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GRAPHS WHICH ARE EDGE-LOCALLY $C_n$

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ABSTRACT. A graph $G$ is called edge-locally $C_n$ (vertex-locally $C_n$) if the neighbourhood of each edge (vertex) of $G$ is formed by a cycle of length $n$. The relationship between edge- and vertex-locally $C_n$ graphs is clarified. This leads to a (geometrical) characterization of edge-locally $C_n$ graphs for $n > 6$. A similar characterization for vertex-locally $C_n$ graphs was given by V i n c e. F r o n č e k proved that for $n$ odd ($n > 3$) there is no edge-locally $C_n$ graph. It is here proved that for all the remaining values of $n$ such a graph exists. Moreover, a complete list of edge-locally $C_n$ graphs for $n \leq 6$ is given. For every $n$ even, $n > 6$, there are infinitely many such graphs.

0. Introduction

Let $G$ be a graph and $u$ one of its vertices. Denote by $G(u)$ the subgraph of $G$ induced by the set of vertices adjacent to $u$. The graph $G$ is called vertex-locally $G_0$ if $G(u) \cong G_0$ for all vertices $u$ of $G$. A graph is called locally homogeneous if it is vertex-locally $G_0$ for some $G_0$. Much attention has been paid to two broad questions related to local homogeneity. The first one, posed by Ž y k o v in 1963 [22], is the following: For which graphs $G_0$ does there exist a (finite) graph $G$ that is vertex-locally $G_0$? (see for instance [1], [2], [3], [12]), while the second one reads as follows: For a given graph $G_0$ characterize all (finite) graphs that are vertex-locally $G_0$ (see [4], [9], [10], [19]). These questions seem to be difficult (see [3]). However, even partial results are of interest since the subject is related to algebraic topology and group theory (see [10], [17], [18], [19]). In particular, it was proved in [2] and also in [5] that for each cycle $C_n$ of length $n \geq 3$ there is a vertex-locally $C_n$ graph. Later, R o n a n [17] showed that there are infinitely many vertex-locally $C_n$ graphs for each $n \geq 6$. Finally, V i n c e [18] characterized graphs which are vertex-locally $C_n$ in terms of groups. The main aim of this paper is to resolve edge versions of these results. All graphs

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1. Edge-locally $C_n$ graphs

Let $e = uv$ be an edge of a graph $G$, and let $G(e)$ denote the subgraph of $G$ induced by the vertices adjacent to $u$ or $v$ but different from $u$, $v$. Call a graph *edge-locally* $G_0$ if $G(e) \cong G_0$ for each edge $e$ of $G$. A graph will be called *edge-locally homogeneous* if it is edge-locally $G_0$ for some $G_0$. The concept of edge-local homogeneity was introduced by Želinka in [21], where he presented examples of edge-locally $C_n$ graphs for $n = 3, 4, 6$ and $8$. On the other hand, he proved that there is no such graph for $n = 5$. This result was improved by Fronček [7], who showed that there is no edge-locally $C_n$ graph for $n$ odd, $n \neq 3$. On the contrary, we prove (see Theorem 5) that for all even $n$ such graphs exist. Denote by $L_{5,5}$ the graph obtained from the complete bipartite graph $K_{5,5}$ by deleting the edges of a perfect matching. Clearly, $K_{5,5}$, $K_{3,3}$ and $L_{5,5}$ are edge-locally $C_3$, $C_4$ and $C_6$, respectively. Another example of an edge-locally $C_6$ graph is the icosahedron, which is in addition vertex-locally $C_5$.

**Theorem 1.** Let $G$ be a connected graph and let $G$ contains a triangle. Then $G$ is edge-locally $C_n$ ($n \geq 3$) if and only if either $n \geq 6$ is even and $G$ is vertex-locally $C_k$ without an induced $C_4$, where $k = (n + 4)/2$, or $n = 3$ and $G \cong K_5$.

**Proof.**

($\Leftarrow\Rightarrow$) Let $G$ be a graph that is vertex-locally $C_k$ for some $k \geq 4$, and which does not contain $C_4$ as an induced subgraph. Consider an arbitrary edge $wx$ in $G$. Since $G(w)$ and $G(x)$ are both cycles of length $k$, their edges give rise to a Hamiltonian cycle $C$ of length $2k - 4$ in $G(wx)$. It remains to prove that $C$ is induced. Suppose there is a chord $uv$ in $C$. Since $G$ is vertex-locally $C_k$, $u$ and $v$ cannot both be in either $G(w)$ or in $G(x)$. Thus we may assume that $u \in G(x) - G(w)$ and $v \in G(w) - G(x)$. Hence $(uvw)$ is an induced cycle of length 4 in $G$, which is a contradiction.

($\Rightarrow\Leftarrow$) Let $G$ be edge-locally $C_n$ ($n \geq 3$). To finish the proof, we split the discussion into two separate claims.
Claim 1. Let $K_4 \subseteq G$, then $G \cong K_5$.

Proof. Denote by $u, v, x, y$ the vertices of $K_4 \subseteq G$. Since $G$ is edge-locally $C_n$ and $n \geq 3$, we have $|V(G)| \geq 5$. Moreover, $G$ is connected, so there is a vertex $z$ in $G$ distinct from $u, v, x, y$, but adjacent to at least one of them, say $u$. Since the triangle $(vxy)$ belongs to $G(uz)$, we have $n = 3$. The vertices $y, z$ and $v$ belong to $G(uv)$, so $z$ is adjacent with the vertices $y$ and $v$, too. Analogously $z, x, y \in G(uv)$, and hence $z$ is adjacent with $x$. Thus the subgraph of $G$ induced by the vertices of $S = \{u, v, x, y, z\}$ is isomorphic to $K_5$. Suppose the subgraph of $G$ induced by $S$ is proper. By the connectivity of $G$, it follows that there is an edge $tw$ in $G$ joining a vertex $t \in T$ with a vertex $w$ not in $T$. Consequently, $K_4 \subseteq G(tw)$, a contradiction. Thus $G \cong K_5$.

Claim 2. If $K_4 \not\subseteq G$, then $G$ is vertex-locally $C_k$ without an induced $C_4$, where $k = (n + 4)/2$.

Proof. First we show that if $uw$ is a common edge for two triangles $(uwv)$ and $(uwz)$, then $G(u)$ and $G(w)$ are cycles. Denote by $C$ the cycle of length $n$ in $G(uw)$. Set $x_1 = x$ and let $x_1, x_2, \ldots, x_n$ be the vertices of $C$ in this order. Clearly, $v = x_j$ for some $j \in \{1, 2, \ldots, n\}$. Since $K_4 \not\subseteq G$, then $x_2 \neq x_j$ and $x_{j+1} \neq x_1$. Now we prove that each of the vertices $x_2, \ldots, x_{j-1}$ is adjacent to exactly one vertex of $uw$. Suppose, on the contrary, that $x_2$ is adjacent to both $u$ and $w$. Then the subgraph of $G$ induced by the vertices $x_1, x_2, w, u$ is isomorphic to $K_4$, a contradiction. Without loss of generality, we may assume $x_2$ is adjacent to $u$. Assume there is a vertex $x_i, 3 \leq i \leq j - 1$, adjacent to $w$. Choose $x_i$ to be the first one in order. Then the vertices $x_{i-1}, x_i, x_j$ and $u$ belong to $G(x_iw)$. But $u$ is adjacent to the vertices $x_1, x_j, x_{i-1}$, thus $u$ is of degree at least 3 in $G(x_iw)$, a contradiction. Thus each vertex of $x_2, \ldots, x_{j-1}$ is adjacent to $u$ but not to $w$. Suppose $x_{j+1}$ is adjacent to $u$. Then the vertices $x_{j-2}, w, x_{j+1}$, $u$ belong to $G(x_{j-1}x_j)$, and $u$ has degree at least 3 in $G(x_{j-1}x_j)$, a contradiction. Thus $x_{j+1}$ is adjacent to $w$ but not to $u$. Now, as above, we are able to prove that each of the vertices $x_{j+1}, \ldots, x_{n-1}$ is adjacent to $w$ but not to $u$. As a result $G(u) \cong (x_1, \ldots, x_j, u)$ and $G(w) \cong (x_j, \ldots, x_n, x_1, u)$. Now we show that for every vertex $y$ in $G$, $G(y)$ is a cycle of the same length. By the assumption, there is a triangle $(uvw)$ in $G$. Since $G(uw) \cong C_n$ there is a vertex $x$ adjacent to $w$ and also to $u$ or $v$, say $u$. Hence $uw$ is a common edge for triangles $(uvw)$ and $(uxv)$. Consequently, $G(u)$ and $G(w)$ are cycles. Since $G$ is connected, there is a path $y_1, y_2, \ldots, y_m$ joining the vertices $u$ and $y$. We have proved that $G(y_1) = G(u)$ is a cycle. Suppose $G(y_i)$ is a cycle $(1 \leq i < m)$. Then $y_iy_{i+1}$ is a common edge for two triangles in $G$, whence $G(y_{i+1})$ is a cycle, too. Thus also $G(y_m) = G(y)$ is a cycle, and we are done. To complete the proof of Claim 2, we must show that each vertex in $G$ has degree $k$. Since $G(x)$ is a cycle for each vertex $x$ in $G$, $x$ is incident
with a triangle \( (xyz) \) in \( G \). Using the fact \( G(xy) \cong G(yz) \cong G(xz) \cong C_n \) we obtain
\[
\begin{align*}
\deg(x) + \deg(y) &= n + 4, \\
\deg(y) + \deg(z) &= n + 4, \\
\deg(x) + \deg(z) &= n + 4.
\end{align*}
\]
This system of linear equations has a unique solution \( \deg(x) = \deg(y) = \deg(z) = (n + 4)/2 = k \). Obviously, \( n \) is even and \( n \geq 6 \). Also \( G \) cannot have an induced \( C_4 \), since we would have a chord in any \( G(uv) \) otherwise. \( \square \)

Note that some ideas of the proof were extracted from Fronček’s paper [7], namely, from a proof of the following statement.

**Proposition 2.** Let \( G \) be edge-locally \( C_n \) graph. Then either \( n \) is even or \( n = 3 \).

From Theorem 1, we have the following characterization of edge-locally \( C_n \) graphs for small values of \( n \).

**Corollary 3.** Let \( G \) be a connected graph. Then
\[
\begin{align*}
(\text{a}) & \; G \text{ is edge-locally } C_3 \text{ if and only if } G \cong K_5, \\
(\text{b}) & \; G \text{ is edge-locally } C_4 \text{ if and only if } G \cong K_{3,3}, \\
(\text{c}) & \; G \text{ is edge-locally } C_6 \text{ if and only if } G \cong L_{5,5} \text{ or } G \text{ is the icosahedron}.
\end{align*}
\]

**Proof.** The statement follows immediately by the Theorem 1 and the well-known fact that for \( k \leq 5 \) there are exactly three vertex-locally \( C_k \) graphs, namely, the icosahedron, the octahedron and the tetrahedron. Of these three only the icosahedron is edge-locally \( C_n \), namely for \( n = 6 \). Similarly, it is not difficult to check that the only edge-locally \( C_n \) graphs without triangles for \( n \leq 6 \) are \( K_{3,3} \) and \( L_{5,5} \).

By Theorem 1, edge-locally \( C_n \) graphs containing triangles are with one exception just vertex-locally \( C_k \) graphs \( (k = (n+4)/2) \) without an induced \( C_4 \). It is not obvious that such a graph exists for each \( k > 6 \). However, as we prove in Theorem 6, there are infinitely many such graphs for each \( k > 6 \).

**Remark.** Corollary 3 may be viewed as a partial solution of the more general problem concerning the relationship between edge-local homogeneity and vertex-local homogeneity of a graph. This question is also related to methods of constructing edge-locally homogeneous graphs and vertex-locally homogeneous graphs. There is a well-known theorem (see [11]) establishing that each edge-transitive graph is either vertex-transitive or it is bipartite. These constructions of locally homogeneous graphs may be generalized in the way described in [18]. One can prove that all edge-locally homogeneous graphs obtained in this way
are either vertex-locally homogeneous or bipartite. The mentioned facts suggest to us the following question.

**Problem.** Is it true that each edge-locally homogeneous graph is either vertex-locally homogeneous or is bipartite?

### 2. Vertex-locally $C_k$ graphs

Here we summarize some known results on vertex-locally $C_k$ graphs. In the rest of the paper, it is assumed that the reader is familiar with the terminology and basic results of topological graph theory. For undefined terms, the reader is referred to [8] or [20]. The concept of vertex-locally $C_k$ graphs is closely related to the theory of 2-cell embeddings of graphs into closed surfaces. With each graph $G$ we may associate a simplicial complex $K(G)$ in which the simplices are complete subgraphs and the incidence relation is subgraph inclusion. The following proposition was already known to Brown and Connelly [2] and shows that topological graph theory is useful in the study of vertex-locally $C_k$ graphs.

**Proposition 4.** Graph $G$ is vertex-locally $C_k$ ($k \geq 3$) if and only if $K(G)$ is a $k$-valent triangulation of a closed surface such that each triangle of $G$ forms a boundary of a face in $K(G)$.

The proof of Proposition 4 can be found in [16]. Using a technique of topological surgery Brown and Connelly proved in [2] that for each $k \geq 3$ there exists a vertex-locally $C_k$ graph. By Theorem 1, each edge-locally $C_n$ graph ($n > 3$) containing a triangle is vertex-locally $C_k$ as well, where $k = (n + 4)/2$. However, there are vertex-locally $C_k$ graphs which are not edge-locally $C_n$ for they contain an induced $C_4$. Unfortunately, this is the case with all graphs constructed by Brown and Connelly in [2]. In Section 3, we shall give an alternative method for constructing vertex locally $C_k$ graphs. Since these graphs will be free of induced 4-cycles, Proposition 2 and Corollary 3 give the following theorem.

**Theorem 5.** An edge-locally $C_n$ graph exists if and only if $n = 3$ or $n$ is even, $n > 4$.

It is well-known that each closed surface of negative Euler characteristic admits a $n$-fold cover for each $n \geq 2$. Consequently, for an arbitrary vertex-locally $C_k$ graph and for each $n \geq 2$ there exists an $n$-fold covering graph which is also vertex-locally $C_k$. Hence we have: For each $k \geq 6$ there are infinitely many vertex-locally $C_k$ graphs. This fact was observed by Ronan in [17]. Arguing in the same way as Ronan we are able to prove the following theorem.
THEOREM 6. For each even \( n \geq 8 \) there are infinitely many (connected!) edge-locally \( C_n \) graphs.

Finally, Vince [18] gave the following geometrical characterization of vertex-locally \( C_k \) graphs. Let \( T_k \) \((k \geq 3)\) be the regular tessellations of a simply connected surface \( S \) into triangles with \( k \) triangles incident to each vertex. Such tessellations are discussed in Coxeter and Moser [6; p. 52]. The surface \( S \) is the sphere, the plane or the unit disk (hyperbolic plane) as \( k < 6 \), \( k = 6 \) or \( k > 6 \), respectively. In particular, these tessellations are exactly the tetrahedron, the octahedron and the icosahedron for \( k = 3,4,5 \). But for \( k \geq 6 \) the respective underlying graphs are infinite. Denote by \( \Gamma_k \) \((k \geq 3)\) the automorphism group of \( T_k \). The group \( \Gamma_k \) is the well-known triangle group with presentation

\[
\Gamma_k = \langle i,j,l; i^2 = j^2 = l^2 = (ij)^3 = (jl)^k = (li)^2 = 1 \rangle.
\]

Call a subgroup \( B \) of \( \Gamma_k \) properly discontinuous if the vertices \( x \) and \( \varphi(x) \) are at a distance of at least 4 for each \( \varphi \in B \) and \( x \in V(T_k) \). Vince proved the following theorem in [18].

**THEOREM 7.** A (finite) graph \( G \) is vertex-locally \( C_k \) if and only if there exists a properly discontinuous subgroup \( B \leq \Gamma_k \) of finite index such that \( K(G) \cong T_k/B \).

Theorem 7 shows that the question of existence of any vertex-locally \( C_k \) graph is equivalent to the question of existence of the corresponding subgroup of \( \Gamma_k \). Now let us turn our attention to edge-locally \( C_n \) graphs for \( n \geq 6 \). Call a subgroup \( B \) of \( G \) strongly discontinuous if, for each \( \varphi \in B \) and \( x \in V(T_k) \), the vertices \( x \) and \( \varphi(x) \) are at a distance of at least 5. Combining Theorems 7 and 1 we obtain:

**THEOREM 8.** Let \( k \geq 6 \) and let \( G \) be a graph containing a triangle. The graph \( G \) is edge-locally \( C_{2k-4} \) if and only if there is a strongly discontinuous subgroup \( B \) of finite index such that \( K(G) \cong T_k/B \).

### 3. Existence of edge-locally \( C_n \) graphs

In this section, we prove Theorem 5. By Theorem 1 and Corollary 3, it is sufficient to prove the existence of vertex-locally \( C_k \) graphs without an induced \( C_4 \) for all \( k \geq 6 \). As already mentioned, examples of vertex-locally \( C_k \) graphs constructed by Brown and Connelly are not appropriate for us since they all contain an induced \( C_4 \). We shall present an alternative construction of \( k \)-valent triangulations which do not contain non-contractible cycles of length \( \leq 4 \). The central idea is that of constructing regular maps from their planar quotients.
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(see [13]). By a map, we mean a graph embedded cellularly into an orientable surface. Obviously, the $k$-valent triangulations we want to construct present a special family of maps. It is well-known that every map $M$ can be described by two permutations $l$ and $r$ acting on the set of arcs $D = D(M)$ of $M$. The arc-reversing involution $l$ sends an arc $x$ to the oppositely directed arc $x^{-1}$, while the rotation $r$ cyclically permutes arcs emanating from the same vertex. The vertices of the map correspond to the orbits of $r$, the edges correspond to the orbits of $l$, and the orbits of $rl$ determine the boundaries of the faces of $M$. Thus any couple $r$, $l$ of permutations, $l$ being involutory, determines the map $M = (D; r, l)$ uniquely. The action of $\langle r, l \rangle$ is transitive, but in general, it is not regular. There is an associated map $M^\# = (D^\#; R, L)$ defined as follows: $D^\# = \langle r, l \rangle$, $R(x) = rx$, $L(x) = lx$ for every $x \in D^\#$. We call $M^\#$ the generic map of $M$. One can easily verify that right translations by elements of $D^\#$ form map automorphisms of $M^\#$. Thus $|\text{Aut} M^\#| \geq |D^\#|$. On the other hand, no map can have more automorphisms than arcs. Therefore, the map automorphism group acts regularly on the arc-set of $M^\#$, and consequently, $M^\#$ is regular. It follows that if the original map $M$ is regular, then $M$ and $M^\#$ coincide; otherwise $M \neq M^\#$. The reader may find more information on regular maps and generic maps in [15] and [14].

In what follows, we shall consider graphs having three kinds of edges, links (possibly parallel), loops and semiedges. A link or a loop gives rise to two oppositely directed arcs, while there is just one arc associated with a semiedge. The arcs coming from semiedges are fixed by the arc-reversing involution. Now we are ready to describe the construction.

Construction.

Let us consider the following sequence $H_k$ ($k = 1, 2, 3, \ldots$) of planar maps. The maps $H_1$, $H_2$, $H_3$ and $H_4$ are depicted on Fig. 1. For $k \geq 5$ the map $H_k$ is formed from $H_{k-3}$ by adding a semiedge and a loop in the way depicted on Fig. 1.

![Figure 1](image-url)
Let \( r \) and \( l \) be the rotation and the arc-reversing involution associated with the map \( H_k \). Then the orders of \( r \) and \( rl \) are \( k \) and 3 (\( k \geq 3 \)), respectively. Consequently, the generic map \( H_k^# \) is a \( k \)-valent triangulation. The problem is, however, to guarantee the contractibility of every cycle of length \( \leq 4 \). To do this, a more sophisticated construction is needed.

First of all, we exclude the case \( k = 6 \) since it is easy to construct infinitely many 6-valent triangulations of the torus which are edge-locally \( C_8 \) (see Fig. 2).

![Figure 2.](image)

Denote by \( \tau_k \) the triangulation of a disc arising from the infinite \( k \)-valent triangular tessellation \( T_k \) of the plane by taking the part of \( T_k \) induced by the vertices at a distance \( \leq 3 \) from a chosen vertex \( v \). Let \( \tau'_k \) and \( \tau''_k \) be two copies of \( \tau_k \), and let \( M_k \) be a spherical map obtained by gluing the boundary cycles of \( \tau'_k \) and \( \tau''_k \) according to some isomorphism \( \varphi: \tau'_k \rightarrow \tau''_k \). We call the cycle arising from the boundary cycles the equator, and the two images \( v' \) and \( v'' \) of the vertex \( v \) poles. Obviously, \( M_k \) is a spherical triangulation which is \( k \)-valent, with the exception of the vertices on the equator, where a sequence of \( k - 5 \) consecutive vertices of valency 4 alternates with a single vertex of valency 6. We shall now modify the triangulation \( M_k \) locally such that a \( k \)-valent triangulation \( N_k \) is obtained. First we replace every edge on the equator by two parallel edges. Then we draw either the map \( H_{k-8} \) or \( H_{k-6} \) inside each digon, depending on whether the ‘left’ vertex in the digon has valency 8 or 6, respectively. The solution in the general case \( k \geq 9 \) is depicted on Fig. 3.
The cases $k = 7$ and $k = 8$ have to be solved separately, see Fig. 4 and Fig. 5.
We claim that the generic triangulation $N_k^#$ is vertex- and edge-locally cyclic. The argument is geometrical. There is a natural (branched) covering projection $\pi: N_k^# \to N_k$ (see [13], [15]). But the neighbourhood up to distance 2 of a preimage of the pole $v'$ ($v''$) is mapped one to one. Therefore, the neighbourhood of every preimage $w \in \pi^{-1}(v')$ is formed by a cycle of length $k$, and the neighbourhood of a preimage of an edge incident with $v'$ is formed by a cycle of length $2k - 4$. Since the generic map $N_k^#$ is vertex- and edge-transitive, the property extends to every vertex and edge of the triangulation $N_k^#$. 

Remark. Let us call the edge-width of a non-spherical triangulation $T$ to be the length of a shortest non-contractible cycle in $T$. Clearly, the presented construction produces $k$-valent triangulations of edge-width at least two for any $k \geq 7$. A generalization of the above construction done in [13] allows us to construct $k$-valent triangulations of arbitrarily large edge-width for every $k \leq 7$.

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