\mathscr{K}, t)$ is the family of all tight *MSF*'s by $Mo(\mathscr{K}, \sigma)$ is denoted the class of all σ-additive *MSF*'s and $Mo(\mathscr{K}, \tau)$ is the notation for the

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space of τ-smooth *MSF*'s. The definition of these set functions is available below.

The second part of the paper is concerned with the possibility of the extension of *MSF*'s from the set lattice $\mathscr{K}$ to set structures (rings, algebras) containing $\mathscr{K}$. We obtain several generalizations of classical extension theorems (see [2, 3, 6, 12, 15]). The results are formulated in terms of Riesz isomorphisms and imbeddings between spaces of *MSF*'s and spaces of (signed) measures.

Application of the foregoing results to the construction of measures on topological spaces and answer to the above formulated question is studied in the last section of the paper. In particular we can say that if X is a topological space and if $Mo(\mathscr{F}_0(X), t)$ is the space of all tight set functions on $\mathscr{F}_0(X)$, then $Mo(\mathscr{F}_0(X), t)$ can be identified with the dual of the Banach space $C_b(X)$ of all bounded real continuous functions on X with the supremum norm.

Preliminaries

Let us consider an arbitrary non-empty set X. The system of all subsets of X will be denoted by $\exp X$. By a *paving* we shall understand an arbitrary non-empty subclass of sets from $\exp X$. The class $\mathscr{K} \subset \exp X$ is said to be a *set lattice* if it is closed under the formation of finite unions, intersections and $\emptyset$ is a member of $\mathscr{K}$. The set lattice $\mathscr{E} \subset \exp X$ closed under the formation of finite unions and complements is said to be a *ring*. If moreover $X \in \mathscr{E}$, we shall speak about an *algebra*. The ring generated by the class $\mathscr{K}$ will be denoted $\mathscr{E}(\mathscr{K})$. The ring (algebra) closed under the formation of countable unions is said to be a σ*-ring* (σ*-algebra*). The σ-ring generated by the class $\mathscr{K}$ will be denoted $\mathscr{E}_\sigma(\mathscr{K})$.

Throughout this paper all discussed set-functions, i.e. all mappings m: $\mathscr{K} \to \mathbf{R}$, where $\mathscr{K} \subset \exp X$ and $\mathbf{R} = (-\infty, \infty)$, are assumed to be bounded (there is a constant $c \in \mathbf{R}$ such that $|mK| < c$ for any $K \in \mathscr{K}$). The set function m on $\mathscr{K}$ is said to have *finite variation* with respect to $\mathscr{K}$, shortly *wrt* $\mathscr{K}$, if there is a constant $c \in \mathbf{R}$ with the property $\sum_{i=1}^{n} |mK_i - mK_{i-1}| < c$ for all strings $K_0 \subset K_1 \subset \ldots \subset K_n$ consisting of sets from $\mathscr{K}$.

The set function $m: \mathscr{E} \to \mathbf{R}$ is called *regular wrt* $\mathscr{K} \subset \mathscr{E}$ if for each $E \in \mathscr{E}$ the equality $mE = \lim_{K \in \mathscr{K}_0} mK$ holds. Here $\mathscr{K}_0 = \{K\colon K \subset E,\ K \in \mathscr{K}\}$ is directed by inclusion. If m is a *monotone* function, i.e. if the inclusion $E_1 \subset E_2$ implies that $mE_1 \leqslant mE_2$ whenever $E_1, E_2 \in \mathscr{E}$, the regularity condition can be rewritten into the form $mE = \sup\{mK\colon K \subset E,\ K \in \mathscr{K}\}$.

The set function $m: \mathscr{K} \to \mathbf{R}$ is said to be *tight wrt* $\mathscr{K}$ if $\mathscr{K}$ is a set lattice, $m\emptyset = 0$ and $mK_1 - mK_2 = \lim_{K \in \mathscr{K}_0} mK$ whenever $K_1 \supset K_2$ are in $\mathscr{K}$. Here $\mathscr{K}_0 =$

$= \{K: K \subset K_1 - K_2, K \in \mathscr{K}\}$. For monotone m the tightness condition becomes the fashion $mK_1 - mK_2 = \sup\{mK: K \in \mathscr{K}_0\}$, compare with [9, 15, 16].

By a *modular* set function (*MSF*) is understood a set function defined on a set lattice $\mathscr{K} \subset \exp X$ and with the property $m\emptyset = 0$, $mK_1 \cup K_2 + mK_1K_2 = mK_1 + mK_2$ for all $K_1, K_2 \in \mathscr{K}$.

A *MSF* defined on a ring is called *measure*. The *MSF* m defined on $\mathscr{K} \subset \exp X$ is said to be *σ-additive wrt* $\mathscr{K}$ if it has finite variation and $mK = \sum_{i-1}^{\infty} (mK_i - m\acute{K}_{i-1})$ whenever $K = \bigcup_{i=1}^{\infty} K_i - \acute{K}_{i-1} \in \mathscr{K}$ and $K_1 - \acute{K}_1$, $K_2 - \acute{K}_2$, ..., $K_i - \acute{K}_i$, ... is such a sequence of pairwisse disjoint sets that $K_i \supset \acute{K}_i$ and K', $K_i \in \mathscr{K}$ for all $i = 1, 2, \ldots$.

Let $\mathscr{K}_0 \subset \mathscr{K}$ be directed by inclusion and filtering downwards to $K_0 \in \mathscr{K}$, i.e. $K_0 = \bigcap_{K \in \mathscr{K}_0} K$. The *MSF* m on $\mathscr{K}$ with finite variation is said to be *τ-smooth at* K_0 if to each $\varepsilon > 0$ there is $K \in \mathscr{K}_0$ such that for each string $\{K_i\} \subset \mathscr{K}$, $K_0 \subset K_1 \subset \ldots \subset K_n \subset K$ $\sum_{i=1}^{n} |mK_i - mK_{i-1}| < \varepsilon$. If m is τ-smooth at each $K \in \mathscr{K}$, it is shortly *τ-smooth wrt* $\mathscr{K}$.

The sets $Mo(\mathscr{K})$, $Mo(\mathscr{K}, t)$, $Mo(\mathscr{K}, \sigma)$ and $Mo(\mathscr{K}, \tau)$ mentioned in the introduction are now well defined.

Example 1. Let $X = [0, 1]$, $\mathscr{K}(X) = \left\{\bigcup_{i=1}^{n} [a_i, b_i] : [a_i, b_i] \subset X \text{ are non-overlapping}, n \in \mathbf{N}\right\}$. Then each real function $F(x)$ with finite variation on X defines by the relations $m\emptyset = 0$,

$$m \bigcup_{i-1}^{n} [a_i, b_i] = \sum_{i=1}^{n} (F(b_i) - F(a_i)), \quad \bigcup_{i=1}^{n} [a_i, b_i] \in \mathscr{K}(X)$$

a *MSF* with finite variation. Conversely, each $m \in Mo(\mathscr{K}(X))$ defines by the relation $F(x) = m[0, x]$ a real function with finite variation on X.

In this example $Mo(\mathscr{K}, t) = Mo(\mathscr{K}, \sigma) = Mo(\mathscr{K}, \tau)$ and $m \in Mo(\mathscr{K}, t)$ if and only if F is right-hand continuous. Here $\mathscr{K} = \mathscr{K}(X)$. Taking $F(0) = 0$ and $F(x) = x \cos \frac{1}{x}$ elsewhere we can construct an example of a bounded *MSF* which is not in $Mo(\mathscr{K})$.

Example 2. Let X, $\mathscr{G}(X)$ be a Hausdorff space and $\mathscr{K}(X)$ be the class of all compact subsets of X. Then each bounded monotone set function on $\mathscr{K}(X)$, with properties $m\emptyset = 0$, $mK_1 \cup K_2 \leqslant mK_1 + mK_2$ whenever $K_1, K_2 \in \mathscr{K}(X)$, $mK_1 \cup K_2 = mK_1 + mK_2$ whenever $K_1, K_2 \in \mathscr{K}(X)$ are disjoint and $mK = \inf\{m_*G: G \supset K, G \in \mathscr{G}(X)\}$, where $m_*G = \sup\{m\acute{K}: \acute{K} \subset G, \acute{K} \in \mathscr{K}(X)\}$, is tight, σ-additive and τ-smooth.

As to the proof see [16; lemma 2.1. and preliminaries].

§1. General Modular Set Functions

Throughout this section X is a non-empty set, $\mathscr{K} \subset \exp X$ is a set lattice and both are fixed. Since for each *MSF* m on $\mathscr{K}$ and for arbitrary real constant c the function cm, defined for each $K \in \mathscr{K}$ by $(cm)K = c\,mK$, is a well-defined *MSF* and because the same can be said about the set function $m_1 + m_2$ defined on $\mathscr{K}$ by $(m_1 + m_2)K = m_1 K + m_2 K$, if m_1 and m_2 are *MSF*'s on $\mathscr{K}$, $Mo(\mathscr{K})$ can be thought of as a linear (vector) space over **R**. Moreover,

Theorem 1.1. *The space $Mo(\mathscr{K})$ can be endowed by an operation of (partial) ordering changing it into a Riesz space.*

The proof is a natural consequence of the following two lemmas due to G. Birkhoff.

Lemma 1.2. *For arbitrary $K \in \mathscr{K}$ let Γ_K be the system of all strings γ: $\emptyset = K_0 \subset K_1 \subset \ldots \subset K_n = K$, $K_i \in \mathscr{K}$ for all $i = 1, \ldots, n$. To any $m \in Mo(\mathscr{K})$ let m^+ and m^- be set functions defined on $\mathscr{K}$ by the relations*

$$m^+ K = \sup_{\gamma \in \Gamma_K} \sum_{i=1}^{n} \max(mK_i - mK_{i-1}, 0) \quad m^- K = -\inf_{\gamma \in \Gamma_K} \sum_{i=1}^{n} \min(mK_i - mK_{i-1}, 0).$$

Then m^+ and m^- are non-negative monotone functions from $Mo(\mathscr{K})$ and $m = m^+ - m^-$.

Lemma 1.3. *Let $\leqslant$ be the relation of partial ordering defined on $Mo(\mathscr{K})$ by the relation*

$$m \leqslant \bar{m} \text{ if and only if } m - \bar{m} \text{ is monotone.}$$

Here $m, \bar{m} \in Mo(\mathscr{K})$. Then $Mo(\mathscr{K})$ is a lattice and for every $m \in Mo(\mathscr{K})$ $m \vee 0 = m^+$ and $m \wedge 0 = -m^-$. $\vee$ and $\wedge$ are the lattice operations of maximum and minimum defined on $Mo(\mathscr{K})$ by means of the relation $\leqslant$. The function identically equal to zero on $\mathscr{K}$ is denoted by 0.

As to the proofs see [3, X; §6; theorem 10]. If we want state $Mo(\mathscr{K}, t)$ as a Riesz subspace of $Mo(\mathscr{K})$, we must know that all such functions are in $Mo(\mathscr{K})$ (see lemma 1.4 and 1.5) and $Mo(\mathscr{K})$ must be proved as closed *wrt* the lattice operations induced from $Mo(\mathscr{K})$ (lemma 1.6).

Lemma 1.4. *If m is tight wrt $\mathscr{K}$, then m is a MSF on $\mathscr{K}$.*

Proof. If m is tight *wrt* $\mathscr{K}$ and if $K_1, K_2 \in \mathscr{K}$, then

$$mK_1 \cup K_2 - mK_1 = \lim_{K \in \mathscr{K}_0} mK = mK_2 - mK_1 K_2,$$

hence, the assertion holds. □

Lemma 1.5. *If m is tight wrt $\mathscr{K}$, then m has finite variation wrt $\mathscr{K}$.*

Proof. Let m be tight *wrt* $\mathscr{K}$ and bounded by the constant c. If $K_0 \subset \subset K_1 \subset \ldots \subset K_n$ is an arbitrary string of sets in $\mathscr{K}$ and if $\varepsilon > 0$ is some real number, then we can find a sequence $\{\acute{K}_i\} \subset \mathscr{K}$ with properties $\acute{K}_i \subset K_i - K_{i-1}$ and $|mK_i - mK_{i-1} - m\acute{K}_i| < \frac{\varepsilon}{n}$ for all $i \in I = \{1, \ldots, n\}$. Denoting $I^+ = \{i: m\acute{K}_i \geqslant \geqslant 0, i \in I\}$, $I^- = \{i: m\acute{K}_i < 0, i \in I\}$ we can write

$$\sum_{i\in I} |mK_i - mK_{i-1}| - \varepsilon < \sum_{i \in I} |m\acute{K}_i| = m \bigcup_{i \in I^+} \acute{K}_i - m \bigcup_{i \in I^-} \acute{K}_i < 2c.$$

Since ε can be made arbitrarily small the proof is complete. □

Lemma 1.6. *The MSF m is tight wrt $\mathscr{K}$ if and only if m^+ and m^- are tight wrt $\mathscr{K}$.*

Proof. If m^+ and m^- are tight *wrt* $\mathscr{K}$, then we can prove, by virtue of the topological properties of **R** and the tightness definition, that $m^+ - m^-$ (i.e. m, by lemma 1.2) is tight as well.

Now let m be tight and let $K_1, K_2 \in \mathscr{K}$. To a chosen $\varepsilon > 0$ there is a string $\{\acute{K}_i\} \subset \mathscr{K}$ such that $\emptyset = \acute{K}_0 \subset \acute{K}_1 \subset \ldots \subset \acute{K}_n = K_1$ and

$$m^+K_1 < \sum_{i-1}^{n} \max(m\acute{K}_i - m\acute{K}_{i-1}, 0) + \varepsilon.$$

Since

$$\acute{K}_0K_2 \subset \acute{K}_1K_2 \subset \ldots \subset \acute{K}_nK_2 \subset K_2 \subset \acute{K}_0 \cup K_2 \subset \acute{K}_1 \cup K_2 \subset \ldots \subset \acute{K}_n \cup K_2 = K_1,$$

$$m^+K_1 < \sum_{i=1}^{n} \max(m\acute{K}_i \cup K_2 - m\acute{K}_{i-1} \cup K_2, 0) + \sum_{i=1}^{n} \max(m\acute{K}_iK_2 - m\acute{K}_{i-1}K_2, 0) + \varepsilon$$

$$\leqslant \sum_{i=1}^{n} \max(m\acute{\acute{K}}_i, 0) + m^+K_2 + 2\varepsilon,$$

where $\acute{\acute{K}}_i \in \mathscr{K}$ are suitably chosen sets with the property $\acute{\acute{K}}_i \subset (\acute{K}_i \cup K_2) - (\acute{K}_{i-1} \cup K_2)$ for all $i = 1, \ldots, n$. We have used the relations

$$\acute{K}_{i-1} \subset \acute{K}_{i-1} \cup \acute{K}_iK_2 = (\acute{K}_{i-1} \cup K_2)\acute{K}_i \subset \acute{K}_i$$

$$m\acute{K}_i - m\acute{K}_i(\acute{K}_{i-1} \cup K_2) = m\acute{K}_i \cup K_2 - m\acute{K}_{i-1} \cup K_2$$

$$m\acute{K}_{i-1} \cup \acute{K}_iK_2 - m\acute{K}_{i-1} = m\acute{K}_iK_2 - m\acute{K}_{i-1}K_2$$

which hold for all $i = 1, \ldots, n$ and the tightness of m. Denoting by K the union of all $\acute{\acute{K}}_i$ with $m\acute{\acute{K}}_i > 0$ we obtain a set $K \in \mathscr{K}$ with properties $K \subset K_1 - K_2$ and

$$m^+K_1 < m^+K + m^+K_2 + 2\varepsilon.$$

Since m^+ is a monotone function and $\varepsilon > 0$ can be made arbitrarily small the proof is complete. □

As a natural consequence of the foregoing considerations we obtain

Theorem 1.7. *The class $Mo(\mathcal{K}, t)$ is a subspace of the Riesz space $Mo(\mathcal{K})$ wrt operations induced from $Mo(\mathcal{K})$.*

Next we will writte $|m|$ for $m^+ + m$ if $m \in Mo(\mathcal{K})$. The subspace M of $Mo(\mathcal{K})$ is said to be *normal* if each set $E \subset M$ bounded above (below) has the least upper bound (greatest lower bound) in $Mo(\mathcal{K})$. The mentioned bounds are denoted by $\vee_E m$ and $\wedge_E m$, re pectively. This subspace is called *complete* if the condition $0 \leqslant |\bar{m}| \leqslant |m|$, $m \in M$ leads to the relation $\bar{m} \in M$.

Theorem 1.8. *The space $Mo(\mathcal{K}, t)$ is complete and normal.*

The proof is divided into three lemmas 1.9 11. We recall that the monotone set function b defined on $\mathcal{K}$ is said to be *supermodular* if $b\emptyset = 0$ and $bK_1 \cup \cup K_2 + bK_1K_2 \geqslant bK_1 + bK_2$ for all $K_1, K_2 \in \mathcal{K}$. Replacing "$\geqslant$" by "$\leqslant$" and assuming that $bK_1 \geqslant bK_2$ if $K_1 \subset K_2$ and $K_1, K_2 \in \mathcal{K}$ we obtain the definition of a *submodular* set function.

Lemma 1.9. *Let $m, \bar{m} \in Mo(\mathcal{K}, t)$. If b is supermodular on $\mathcal{K}$ and $mK \leqslant bK$, $\bar{m}K \leqslant bK$ for all $K \in \mathcal{K}$, then $m \vee \bar{m}K \leqslant bK$. If b is submodular and $mK > bK$, $\bar{m}K \geqslant bK$ for all $K \in \mathcal{K}$, then $m \wedge \bar{m}K \geqslant bK$.*

Proof. We start from the first assertion. Since $m \vee \bar{m} = m + (m - \bar{m})^+$,

$$m \vee \bar{m}K = \sup_{\gamma \in \Gamma_K} \sum_{i=1}^{n} \max(mK_i - mK_{i-1}, \bar{m}K_i - \bar{m}K_{i-1})$$

for each $K \in \mathcal{K}$. Hence, for a fixed $K \in \mathcal{K}$ and $\varepsilon > 0$ there is a string $\emptyset = K_0 \subset \subset K_1 \subset \dots \subset K_n = K$ in $\mathcal{K}$ for which

$$m \vee \bar{m}K < \sum_{i \in I} \max(mK_i - mK_{i-1}, \bar{m}K_i - \bar{m}K_{i-1}) + \varepsilon,$$

where $I = \{1, \dots, n\}$. Let us denote $I_1 = \{i: mK_i - mK_{i-1} \geqslant \bar{m}K_i - \bar{m}K_{i-1}\}$ and $I_2 = I - I_1$. Choosing in $\mathcal{K}$ sets $\acute{K}_i \subset K_i - K_{i-1}$ and $\acute{\acute{K}}_i \subset K_i - K_{i-1}$ so that $mK_i - mK_{i-1} < m\acute{K}_i + \dfrac{\varepsilon}{n}$ and $\bar{m}K_i - \bar{m}K_{i-1} < \bar{m}\acute{K}_i + \dfrac{\varepsilon}{n}$ we obtain

$$m \vee \bar{m}K - \varepsilon < \sum_{i \in I_1} m\acute{K}_i + \sum_{i \in I_2} \bar{m}\acute{\acute{K}}_i = m \bigcup_{i \in I_1} \acute{K}_i + \bar{m} \bigcup_{i \in I_2} \acute{\acute{K}}_i \leqslant$$

$$b \bigcup_{i \in I_1} \acute{K}_i + b \bigcup_{i \in I_2} \acute{\acute{K}}_i \leqslant b \bigcup_{i \in I_1} \acute{K}_i \cup \bigcup_{i \in I_2} \acute{\acute{K}}_i \leqslant bK.$$

As $\varepsilon > 0$ and can be made arbitrarily small the first assertion is proved. The second one follows from the relations

$$m \wedge \bar{m} = -((-m) \vee (-\bar{m})) \geqslant -(-b) = b.$$

They are true by the just proved result. □

Lemma 1.10. *Let $E \subset Mo(\mathscr{K}, t)$. If E is bounded above (below) by a supermodular (submodular) set function b, then $\vee_E m$ ($\wedge_E m$) is in $Mo(\mathscr{K}, t)$ and $\vee_E mK \leqslant bK$ ($\wedge_E mK \geqslant bK$) for all $K \in \mathscr{K}$.*

If E is bounded above (below) by $\bar{m} \in Mo(\mathscr{K})$, then $\vee_E m$ ($\wedge_E m$) is in $Mo(\mathscr{K}, t)$.

Proof. If $E \subset Mo(\mathscr{K}, t)$ is bounded by a supermodular function b, then the class E_0 consisting of all MSF's of the form $m_1 \vee m_2 \vee \ldots \vee m_n$ where $m_i \in E$, $i = 1, \ldots, n$, is bounded from above by b (lemma 1.9). If $\acute{E}_0 \subset E_0$ is a completely ordered set, then $\vee_{\acute{E}_0} mK = \sup_{m \in \acute{E}_0} mK \leqslant bK$ for all $K \in \mathscr{K}$. Hence, if $\bar{E}$ contains E_0 and the least upperbounds of all completely ordered subsets in E_0, then, due to Zorn's lemma, $\bar{E}$ has a least upper bound and, naturally, it is bounded above by b. Since $\vee_E m = \vee_{\bar{E}} m$, the first assertion holds. The second must be also true because $\wedge_E m = -(\vee_E(-m))$.

Now let E be bounded above by $\bar{m} \in Mo(\mathscr{K})$. Following the just given proof, however, using theorem 1.7 instead of lemma 1.9, we can prove that $\vee_E m$ exists and belongs to $Mo(\mathscr{K}, t)$. The rest of the proof is now clear. □

Lemma 1.10 generalizes theorem 6.1 ii) from [16]. For the relation of supermodular and submodular set functions to the modular ones see [8].

Lemma 1.11. *If m_1 and m_2 are tight and if $m_1 \leqslant m_2$, then each MSF m with the property $m_1 \leqslant m \leqslant m_2$ is tight.*

Proof. Considering $m - m_1$ and noting that if $m - m_1$ is tight then so is m, we reduce the proof of the presented lemma to the case $m_1 = 0$. If $K_1 \supset K_2$ is an arbitrary couple of sets in $\mathscr{K}$ and if $\varepsilon > 0$ is arbitrarily small, then in $\mathscr{K}$ there is $K \subset K_1 - K_2$ for which $m_2 K_1 - m_2 K_2 < m_2 K + \varepsilon$. The function $m_2 - m$ is monotone and $(m_2 - m) K_1 - (m_2 - m) K_2 < (m_2 - m) K + \varepsilon$. Therefore $(m_2 - m) \in Mo(\mathscr{K}, t)$ and, consequently, $m \in Mo(\mathscr{K}, t)$. □

Let us recall that $m \in Mo(\mathscr{K})$ is said to be *disjoint* from $\bar{m} \in Mo(\mathscr{K})$ if $|m| \wedge |\bar{m}| = 0$. Two subspaces M and $\acute{M}$ form a *direct decomposition* of $Mo(\mathscr{K})$ if $|m| \wedge |\bar{m}| = 0$ for all $m \in M$ and $\bar{m} \in \acute{M}$ and $Mo(\mathscr{K})$ is the direct sum of M and $\acute{M}$. Let $Mo(\mathscr{K}, s)$ be the subspace of $Mo(\mathscr{K})$ defined by the relation $m \in Mo(\mathscr{K}, s)$ if and only if $|m| \wedge |\bar{m}| = 0$ for all $\bar{m} \in Mo(\mathscr{K}, t)$.

Theorem 1.12. *The spaces $Mo(\mathscr{K}, t)$ and $Mo(\mathscr{K}, s)$ form a direct decomposition of $Mo(\mathscr{K})$. Moreover, each $m \in Mo(\mathscr{K})$ may be written as $m = \bar{m} + s$, where $\bar{m} \in Mo(\mathscr{K}, t)$, $s \in Mo(\mathscr{K}, s)$ and $|m| \geqslant |\bar{m}|$.*

Proof of this theorem, including the fact that $Mo(\mathscr{K}, s)$ is a complete normal subspace of $Mo(\mathscr{K})$, follows from the completness and normality of $Mo(\mathscr{K}, t)$ by virtue of [4; theorem 3].

Theorem 1.13. *The class $Mo(\mathscr{K}, \sigma)$ is a subspace of the Riesz space $Mo(\mathscr{K})$.*

Proof. By the definition $Mo(\mathscr{K}, \sigma) \subset Mo(\mathscr{K})$ and $Mo(\mathscr{K}, \sigma)$ is a linear space *wrt* the operations induced from $Mo(\mathscr{K})$. The rest of the proof follows from lemma 1.14. □

Lemma 1.14. *If m is a σ-smooth MSF, then m^+ is a σ-smooth MSF.*

Proof. To prove it let $K = \bigcup_{n=1}^{\infty} \acute{K}_n - \acute{\acute{K}}_n \in \mathscr{K}$, where the sets representing K are pairwise disjoint, $\acute{K}_n \supset \acute{\acute{K}}_n$ are in $\mathscr{K}$ and $n = 1, 2, \ldots$. By [12] m has a unique extension to a measure $\bar{m}$ on $\mathscr{E}(\mathscr{K})$. Hence, if $\varepsilon > 0$ is a given number, there is a string $\emptyset = K_0 \subset K_1 \subset \ldots \subset K_n = K$ such that

$$m^+K < \sum_{i=1}^{n} \max(mK_i - mK_{i-1}, 0) + \varepsilon$$

and denoting $\{i: mK_i > mK_{i-1}\}$ by I^+ we obtain that

$$m^+K - \varepsilon < \sum_{i \in I^+} (mK_i - mK_{i-1}) = \sum_{i \in I^+} \left(\sum_{n=1}^{\infty} (mK_i\acute{K}_n - mK_i\acute{\acute{K}}_n) - \sum_{n=1}^{\infty} (mK_{i-1}\acute{K}_n - mK_{i-1}\acute{\acute{K}}_n) \right) = \sum_{n=1}^{\infty} \bar{m} \bigcup_{i \in I^+} (K_i - K_{i-1})(\acute{K}_n - \acute{\acute{K}}_n) \leqslant$$

$$\sum_{n=1}^{\infty} \bar{m}^+ \acute{K}_n - \acute{\acute{K}}_n = \sum_{n=1}^{\infty} (m^+\acute{K}_n - m^+\acute{\acute{K}}_n).$$

Consequently, $m^+K \leqslant \sum_{i=1}^{\infty} (m^+\acute{K}_i - m^+\acute{\acute{K}}_i)$. The reverse inequality follows from the relations

$$\sum_{n=1}^{n_0} (m^+\acute{K}_n - m^+\acute{\acute{K}}_n) = \bar{m}^+ \bigcup_{n=1}^{n_0} \acute{K}_n - \acute{\acute{K}}_n \leqslant \bar{m}^+K = m^+K$$

which hold for each finite n_0. □

Theorem 1.15. *The class $Mo(\mathscr{K}, \sigma)$ is a complete normal Riesz space.*

Proof. It follows from lemmas 1.16 and 1.17. □

Lemma 1.16. *If $0 \leqslant |\bar{m}| \leqslant |m|$ and if m is σ-additive wrt $(\mathscr{K})$, then $\bar{m}$ is σ-additive wrt $\mathscr{K}$.*

Proof. As in the proof of lemma 1.11 we can assume that $0 \leqslant \bar{m} \leqslant m$. From the proof of lemma 1.14 we know that if $K = \bigcup_{i=1}^{\infty} \acute{K}_i - \acute{\acute{K}}_i$, where the sets $\acute{K}_i - \acute{\acute{K}}_i$ are disjoint, $\acute{K}_i \supset \acute{\acute{K}}_i$ and $\acute{K}_i$, $\acute{\acute{K}}_i \in \mathscr{K}$ for all $i = 1, 2, \ldots$ then $\bar{m}K \geqslant$ $\geqslant \sum_{i=1}^{\infty} (\bar{m}\acute{K}_i - \bar{m}\acute{\acute{K}}_i)$. Hence, using the σ-additivity of m we can derive that

$$0 \leqslant \bar{m}K - \sum_{i=1}^{\infty} (\bar{m}\acute{K}_i - \bar{m}\acute{\acute{K}}_i) = (\bar{m} - m) K - \sum_{i=1}^{\infty} [(\bar{m} - m) \acute{K}_i - (\bar{m} - m) \acute{\acute{K}}_i] \leqslant 0,$$

thus the equality is true and $\bar{m}$ is σ-additive. □

Lemma 1.17. *If $E \subset Mo(\mathscr{K}, \sigma)$ is bounded above (below) by $\bar{m} \in Mo(\mathscr{K}, \sigma)$, then $\vee_E m$ ($\wedge_E m$) is in $Mo(\mathscr{K}, \sigma)$.*

Proof. We can follow the proof of lemma 1.10. Let E_0 be created like in the proof of 1.10. If $\acute{E}_0 \subset E_0$ is completely ordered, then $\vee_{\acute{E}_0} mK = \sup_{m \in \acute{E}_0} mK$ for each $K \in \mathscr{K}$.

Let $\bar{m} = \vee_{\acute{E}_0} m$. We can and do assume that $\bar{m} \geqslant 0$. Let $K = \bigcup_{n=1}^{\infty} \acute{K}_n - \acute{\acute{K}}_n$ be a representation of K by disjoint sets $\acute{K}_n - \acute{\acute{K}}_n$, where $\acute{K}_n \supset \acute{\acute{K}}_n$ and $\acute{K}_n$, $\acute{\acute{K}}_n \in \mathscr{K}$ for all $n = 1, 2, \ldots$. Like above we can prove that $\bar{m}K \geqslant \sum_{n=1}^{\infty} (\bar{m}\acute{K}_n - \bar{m}\acute{\acute{K}}_n)$. On the other hand for each $m \in \acute{E}_0$

$$\sum_{n-1}^{\infty} (\bar{m}\acute{K}_n - \bar{m}\acute{\acute{K}}_n) = \sum_{n=1}^{\infty} ((\bar{m} - m) \acute{K}_n - (\bar{m} - m) \acute{\acute{K}}_n) + mK \geqslant mK.$$

Hence, $\sum_{n=1}^{\infty} (\bar{m}\acute{K}_n - \bar{m}\acute{\acute{K}}_n) \geqslant \bar{m}K$ and the equality holds.

The rest of the proof resembles that of 1.10. □

The set function $p \in Mo(\mathscr{K})$ is said to be *purely additive* if $|p| \wedge |m| = 0$ for all $m \in Mo(\mathscr{K}, \sigma)$. Denoting the class of all purely additive set functions by $Mo(\mathscr{K}, p)$ we can state

Theorem 1.17. *The spaces $Mo(\mathscr{K}, \sigma)$ and $Mo(\mathscr{K}, p)$ form a direct decomposition of $Mo(\mathscr{K})$. Moreover, each $m \in Mo(\mathscr{K})$ may be written in the fashion $m = \bar{m} + p$, where $\bar{m} \in Mo(\mathscr{K}, \sigma)$ and $p \in Mo(\mathscr{K}, p)$.*

Proof. See [4; theorem 3]. □

Theorem 1.18. *The class $Mo(\mathscr{K}, \tau)$ is a subspace of the Riesz space $Mo(\mathscr{K})$.*

Proof. The definition of τ-smooth functions implies that $Mo(\mathscr{K}, \tau)$ is a linear subspace of $Mo(\mathscr{K})$. As to the ordering see lemma 1.19. □

Lemma 1.19. *The following conditions are equivalent*: $|m| \in Mo(\mathscr{K}, \tau)$, $m \in Mo(\mathscr{K}, \tau)$, m^+ *and* m^- *are in* $Mo(\mathscr{K}, \tau)$.

Proof. Let $\mathscr{K}_0 \subset \mathscr{K}$ filter downwards to $K_0 \in \mathscr{K}$. If $|m| \in Mo(\mathscr{K}, \tau)$ then, by the monotony of $|m|$ and the definition of τ-smoothness, to each $\varepsilon > 0$ there is $\bar{K} \in \mathscr{K}_0$ such that $|m|\bar{K} - |m|K_0 < \varepsilon$. Since

$$|m|\bar{K} - |m|K_0 = \sup \sum_{i-0}^{n} |mK_i - mK_{i-1}| < \varepsilon,$$

where the supremum is taken over all strings $\{K_i\} \subset \mathscr{K}$ with the property $K_0 \subset K_1 \subset \ldots \subset K_n = \bar{K}$, we can conclude that m is τ-smooth.

Conversely, if $m \in Mo(\mathscr{K}, \tau)$ and if $\varepsilon > 0$ is a given number, the definition of τ-smoothness requires the existence of a set $K \in \mathscr{K}_0$ with the property $\sum_{i=0}^{n} |mK_i - mK_{i-1}| < \varepsilon$ for each string $\{K_i\} \subset \mathscr{K}$, where $K_0 \subset K_1 \subset \ldots \subset K_n \subset K$. Thus

$$|m|K - |m|K_0 = \sup \sum_{i-0}^{n} |mK_i - mK_{i-1}| < \varepsilon.$$

The τ-smoothness of $|m|$ is now a consequence of its monotony.

The rest of the proof is a simple consequence of the monotony of m^+, m^- and $|m|$. □

Theorem 1.20. *The space* $Mo(\mathscr{K}, \tau)$ *is a complete normal Riesz space.*

Proof. It is divided into lemma 1.21 and lemma 1.22. □

Lemma 1.21. *If* m *is a* τ*-smooth MSF on* $\mathscr{K}$, *then each MSF* $\bar{m}$ *on* $\mathscr{K}$ *with the property* $0 \leqslant |\bar{m}| \leqslant |m|$ *is* τ*-smooth.*

Proof. It follows from the lemma 1.19 and from the monotony of $|m|$ and $|\bar{m}|$ because if $0 \leqslant |\bar{m}| \leqslant |m|$, then

$$0 \leqslant |m|K - |\bar{m}|K \leqslant |m|K_0 - |\bar{m}|K_0 + \varepsilon$$

whenever $K, K_0 \in \mathscr{K}$, $K \supset K_0$ and $|m|K < |m|K_0 + \varepsilon$. Thus $|m| - |\bar{m}| \in Mo(\mathscr{K}, \tau)$ and $|\bar{m}|$ must be in $Mo(\mathscr{K}, \tau)$ as well. □

Lemma 1.22. *The space* $Mo(\mathscr{K}, \tau)$ *is complete.*

Proof. The same method like in 1.10 can be used. We restrict our attention to the proof of the relation $\vee_{\acute{E}_0} m \in Mo(\mathscr{K}, \tau)$ if $\acute{E}_0 \subset Mo(\mathscr{K}, \tau)$ is completely ordered.

Let $\mathscr{K}_0 \subset \mathscr{K}$ filter down to $K_0 \in \mathscr{K}$ and put $\bar{m} = \vee_{\acute{E}_0} m$. Let $\varepsilon > 0$ and be fixed. Because $\bar{m}K = \sup_{m \in \acute{E}} mK$ for each $K \in \mathscr{K}$, to an arbitrary fixed $\bar{K} \in \mathscr{K}_0$ we can find $m \in \acute{E}_0$ such that $(\bar{m} - m)\bar{K} < \varepsilon$. Since m is τ-smooth, there is $K \in \mathscr{K}_0$ for which $\sum_{i-1}^{n} |mK_i - mK_{i-1}| < \varepsilon$ whenever $K_0 \subset K_1 \subset \ldots \subset K_n \subset K$ is a string in $\mathscr{K}$. We can assume that $K \subset \bar{K}$. Now, keeping in mind that $\bar{m} - m$ is a monotone function, we obtain

$$\sum_{i=1}^{n} |\bar{m}K_i - \bar{m}K_{i-1}| \leqslant (\bar{m} - m)K - (\bar{m} - m)K_0 + \sum_{i=1}^{n} |mK_i - mK_{i-1}| < 2\varepsilon$$

whenever $K_0 \subset K_1 \subset \ldots \subset K$ is a string from $\mathscr{K}$. The last relation holds for each such string, hence, $m \in Mo(\mathscr{K}, \tau)$. □

Let $Mo(\mathscr{K}, d)$ be the orthogonal complement of $Mo(\mathscr{K}, \tau)$ in $Mo(\mathscr{K})$, i.e. let $Mo(\mathscr{K}, d)$ consists of all *MSF*'s in $Mo(\mathscr{K})$ which are disjoint with each *MSF* in $Mo(\mathscr{K}, \tau)$. Then,

Theorem 1.23. *The spaces $Mo(\mathscr{K}, \tau)$ and $Mo(\mathscr{K}, d)$ form a direct decomposition of $Mo(\mathscr{K})$ and each $m \in Mo(\mathscr{K})$ can be written in the fashion $m = \bar{m} + d$, where $\bar{m} \in Mo(\mathscr{K}, \tau)$ and $d \in Mo(\mathscr{K}, d)$.*

Proof. See [4; theorem 3]. □

§2. Construction and Extension of Measures

Let $\mathscr{K} \subset \exp X$ be a set lattice and $Mc(\mathscr{K})$ be the system of all finite real-valued measures with finite variation on $\mathscr{E}(\mathscr{K})$, where $\mathscr{E}(\mathscr{K})$ is the ring generated by $\mathscr{K}$. Here Mc is the abbreviation of measure-content. The symbol M is reserved for spaces of measures with range of definition on a σ-ring.

Next, $Mc(\mathscr{K}, r)$ will consist of all $\mathscr{K}$-regular measures from $Mc(\mathscr{K})$. The class of all σ-additive (*wrt* $\mathscr{E}(\mathscr{K})$) measures is denoted by $Mc(\mathscr{K}, \sigma)$. The members of $Mc(\mathscr{K}, \sigma)$ are often called premeasures. By $Mc(\mathscr{K}, \tau)$ we denote the class of all measures in $Mc(\mathscr{K})$ which are τ-smooth *wrt* $\mathscr{K}$.

Since each measure is a *MSF*, we can use the results from §1. and change $Mc(\mathscr{K})$ into a Riesz space and $Mc(\mathscr{K}, \sigma)$ with $Mc(\mathscr{K}, \tau)$ into its normal subspaces.

To prove that $Mc(\mathscr{K}, r)$ is a complete normal subspace we use the fact that the least upper bound and greatest lower bound for $m \in Mc(\mathscr{K})$ and zero measure 0 agree with the functions

$$m^+E = \sup\{m\acute{E}: \acute{E} \subset E, \acute{E} \in \mathscr{E}(\mathscr{K})\}, \quad m^-E = -\inf\{m\acute{E}: \acute{E} \subset E, \acute{E} \in \mathscr{E}(\mathscr{K})\},$$

respectively. In order to verify it, it suffices to show that m^+ is an additive set function dominating m and 0 which is less or equal to each measure in $Mc(\mathscr{K})$ dominating m and 0. Note that the assumption "$\mathscr{E}(\mathscr{K})$ is a ring" is here substantial.

Theorem 2.1. *The class $Mc(\mathscr{K}, r)$ is a complete normal subspace of $Mc(\mathscr{K})$.*

Proof. Due to the definition of regularity *wrt* $\mathscr{K}$ $Mc(\mathscr{K}, r)$ is a linear space. The just mentioned representation of m^+ makes it possible to prove that if $m \in Mc(\mathscr{K}, r)$, then $m^+ \in Mc(\mathscr{K}, r)$. Hence $Mc(\mathscr{K}, r)$ is a Riesz space. If $0 \leqslant \bar{m} \leqslant m$ and if $m \in Mc(\mathscr{K}, r)$, then $m - \bar{m} \in Mc(\mathscr{K}, r)$ and, consequently, $\bar{m} \in Mc(\mathscr{K}, r)$. Hence, $M(\mathscr{K}, r)$ is complete. The proof of normality resembles that of lemma 1.10. □

Theorem 2.2. *The following couples of Riesz spaces are isomorphic:*

$$\begin{array}{lll} Mo(\mathscr{K}) & \textit{and} & Mc(\mathscr{K}) \\ Mo(\mathscr{K}, t) & \textit{and} & Mc(\mathscr{K}, r) \\ Mo(\mathscr{K}, \sigma) & \textit{and} & Mc(\mathscr{K}, \sigma) \\ Mo(\mathscr{K}, \tau) & \textit{and} & Mc(\mathscr{K}, \tau). \end{array}$$

Proof. Let "$-$" denote the operator of extension, i.e. the map which to each *MSF* $m \in Mo(\mathscr{K})$ adheres its extension $\bar{m} \in Mc(\mathscr{K})$. By [12; theorem 1.2] each *MSF* m on $\mathscr{K}$ has a unique extension from $\mathscr{K}$ to a measure $\bar{m}$ on $\mathscr{E}(\mathscr{K})$. Consequently, each measure $\bar{m} \in Mc(\mathscr{K})$ restricted to $\mathscr{K}$ determines just one *MSF* $m \in Mo(\mathscr{K})$. Hence the operator of extension defines a one to one map between $Mo(\mathscr{K})$ and $Mc(\mathscr{K})$.

Now we show that if $m_1, m_2 \in Mo(\mathscr{K})$, then $m_1 \leqslant m_2$ if and only if $\bar{m}_1 \leqslant \bar{m}_2$. Let $m_1 \leqslant m_2$. Then $m_2 - m_1 \geqslant 0$ and $m_2 - m_1$ has a unique extension to a monotone measure $\overline{m_2 - m_1} \geqslant 0$, see [12; theorem 1.2]. Since $\overline{m_2 - m_1} + \bar{m}_1$ is an extension of m_2 from $\mathscr{K}$ to $\mathscr{E}(\mathscr{K})$ and since such extension is unique, $\overline{m_2 - m_1} + \bar{m}_1 = \bar{m}_2$. But this means that $\bar{m}_1 \leqslant \bar{m}_2$ because $\overline{m_2 - m_1} \geqslant 0$.

Proof of the reverse implication is obvious, hence, $Mo(\mathscr{K})$ and $Mc(\mathscr{K})$ are isomorphic.

The rest of the proof follows from theorem 2.3. □

Theorem 2.3. *Let m be a MSF on $\mathscr{K}$ and let $\bar{m}$ be the extension of m to $\mathscr{E}(\mathscr{K})$. Then $\bar{m}$ is regular wrt $\mathscr{K}$ if and only if m is tight wrt $\mathscr{K}$, $\bar{m}$ is σ-additive (wrt $\mathscr{E}(\mathscr{K})$) if and only if m is σ-additive, $\bar{m}$ is τ-smooth wrt $\mathscr{K}$ if and only if m is so.*

Proof. To prove the first implication we must recall the construction of $\bar{m}$ extending m from $\mathscr{K}$ to $\mathscr{E}(\mathscr{K})$. It is known (see Halmos [6, pages 25 26]) that to an arbitrary set $E \in \mathscr{E}(\mathscr{K})$ we can find two finite sequences of sets with the properties $K_i \supset \acute{K}_i$ and $K_i, \acute{K}_i \in \mathscr{K}$ for $i = 1, \ldots, n$, $K_i - \acute{K}_i \cap K_j - \acute{K}_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} K_i - \acute{K}_i = E$. Consider $m \in Mo(\mathscr{K})$ and define the value $\bar{m}E$ by the relation $\bar{m}E = \sum_{i=1}^{n} (mK_i - m\acute{K}_i)$. The definition of $\bar{m}E$ can be proved as independent of the representation of E by the sets from $\mathscr{K}$. The additivity of $\bar{m}$ and other details are proved in [12].

From the definition of $\bar{m}$ it can be easily derived that if $m \in Mo(\mathscr{K}, t)$, then $\bar{m} \in Mc(\mathscr{K}, r)$.

If $\bar{m} \in Mc(\mathscr{K}, \sigma)$, then the σ-additivity property *wrt* $\mathscr{E}(\mathscr{K})$ is equivalent with the condition: if $E \in \mathscr{E}(\mathscr{K})$ and if $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n \in \mathscr{E}(\mathscr{K})$ for all $n = 1, 2, \ldots$ and they are pairwisse disjoint, then $\bar{m}E = \sum_{n=1}^{\infty} \bar{m}E_n$. Especially if $E = K_1 - K_2$, where $K_1 \supset K_2$ and $K_1, K_2 \in \mathscr{K}$, then the same condition can be rewritten as $\bar{m}K_1 = \sum_{n=0}^{\infty} \bar{m}E_n$, where $E_0 = K_2$. Taking into account the construction of $\bar{m}$ we can see that $\bar{m}K_1 = \sum_{n=0}^{\infty} \bar{m}E_n$ if and only if m, the restriction of $\bar{m}$ from $\mathscr{E}(\mathscr{K})$ to $\mathscr{K}$, is σ-additive. Since each $E \in \mathscr{E}(\mathscr{K})$ has a representation $E = \bigcup_{i\ 1}^{n} K_i - \acute{K}_i$, where $K_i - \acute{K}_i$ are disjoint, $K_i \supset \acute{K}_i$ and $K_i, \acute{K}_i \in \mathscr{K}$, the second part of the assertion is proved.

The relation $m \in Mo(\mathscr{K}, \tau)$ if and only if $\bar{m} \in Mc(\mathscr{K}, \tau)$ holds trivially. □

Let $M(\mathscr{K})$ be the space of all measures on $\mathscr{E}_\sigma(\mathscr{K})$ with finite variation *wrt* $\mathscr{E}_\sigma(\mathscr{K})$. The class of $\mathscr{K}$-regular measures in $M(\mathscr{K})$ is denoted by $M(\mathscr{K}, r)$. $M(\mathscr{K}, \sigma)$ denotes the class of all σ-additive (*wrt* $\mathscr{E}_\sigma(\mathscr{K})$) measures in $M(\mathscr{K})$ and $M(\mathscr{K}, \tau)$ is the class of all measures in $M(\mathscr{K})$ τ-smooth *wrt* $\mathscr{K}$. As in the case of the measure-contents, all this spaces may be considered as Riesz spaces.

Theorem 2.4. *There is a Riesz isomorphism between the spaces*

$$Mo(\mathscr{K}, \sigma) \quad \textit{and} \quad M(\mathscr{K}, \sigma).$$

If each countable decreasing sequence of sets from $\mathscr{K}$ has the intersection in $\mathscr{K}$, then there is a Riesz isomorphism between

$$Mo(\mathscr{K}, \sigma) \cap Mo(\mathscr{K}, t) \quad \textit{and} \quad M(\mathscr{K}, \sigma) \cap M(\mathscr{K}, r)$$
$$Mo(\mathscr{K}, \tau) \cap Mo(\mathscr{K}, t) \quad \textit{and} \quad M(\mathscr{K}, \tau) \cap M(\mathscr{K}, r).$$

Proof. By theorem 2.5 each $m \in Mo(\mathscr{K}, \sigma)$ has a unique extension from $\mathscr{K}$ to $\mathscr{E}_\sigma(\mathscr{K})$. Consequently, each $\bar{m} \in M(\mathscr{K}, \sigma)$ determines just one *MSF* in $Mo(\mathscr{K}, \sigma)$. The equivalence $m_1 \leqslant m_2$ if and only if $\bar{m}_1 \leqslant \bar{m}_2$, where $m_1, m_2 \in Mo(\mathscr{K}, \sigma)$ and $\bar{m}_1, \bar{m}_2 \in M(\mathscr{K}, \sigma)$ are the extensions of m_1, m_2, can be proved in the same manner as in theorem 2.2. This proves the isomorphism of $Mo(\mathscr{K}, \sigma)$ and $M(\mathscr{K}, \sigma)$.

The second and third part of the theorem follows now from theorem 2.5 and corollary 2.6, respectively. □

Theorem 2.5. *If m is a monotone MSF on $\mathscr{K}$, then m has a σ-additive extension from $\mathscr{K}$ to $\mathscr{E}_\sigma(\mathscr{K})$ if and only if it is σ-additive wrt $\mathscr{K}$. If in addition $\mathscr{K}$ is closed under countable intersections, then the extension is regular wrt $\mathscr{K}$ when m is tight.*

Proof. If m is a monotone MSF on $\mathscr{K}$, then, by theorem 2.3, it has a unique σ-additive extension to $\mathscr{E}(\mathscr{K})$. Now the well-known Carathéodory's extension theorem states that $\bar{m}$ has a unique σ-additive extension from $\mathscr{E}(\mathscr{K})$ to $\mathscr{E}_\sigma(\mathscr{K})$ (see [11; §16]). Since if $\bar{m}$ is σ-additive, the restriction of $\bar{m}$ to must be also σ-additive, the first part of the theorem is proved.

Let $\mathscr{K}$ be closed under formation of countable intersections and let $\mathscr{E}$ be the system of all $E \in \mathscr{E}_\sigma(\mathscr{K})$ with the property $\bar{m}E = \sup\{mK\colon K \subset E,\ K \in \mathscr{K}\}$. We are going to show that $\mathscr{E}$ is a monotone class.

Let us consider a sequence $\{E_n\} \subset \mathscr{E}$ filtering upwards to $E \in \mathscr{E}_\sigma(\mathscr{K})$. As $\bar{m}$ is σ-additive, $\bar{m}E = \lim \bar{m}E_n$. Since the sequence $\{\bar{m}E_n\}$ is increasing and $\bar{m}E = \sup\{\bar{m}K\colon K \subset E_n,\ K \in \mathscr{K}\}$, $\bar{m}E = \sup\{\bar{m}K\colon K \subset E,\ K \in \mathscr{K}\}$. Hence, $E \in \mathscr{E}$.

Let $\{E_n\} \subset \mathscr{E}$ filter downwards to $E \in \mathscr{E}_\sigma(\mathscr{K})$. We can assume $\bar{m}E_1$ finite. If $\varepsilon > 0$, then, since $\{E_n\} \subset \mathscr{E}$, there is a sequence $\{K_n\} \subset \mathscr{K}$ such that $K_n \subset E$ and $\bar{m}E_n < \bar{m}K_n + \varepsilon 2^{-n}$ for all $n = 1, 2, \dots$. By means of the mathematical induction it can be shown that

$$\bar{m}E_n < \bar{m}\bigcap_{i=1}^{n} K_i + \varepsilon \sum_{i=1}^{n} 2^{-i} \quad n = 1, 2, \dots .$$

As a consequence of σ-additivity $\lim \bar{m}K_n = \bar{m}K$, where $K = \bigcap_{n=1} K_n$. Hence, $\bar{m}E < \bar{m}K + \varepsilon$ and $K \subset E$ is in $\mathscr{K}$ by the assumption. Since ε was arbitrary, $E \in \mathscr{E}$. By theorem 2.3 $\mathscr{E}(\mathscr{K}) \subset \mathscr{E}$, thus the proof follows from the theorem about monotone classes. □

A similar result was obtained for compact lattices in [9; theorem 1.11] and for arbitrary lattices in [16; theorem 1].

Corollary 2.6. *If $\mathscr{K}$ is closed under countable intersections, then each tight and τ-smooth monotone set function has a unique extension to a regular measure on $\mathscr{E}_\sigma(\mathscr{K})$ which is τ-smooth wrt $\mathscr{K}$.*

Proof. It suffices to show that if $\{E_n\} \subset \mathscr{E}(\mathscr{K})$ filters downwards to 0 and if $\bar{m}E_1 < \infty$, then τ-smoothness of $\bar{m}$ guarantees that $\lim \bar{m}E_n = \bar{m}0 = 0$, where $\bar{m}$ is the $\mathscr{K}$-regular extension of m from $\mathscr{K}$ to $\mathscr{E}(\mathscr{K})$. This condition is namely equivalent to the σ-additivity of $\bar{m}$ wrt $\mathscr{E}(\mathscr{K})$ and it makes theorem 2.5 applicable. But it suffices to follow the second part of the proof of theorem 2.5. □

Remark 2.7. If $\mathscr{K}$ is not closed under countable intersections, we cannot in 2.6 guarantee the regularity of the extension although the extension itself will always exist.

Remark 2.8. Throughout this paper we deal only with bounded *MSF*'s. However, the definitions of modularity, regularity, tightness, σ-additivity and τ-smoothness are sensefull also for unbounded set functions with values in $\mathbf{R}$. For such *MSF*'s theorems 2.3 and 2.5 as well as corollary 2.6 stay to be true.

The assumption of finite variation *wrt* $\mathcal{K}$ can be replaced by the assumption of locally finite variation. A (not necessarily bounded) *MSF* m on $\mathcal{K}$ has a locally finite variation (*wrt* $\mathcal{K}$) if to each $K \in \mathcal{K}$ there is a constant c_K such that $\sum_{i=1}^{n} |mK_i - mK_{i-1}| < c_K$ for each string $\emptyset = K_0 \subset K_1 \subset \ldots \subset K_n = K$ consisting of sets from $\mathcal{K}$. Now m^+ and m^- can be defined in the usual way and the foregoing assertions stay to be true.

Example 3. Let $X = \mathbf{R}$ and $\mathcal{K}(X) = \left\{ \bigcup_{i=1}^{n} [a_i, b_i] : [a_i, b_i] \right\} \subset -(\infty, 0) \cup \cup (0, \infty)$, where $[a_i, b_i]$ are non-overlapping, $n \in \mathbf{N}$. If $F\colon \mathbf{R} \to \mathbf{R}$ has finite variation on each $[a, b]$ not containing zero, then m, defined as in example 1, has locally finite variation *wrt* $\mathcal{K}(X)$. E.g. $F(x) = x \cos \frac{1}{x}$ or $F(x) = \ln |x|$ for $x \neq 0$ with $F(0) = 0$ define *MSF*'s with locally finite variations *wrt* $\mathcal{K}(X)$.

Theorem 2.9. *Let* $\mathcal{F} \subset \exp X$ *be a lattice containing the lattice* $\mathcal{K}$. *Then*

a) $Mo(\mathcal{K})$ *can be imbedded into* $Mo(\mathcal{F})$;

b) $Mo(\mathcal{K}, t)$ *can be imbedded into* $Mo(\mathcal{F}, t)$;

c) *if for each* $K \in \mathcal{K}$ *and* $F \in \mathcal{F}$ $KF \in \mathcal{K}$, *then* $Mo(\mathcal{K}, t)$ *and* $Mo(\mathcal{F}, t)$ *are isomorphic*;

d) *if each* $F \in \mathcal{F}$ *is an intersection of a countable decreasing sequence of sets from* $\mathcal{K}$, *then* $Mo(\mathcal{K}, \sigma) \cap Mo(\mathcal{K}, t)$ *and* $Mo(\mathcal{F}, \sigma) \cap Mo(\mathcal{F}, t)$ *are isomorphic*;

e) *if to each* $F \in \mathcal{F}$ *there is a decreasing net* $\mathcal{K}_0 \subset \mathcal{K}$ *such that* $F = \bigcap_{K \in \mathcal{K}_0} K$, *then* $Mo(\mathcal{K}, \tau) \cap Mo(\mathcal{K}, t)$ *and* $Mo(\mathcal{F}, \tau) \cap Mo(\mathcal{F}, t)$ *are isomorphic.*

Proof. To prove 2.9 a), b) means to show that if $m \in Mo(\mathcal{K})$ ($m \in Mo(\mathcal{K}, t)$), then m has at least one extension to $\bar{m} \in Mo(\mathcal{F})$ ($m \in Mo(\mathcal{F}, t)$) and we can achieve that if $m_1, m_2 \in Mo(\mathcal{K})$ have the property $m_1 \leqslant m_2$, then their extensions can be chosen in such a way that $\bar{m}_1 \leqslant \bar{m}_2$. In the cases c) — e) we must moreover show the unicity of such an extension.

The proof can be performed by the transfinite induction. Lemma 2.10 plays the key role in this process. The details are omitted. □

Lemma 2.10. *Let* $\mathcal{K} \subset \exp X$ *be a set lattice and let* $F \subset X$ *be not contained in* $\mathcal{K}$. *If* $\mathcal{F}$ *is the smallest lattice containing* F *with* $\mathcal{K}$ *and if* $m \in Mo(\mathcal{K})$, *then*

a_0) m *can be extend to a MSF* $\bar{m}_0$ *on* $\mathcal{F}$.

If in addition m is tight wrt $\mathscr{K}$, then

b$_0$) *$\bar{m}_0$ can be constructed tight wrt $\mathscr{F}$;*

c$_0$) *if $KF \in \mathscr{K}$ for each $K \in \mathscr{K}$, then $\bar{m}_0$ is tight wrt $\mathscr{K}$;*

d$_0$) *if there is a decreasing sequence $\{K_n\} \subset \mathscr{K}$ such that $F = \bigcap_{n=1}^{\infty} K_n$ and if m is σ-additive wrt $\mathscr{K}$, then $\bar{m}_0$ is tight and σ-additive wrt $\mathscr{F}$;*

e$_0$) *if there is a decreasing net $\mathscr{K}_0 \subset \mathscr{K}$ such that $F = \bigcap_{K \in \mathscr{K}_0} K$ and if m is τ-smooth wrt $\mathscr{K}$, then $\bar{m}_0$ is tight and τ-smooth wrt $\mathscr{F}$.*

Proof. Let $m \in Mo(\mathscr{K})$ be monotone. We assume that $X \in \mathscr{K}$. Otherwise we can replace $\mathscr{K}$ by $\mathscr{K} \cup \{X\}$ and m by the MSF extended to X by $mX = \sup\{mK : K \in \mathscr{K}\}$. Such a definition guarantees the tightness of the extension if m is tight as well as τ-smoothness and σ-additivity if m has these properties.

By $\bar{m}$ we denote the extension of m from $\mathscr{K}$ to a measure on $\mathscr{E}(\mathscr{K})$. By theorem 2.3 if $m \in Mo(\mathscr{K}, t)$ $(Mo(\mathscr{K}, \sigma), Mo(\mathscr{K}, \tau))$, then $\bar{m} \in Mc(\mathscr{K}, r)$ $(Mc(\mathscr{K}, \sigma), Mc(\mathscr{K}, \tau))$. Since each $E \in \mathscr{E}(\mathscr{F})$ can be represented as $E_1 F \cup E_2 - F$ for some $E_1, E_2 \in \mathscr{E}(\mathscr{K})$ the relations

i) $\bar{m}_0 E_2 - F = \sup\{\bar{m}\acute{E} : \acute{E} \subset E_2 - F,\ \acute{E} \in \mathscr{E}(\mathscr{K})\}$	for all	$E_2 \in \mathscr{E}(\mathscr{K})$
ii) $\bar{m}_0 E_1 F = \inf\{\bar{m}\acute{E} : \acute{E} \supset E_1 F,\ \acute{E} \in \mathscr{E}(\mathscr{K})\}$	for all	$E_1 \in \mathscr{E}(\mathscr{K})$
iii) $\bar{m}_0 E_1 F \cup E_2 - F = \bar{m}_0 E_1 F + \bar{m}_0 E_2 - F$	for all	$E_1, E_2 \in \mathscr{E}(\mathscr{K})$

define a measure $\bar{m}_0 \in Mc(\mathscr{F})$ (see [3]). Now $\bar{m}_0$ restricted to $\mathscr{F}$ defines the desired extension of m.

If in addition $m \in Mo(\mathscr{K}, t)$, then, since $\bar{m}_0 E_2 - F = \sup\{\bar{m}\acute{E} : \acute{E} \subset E_2 - F,\ \acute{E} \in \mathscr{E}(\mathscr{K})\}$ and $\bar{m}\acute{E} = \sup\{\bar{m}K : K \subset E_2 - F,\ K \in \mathscr{K}\}$ for each $\acute{E} \in \mathscr{E}(\mathscr{K})$ $\bar{m}_0 E_2 - F = \sup\{\bar{m}K : K \subset E_2 - F,\ K \in \mathscr{K}\}$. Similarly $\bar{m}_0 E_1 F = \sup\{\bar{m}_0 KF : KF \subset E_1 F, K \in \mathscr{K}\}$, because $\bar{m}_0 E_1 = \sup\{\bar{m}_0 K : K \subset E_1, K \in \mathscr{K}\}$ and $\bar{m}_0$ is monotone and additive. As

$$\sup \bar{m}_0 F \leqslant \bar{m}_0 E = \bar{m}_0 E_1 F + \bar{m}_0 E_2 - F \leqslant \sup \bar{m}_0 F,$$

where the supremum is taken over all $F \subset E$, $F \in \mathscr{F}$, we can conclude that $\bar{m}_0 \in Mc(\mathscr{F}, r)$. $\bar{m}_0$ restricted to $\mathscr{F}$ is a MSF from $Mo(\mathscr{F}, t)$.

Under the assumptions of c$_0$), the point ii) of the definition of $\bar{m}_0$ yields $\bar{m}_0 KF = \bar{m}KF$. Hence, following the foregoing decisions, $\bar{m}_0 E = \sup\{\bar{m}K : K \subset E, K \in \mathscr{K}\}$. Since $\bar{m}K = mK$ for $K \in \mathscr{K}$, $\bar{m}_0$ and the restriction of $\bar{m}_0$ to $\mathscr{F}$ are regular *wrt* $\mathscr{K}$.

As to the proof of d$_0$) and e$_0$) see [16; §5].

Since each $m \in Mo(\mathscr{K})$ can be written as $m = m^+ - m^-$, where $m^+, m^- \in Mo(\mathscr{K})$ are monotone and tight, σ-additive or τ-smooth if m has these properties (see §1) the theorem is proved. □

Remark 2.11. *Let $\mathscr{K} \subset \mathscr{F} \subset \exp X$ be set lattices with the property $KF \in \mathscr{K}$ for* *all* $K\in\mathscr{K}$ *and* $F\in\mathscr{F}$. *If* $X\in\mathscr{F}$ *and if* $m\in Mc(\mathscr{F}, r)$, *then* $m\in Mc(\mathscr{K}, r)$ *if and only if* $mX = \sup\{mK\colon K\in\mathscr{K}\}$.

Proof. We assume $m \geqslant 0$. Let $\varepsilon > 0$ and $mX < mK + \varepsilon$. Then for each $F\in\mathscr{F}$ $mF = mFK + mF - K \leqslant mFK + \varepsilon$ and since $FK\in\mathscr{K}$, tending with ε to zero, we obtain that $mF = \sup\{mK\colon K\subset F,\ K\in\mathscr{K}\}$. The rest of the proof is obvious. □

In order to demonstrate an application of the foregoing methods we give an outline of the proof of the known Kolmogoroff consistency theorem. The idea is taken from [9]. For generalization of this theorem see [9, 13, 18].

Example 4. Let $X = C[0, \infty)$ and I be the set of all finite subsets of $[0, \infty)$ directed by inclusion. For each $i\in I$ of the form $(t_1, \ldots, t_k)$ let $X_i = \mathbf{R}^k$. X is considered with the topology of uniform convergence on compact sets, X_i are provided by the Euclidean topology, $\mathscr{B}(X)$ and $\mathscr{B}(X_i)$ are the Borel σ-algebras generated by these topologies. On each $\mathscr{B}(X_i)$ let us consider a probability measure p_i connected with p_j for $i \leqslant j$ by the relation $p_i E = p_j \pi_{ij}^{-1} E$ for all $E\in\mathscr{B}(X_i)$. Here π_{ij} is the projection of X_j onto X_i. If π_i is the projection of X onto X_i, then a measure p on $\mathscr{B}(X)$, such that $p_i E = p_i \pi_i^{-1} E$ for each $E\in\mathscr{B}(X_i)$ and $i\in I$, exists if and only if

$$* \qquad \underset{\varepsilon>0}{\forall}\ \underset{K\subset X,\ K \text{ compact}}{\exists}\ \underset{i\in I}{\forall}\ p_i X_i - \pi_i K < \varepsilon.$$

Proof. X is a complete separable metric space. Hence, each σ-additive measure on $\mathscr{B}(X)$ is a Radon measure (see [11, theorem 19.18]) and this implies $*$ if p, for which $p_i E = p_i \pi_i^{-1} E$ for all $E\in\mathscr{B}(X_i)$ and $i\in I$, exists.

Conversely, X_i are σ-compact and their open subsets are F_σ, thus p_i are Radon measures for all $i\in I$. If we put $\mathscr{E} = \{E\colon E = \pi_i^{-1} B$ for some $i\in I$ and $B\in\mathscr{B}(X_i)\}$ and if

$$pE = \lim_{i\in I} p_i \pi_i E \quad \text{for} \quad E\in\mathscr{E},$$

then p is a well-defined measure-content on $\mathscr{E}$ and, since π_i are closed maps and p_i are Radonian, p is regular *wrt* the set lattice $\mathscr{F}$ of all closed sets in $\mathscr{E}$. If $*$ holds, and if $\mathscr{F}_0 \subset \mathscr{E}$ are closed sets filtering downwards to $\emptyset$ then their complements filter upwards to X. Consequently, if $\varepsilon > 0$ and $p_i X_i - \pi_i K < \varepsilon$ for all $i\in I$, where K is compact, there is $F\in\mathscr{F}_0$ such that $K\subset F^c$ and $p_i \pi_i F < \varepsilon$ for all $i\in I$. Hence, $\inf_{F\in\mathscr{F}_0} pF = 0$, p is τ-smooth and can be extended to $\mathscr{B}(X)$ (corollary 2.6). □

It is hard to say which results from §2 are to be considered original and which only a paraphrase of the known ones. Lattice properties of measures were extensively studied in [4], less explicitly in [17, 19]. Important extension theorems are in [3] (here see other references) and in [7, 12]. The extension of monotone tight MSF's was widely studied in [15, 16], where there are, e.g., the second part of theorem 2.5 and corollary 2.6.

The ideas used in the proof of theorem 2.3 and lemma 2.10. I met (after finishing of the original paper) in [2], for extension theorems related to 2.9 and 2.10 see [2, 8, 10].

Besides the mentioned papers on which I build my work directly (see references) there is a number of others dealing with this subject (see references in [1, 2, 9, 14]).

§3. Measures on Topological Spaces

In the rest of the paper X, $\mathcal{G}(X)$ denotes a topological space. The class $\mathcal{G}(X)$ of all open sets determines the lattices $\mathcal{F}(X)$ and $\mathcal{K}(X)$ of all closed and closed compact subsets of X, respectively, as well as the classes $\mathcal{F}_0(X)$ and $\mathcal{K}_0(X)$ (see introduction). The algebra $\mathcal{B}_0(X)$ generated by $\mathcal{F}_0(X)$ is known as the algebra of Baire sets, the σ-algebra $\mathcal{B}(X)$ generated by $\mathcal{F}(X)$ is known as the Borel σ-algebra.

The system of all measures on $\mathcal{B}(X)$ ($\mathcal{B}_0(X)$) with finite variation will be denoted by $\mathcal{M}(X)$ ($\mathcal{M}_0(X)$). Of course $\mathcal{M}(X)$ ($\mathcal{M}_0(X)$) can be considered as a Riesz space (see §2). The classes of all τ-smooth, σ-additive, regular and tight measures from $\mathcal{M}(X)$ ($\mathcal{M}_0(X)$) are denoted by $\mathcal{M}(X, \tau)$ ($\mathcal{M}_0(X, \tau)$), $\mathcal{M}(X, \sigma)$ ($\mathcal{M}_0(X, \sigma)$), $\mathcal{M}(X, r)$ ($\mathcal{M}_0(X, r)$) and $\mathcal{M}(X, t)$ ($\mathcal{M}_0(X, t)$), respectively (for definitions see, e.g., [16, 17]).

The measures in $\mathcal{M}(X)$ are often called Borel measures while the members in $\mathcal{M}_0(X, r)$ are known as Baire measures.

Theorem 3.1. *There is a Riesz isomorphism between the couples of spaces*

$$\begin{array}{lll} Mo(\mathcal{F}_0(X)) & \textit{and} & \mathcal{M}_0(X) \\ Mo(\mathcal{F}_0(X), \sigma) & \textit{and} & \mathcal{M}_0(X, \sigma) \\ Mo(\mathcal{F}_0(X), \tau) & \textit{and} & \mathcal{M}_0(X, \tau) \\ Mo(\mathcal{F}_0(X), t) & \textit{and} & \mathcal{M}_0(X, r) \\ Mo(\mathcal{K}_0(X), t) & \textit{and} & \mathcal{M}_0(X, t). \end{array}$$

Proof. Theorem 3.1 is only a paraphrase of theorem 2.2. As to the last relation see theorem 2.9 c). □

As a consequence of theorem 3.1 and the well-known representation theorems in [17] we can say that there is an isomorphism between the class of all bounded continuous functionals on $C_b(X)$ (the space of all bounded continuous functions on X provided with the supremum norm) and the class $Mo(\mathcal{F}_0(X), t)$. In particular, spaces of all σ-smooth, τ-smooth and tight continuous functionals on $C_b(X)$ (see [17]) are isomorphic with the spaces of tight MSF's in $Mo(\mathscr{F}_0(X), \sigma)$, $Mo(\mathscr{F}_0(X), \tau)$ and $Mo(\mathscr{K}_0(X))$.

Theorem 3.2. *There is a Riesz isomorphism between the couples of spaces*

$Mo(\mathscr{F}(X), \sigma)$	*and*	$\mathscr{M}(X, \sigma)$
$Mo(\mathscr{F}(X), \sigma) \cap Mo(\mathscr{F}(X), t)$	*and*	$\mathscr{M}(X, \sigma) \cap \mathscr{M}(X, r)$
$Mo(\mathscr{F}(X), \tau) \cap Mo(\mathscr{F}(X), t)$	*and*	$\mathscr{M}(X, \tau) \cap \mathscr{M}(X, r)$
$Mo(\mathscr{F}(X, t))$	*and*	$\mathscr{M}(X, t)$

Proof. See theorem 2.4 and theorem 2.9 c). □

Theorem 3.3. *If X is a Hausdorff space and if to each $K \in \mathscr{K}(X)$ there is a class $\mathscr{K}_0 \subset \mathscr{K}_0(X)$ such that $\mathscr{K}_0$ filters downwards, to K, then*

$$Mo(\mathscr{K}_0(X), t) \quad \text{and} \quad \mathscr{M}(X, t)$$

are isomorphic. If X is completely regular, then

$$Mo(\mathscr{F}_0(X), \tau) \quad \text{and} \quad \mathscr{M}(X, \tau)$$

are isomorphic. Generally, $Mo(X)$ can be imbedded into $\mathscr{M}(X)$, $Mo(\mathscr{K}_0(X), t)$ into $\mathscr{M}(X, t)$ and $Mo(\mathscr{F}_0(X), r)$ into $\mathscr{M}(X, r)$.

Proof. See theorem 2.9. Note, that here τ-smoothness with complete regularity guarantees the tightness of MSF's in $Mo(\mathscr{F}_0(X), \tau)$, see [15, 16].

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