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ON CONDITIONAL EXPECTATIONS OF VECTOR VALUED VARIABLES

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As proved in [1], the property of convergence of conditional expectations holds also for vector valued variables and is used for proving the martingale convergence theorem for such functions. We shall give a proof of convergence of conditional expectations, which is based on commutation of a continuous linear operator and a linear operator of conditional expectation.

We shall assume that we are given a probability space $(\Omega, \mathcal{F}, P)$ and a separable Banach space X. We shall use notions of step, measurable and integrable functions defined in [3].

Lemma 1. *Let f: $\Omega \to X$ be an integrable function and $\mathcal{C} \subset \mathcal{F}$ be a σ-algebra. If Y is a separable Banach space and T: $X \to Y$ is a continuous linear operator, then* $E^{\mathcal{C}}(T(f)) = T(E^{\mathcal{C}}(f))$, a.e.

Proof. If a function g: $\Omega \to X$ is integrable, then $T(g)$ is also integrable and $\int T(g)\,dP = T(\int g\,dP)$. Since $T(\Theta) = \Theta'$, it is easy to see that for every set $C \in \mathcal{C}$ we have

$$\int_C E^{\mathcal{C}}(T(f))\,dP = \int_C T(f)\,dP = \int T(f\chi_C)\,dP =$$

$$= T\left(\int_C E^{\mathcal{C}}(f)\,dP\right) = \int_C T(E^{\mathcal{C}}(f))\,dP.$$

Since $E^{\mathcal{C}}(T(f))$, $T(E^{\mathcal{C}}(f))$ are $\mathcal{C}$ measurable, $E^{\mathcal{C}}(T(f)) = T(E^{\mathcal{C}}(f))$ a.e. by Lemma 3 in [2].

Let $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be an increasing sequence of σ-algebras, i.e. for every n the inclusion $\mathcal{C}_n \subset \mathcal{C}_{n+1} \subset \mathcal{F}$ is valid. If we denote the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ as $\bigvee_{n=1}^{\infty} \mathcal{C}_n$, the following lemma holds.

Lemma 2. *Let Y be an m-dimensional normed linear space and $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be an* *increasing sequence of σ-algebras. If $f\colon \Omega \to Y$ is integrable and $\mathscr{C} = \bigvee_{n=1}^{\infty} \mathscr{C}_n$, then* $\lim_{n\to\infty} \int \|E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f)\| \mathrm{d}P = 0$ *and* $E^{\mathscr{C}}(f) = \lim_{n\to\infty} E^{\mathscr{C}_n}(f)$ *a.e.*

Proof. (I) Let $Y = R^m$. If we denote $\Pi_j((y_1, \ldots, y_m)) = y_j$, then this linear functional is continuous and Lemma 1 implies that

$$\Pi_j(E^{\mathscr{C}}(f)) = E^{\mathscr{C}}(\Pi_j(f)), \quad \Pi_j(E^{\mathscr{C}_n}(f)) = E^{\mathscr{C}_n}(\Pi_j(f)), \quad \text{a.e.}$$

for $j = 1, \ldots,$ m and every n. According to theorems about integrable functions with real values the following equality

$$\lim_{n\to\infty} \Pi_j(E^{\mathscr{C}_n}(f)) = \lim_{n\to\infty} E^{\mathscr{C}_n}(\Pi_j(f)) = E^{\mathscr{C}}(\Pi_j(f)) = \Pi_j(E^{\mathscr{C}}(f))$$

is valid a.e. for $j = 1, \ldots,$ m. Since the convergence in R^m is identical with the coordinate convergence, we have

$$\lim_{n\to\infty} E^{\mathscr{C}_n}(f) = (\Pi_1(E^{\mathscr{C}}(f)), \ldots, \Pi_m(E^{\mathscr{C}}(f))) = E^{\mathscr{C}}(f) \quad \text{a.e.}$$

Similarly, according to theorems about integrable functions with real values

$$\lim_{n\to\infty} \int |E^{\mathscr{C}}(\Pi_j(f)) - E^{\mathscr{C}_n}(\Pi_j(f))| \, \mathrm{d}P = 0 \quad j = 1, \ldots, \mathrm{m},$$

and since the norm in R^m is given by the formula $\|(x_1, \ldots, x_m)\| = \sum_{j=1}^{m} |x_j|$, we have

$$\lim_{n\to\infty} \int \|E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f)\| \, \mathrm{d}P =$$

$$= \lim_{n\to\infty} \int \sum_{j=1}^{m} |\Pi_j(E^{\mathscr{C}}(f)) - \Pi_j(E^{\mathscr{C}_n}(f))| \, \mathrm{d}P =$$

$$= \sum_{j=1}^{m} \lim_{n\to\infty} \int |E^{\mathscr{C}}(\Pi_j(f)) - E^{\mathscr{C}_n}(\Pi_j(f))| \, \mathrm{d}P = 0.$$

(II) By the assumption Y is an m-dimensional normed linear space. It is proved in [4] that there exists such a continuous linear operator $T\colon Y \to R^m$ that T^{-1} exists and is a continuous linear operator. Thus

$$E^{\mathscr{C}}(f) = T^{-1}(E^{\mathscr{C}}(Tf)) = \lim_{n\to\infty} T^{-1}(E^{\mathscr{C}_n}(Tf)) = \lim_{n\to\infty} E^{\mathscr{C}_n}(f) \quad \text{a.e.}$$

by the first part of this proof and Lemma 1. Similarly, the first part of this proof implies that

$$0 \leqslant \lim_{n\to\infty} \int \|E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f)\| \, dP =$$

$$\lim_{n\to\infty} \int \|T^{-1}[T(E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f))]\| \, dP \leqslant$$

$$\leqslant \lim_{n\to\infty} \|T^{-1}\| \int \|E^{\mathscr{C}}(Tf) - E^{\mathscr{C}_n}(Tf)\| \, dP = 0 .$$

Convergence Theorem for Conditional Expectations. *If $f\colon \Omega \to X$ is an integrable function, $\{\mathscr{C}_n\}_{n=1}^{\infty}$ is an increasing sequence of σ-algebras and $\mathscr{C} = \bigvee_{n=1}^{\infty} \mathscr{C}_n$, then*

$E^{\mathscr{C}}(f) = \lim_{n\to\infty} E^{\mathscr{C}_n}(f)$ a.e. and $\lim_{n\to\infty} \int \|E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f)\| \, dP = 0$.

Proof. As the function f is integrable, we can choose step functions $\{f_m\}_{m=1}^{\infty}$ such that $\|f - f_m\| \to 0$ and $\|f_m\| \leqslant 2\|f\|$ for all m. Since X possesses a Hamel basis, the step function f_m takes values in a finite-dimensional subspace of X. If we denote $E^{\mathscr{C}}(f) = \sum_{j=1}^{m} x_j P^{\mathscr{C}}(A_j)$ where $f = \sum_{j=1}^{m} x_j \chi_{A_j}$ and $P^{\mathscr{C}}(A_j)$ is a conditional probability, then we see we can choose variants of conditional expectations $E^{\mathscr{C}}(f_m)$, $E^{\mathscr{C}_n}(f_m)$ $n = 1, 2, \ldots$ so that they take values in the same finite-dimensional subspace as the function f_m. Let us choose such functions for $m = 1, 2, \ldots$ and choose variants of conditional expectations $E^{\mathscr{C}}(f)$, $E^{\mathscr{C}_n}(f)$, $E^{\mathscr{C}}(\|f - f_m\|)$, $E^{\mathscr{C}_n}(\|f - f_m\|)$ for n, $m = 1, 2, \ldots$. Properties of conditional expectations imply that there exists such a set A that $P(A) = 1$ and A has the following properties. If $\omega \in A$, then for every integers n, $m \geqslant 1$ we have

(I) $\|E^{\mathscr{C}}(f)(\omega) - E^{\mathscr{C}_n}(f)(\omega)\| \leqslant$

$$\leqslant \|E^{\mathscr{C}}(f)(\omega) - E^{\mathscr{C}}(f_m)(\omega)\| + \|E^{\mathscr{C}}(f_m)(\omega) - E^{\mathscr{C}_n}(f_m)(\omega)\| +$$

$$+ \|E^{\mathscr{C}_n}(f_m)(\omega) - E^{\mathscr{C}_n}(f)(\omega)\| \leqslant E^{\mathscr{C}}(\|f - f_m\|)(\omega) +$$

$$+ \|E^{\mathscr{C}}(f_m)(\omega) - E^{\mathscr{C}_n}(f_m)(\omega)\| + E^{\mathscr{C}_n}(\|f_m - f\|)(\omega),$$

(II) $\lim_{k\to\infty} E^{\mathscr{C}}(\|f - f_k\|)(\omega) = 0,$

(III) $\lim_{k\to\infty} \|E^{\mathscr{C}}(f_m)(\omega) - E^{\mathscr{C}_k}(f_m)(\omega)\| = 0,$

(IV) $\lim_{k\to\infty} E^{\mathscr{C}_k}(\|f - f_m\|)(\omega) = E^{\mathscr{C}}(\|f - f_m\|)(\omega).$

Since $\|f - f_m\| \leqslant 3\|f\|$, the property II is a consequence of a theorem about the domination of random variables with real values. The property III is a consequence of Lemma 2.

Let $\omega \in A$ and ε be a positive number. Let m_0 be such an integer that $m \geqslant m_0$ implies $E^{\mathscr{C}}(\|f - f_m\|)(\omega) \leqslant \varepsilon$. If $m \geqslant m_0$, then the properties III and I imply that

$$0 \leqslant \limsup_{n \geqslant 1} \|E^{\mathscr{C}}(f)(\omega) - E^{\mathscr{C}_n}(f)(\omega)\| \leqslant$$

$$\leqslant \varepsilon + \lim_{n \to \infty} E^{\mathscr{C}_n}(\|f_m - f\|)(\omega) = \varepsilon + E^{\mathscr{C}}(\|f - f_m\|)(\omega) \leqslant 2\varepsilon$$

and this inequality completes the proof of the convergence a.e. Since $P(A) = 1$, we have

$$\int \|E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f)\| \, dP \leqslant 2 \int \|f - f_m\| \, dP + \int \|E^{\mathscr{C}}(f_m) - E^{\mathscr{C}_n}(f_m)\| \, dP.$$

If $\varepsilon > 0$, then $2\int \|f - f_m\| \, dP < (\varepsilon | 2)$ for sufficiently large m, hence

$$\int \|E^{\mathscr{C}}(f) - E^{\mathscr{C}_n}(f)\| \, dP < \varepsilon$$

for sufficiently large n by Lemma 2.

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О УСЛОВНЫХ МАТЕМАТИЧЕСКИХ ОЖИДАНИЯХ ФУНКЦИЙ С ВЕКТОРНЫМИ ЗНАЧЕНИЯМИ

Франтишек Рублик

Резюме

В работе показывается доказательство теоремы о сходимости условных математических ожиданий для случайной величины со значениями в пространстве Банаха, основано на перестановке оператора условного математического ожидания с непрерывным линейным оператором.