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OSCILLATION AND GLOBAL ATTRACTIVITY IN A NONLINEAR DELAY DIFFERENCE EQUATION

DENGHUA CHENG* — JURANG YAN**

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ABSTRACT. We obtain a necessary and sufficient condition for every positive solution of the nonlinear delay difference equation

$$x_{n+1} = \frac{x_n}{a + bx_{n-k}^p - cx_{n-k}^q}, \qquad n = 0, 1, \dots \tag{$*$}$$

to oscillate about its positive equilibrium. We also obtain conditions under which the positive equilibrium of $(*)$ is globally attractive.

1. Introduction

There have been many papers considering the oscillation and the nonoscillation of nonlinear delay difference equations, see, for example, [1]–[6] and the references cited in [1].

Our aim in this paper is to investigate the oscillation and global attractivity of the nonlinear delay difference equation

$$x_{n+1} = \frac{x_n}{a + bx_{n-k}^p - cx_{n-k}^q}, \qquad n = 0, 1, \dots, \tag{1}$$

where

$$\begin{gathered} a \in (0,1), \quad b, p, q \in (0, \infty), \quad c \in (-\infty, \infty), \quad k \in \mathbb{N}, \\ p > q, \qquad a + b\Big(\frac{cq}{bp}\Big)^{\frac{p}{p-q}} - c\Big(\frac{cq}{bp}\Big)^{\frac{q}{p-q}} > 0. \end{gathered} \tag{2}$$

By a solution of (1) we mean a sequence $\{x_n\}$ of real numbers which is defined for $n \geq -k$ and satisfies (1) for $n = 0, 1, \dots$. It is easy to see under the initial conditions:

$$x_n = A_n > 0, \qquad n = -k, -k+1, \dots, 0, \tag{3}$$

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equation (1) has a unique positive solution satisfying (3).

Equation (1) has a unique positive equilibrium x^*. In Section 2, we establish a necessary and sufficient condition for every positive solution of (1) to oscillate about x^* and in Section 3, we establish a sufficient condition for the global attractivity of x^*.

When $p = 2$ and $q = 1$ V. L. Kocic and G. Ladas [1; pp. 166, 167] investigated a similar equation. Our results in this paper extend and improve their results.

2. Oscillation of equation (1)

In this section, we study the oscillatory behavior of the solution of (1). As usual, a solution $\{x_n\}_{n\geq -k}$ of (1) is said to be oscillatory about x^* if the terms x_n of the sequence are neither eventually greater than x^* nor eventually less than x^*. Otherwise, the solution is called nonoscillatory about x^*.

Before we present the main result we state two lemmas which will be useful in the sequel. The first one is extracted from [1; pp. 6, 7].

LEMMA 1. ([1]) *Consider the delay equation*

$$y_{n+1} - y_n + rf(y_{n-k}) = 0\,, \qquad n = 0, 1, \dots\,, \tag{4}$$

where $r \in (0, \infty)$, $k \in \mathbb{N}$ *and* $f \in C[\mathbb{R}, \mathbb{R}]$. *Assume that*

$$uf(u) > 0 \qquad \text{for} \quad u \neq 0$$

and that

$$\lim_{u \to 0} \frac{f(u)}{u} = 1\,.$$

Suppose also there exists a positive number δ *such that either*

$$f(u) \leq u \qquad \text{for} \quad 0 < u < \delta\,,$$

or

$$f(u) \geq u \qquad \text{for} \quad -\delta < u < 0\,.$$

Then every solution of (4) *oscillates if and only if*

$$r \begin{cases} \geq 1 & \text{if } k = 0\,, \\ > \frac{k^k}{(k+1)^{k+1}} & \text{if } k \geq 1\,. \end{cases}$$

The proof of the next lemma is straightforward and will be omitted.

LEMMA 2. *Assume that* (2) *holds and set*

$$F(x) = a + bx^p - cx^q .$$

Then there is a unique positive number x^* *such that* $F(x^*) = 1$. *Furthermore,*

$$F(x)\begin{cases} < 1 & \text{for } 0 < x < x^* , \\ > 1 & \text{for } x^* < x < \infty . \end{cases} \tag{5}$$

In addition, if $c \le 0$, *then*

$$F(x) \text{ is increasing} \quad \text{for } x > 0 , \tag{6}$$

and if $c > 0$, *then*

$$F(x)\begin{cases} \text{is decreasing} & \text{for } 0 < x < \left(\frac{cq}{bp}\right)^{\frac{1}{p-q}} , \\ \text{is increasing} & \text{for } \left(\frac{cq}{bp}\right)^{\frac{1}{p-q}} < x < \infty . \end{cases} \tag{7}$$

The main result in this section is the following:

THEOREM 1. *Assume that* (2) *holds. Then every positive solution of* (1) *oscillates about* x^* *if and only if*

$$pb(x^*)^p - qc(x^*)^q \begin{cases} \ge 1 & \text{if } k = 0 , \\ > \frac{k^k}{(k+1)^{k+1}} & \text{if } k \ge 1 . \end{cases} \tag{8}$$

P r o o f . The change of variable

$$x_n = x^* \, \mathrm{e}^{y_n}$$

transforms (1) to the difference equation

$$y_{n+1} - y_n + \ln\left[a + b(x^*)^p \, \mathrm{e}^{py_{n-k}} - c(x^*)^q \, \mathrm{e}^{qy_{n-k}}\right] = 0 . \tag{9}$$

Clearly every solution of (1) oscillates about x^* if and only if every solution of (9) oscillates about zero. Set

$$f(u) = \ln\left[a + b(x^* \, \mathrm{e}^u)^p - c(x^* \, \mathrm{e}^u)^q\right] ,$$
$$g(u) = f(u) - \left[pb(x^*)^p - qc(x^*)^q\right] u .$$

If $c \le 0$, then clearly

$$uf(u) > 0 \qquad \text{for} \quad u \ne 0 . \tag{10}$$

Next, assume that $c > 0$. As

$$p > q > 0 \qquad \text{and} \qquad b(x^*)^p - c(x^*)^q = 1 - a > 0 ,$$

it follows that

$$f(u) \geq \ln\left[a + \left(b(x^*)^p - c(x^*)^q\right) \mathrm{e}^{qu}\right] > 0 \qquad \text{for} \quad u > 0\,,$$
$$f(u) \leq \ln\left[a + \left(b(x^*)^p - c(x^*)^q\right) \mathrm{e}^{qu}\right] < 0 \qquad \text{for} \quad u < 0\,.$$

Hence, (10) holds for $c \in (-\infty, \infty)$.

Observe that

$$\begin{aligned}
\frac{dg}{du} &= \frac{pb(x^*)^p\,\mathrm{e}^{pu} - qc(x^*)^q\,\mathrm{e}^{qu}}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} - \left[pb(x^*)^p - qc(x^*)^q\right] \\
&\leq \frac{1}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} \left[pb(x^*)^p\,\mathrm{e}^{pu} - qc(x^*)^q\,\mathrm{e}^{qu}\right. \\
&\quad \left. - \left(pb(x^*)^p - qc(x^*)^q\right)\left(a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}\right)\right] \\
&= \frac{1}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} \left[pb(x^*)^p\,\mathrm{e}^{pu} - qc(x^*)^q\,\mathrm{e}^{qu} - a\left(pb(x^*)^p - qc(x^*)^q\right)\right. \\
&\quad - \left(pb(x^*)^p - qc(x^*)^q\right)b(x^*)^p\,\mathrm{e}^{pu} + \left(pb(x^*)^p - qc(x^*)^q\right)c(x^*)^q\mathrm{e}^{qu} \\
&= \frac{1}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} \left[\left(pb(x^*)^p - b(x^*)^p\left(pb(x^*)^p - qc(x^*)^q\right)\right)\mathrm{e}^{pu}\right. \\
&\quad \left. - \left(qc(x^*)^q - c(x^*)^q\left(pb(x^*)^p - qc(x^*)^q\right)\right)\mathrm{e}^{qu} - a\left(pb(x^*)^p - qc(x^*)^q\right)\right] \\
&\leq \frac{1}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} \left[\left(pb(x^*)^p - \left(pb(x^*)^p - qc(x^*)^q\right)b(x^*)^p - qc(x^*)^q\right.\right. \\
&\quad \left.\left. + \left(pb(x^*)^p - qc(x^*)^q\right)c(x^*)^q\right)\mathrm{e}^{qu} - a\left(pb(x^*)^p - qc(x^*)^q\right)\right] \\
&\leq \frac{1}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} \left[pb(x^*)^p - qc(x^*q)\left(1 - b(x^*)^p + c(x^*)^q\right)\mathrm{e}^{qu}\right. \\
&\quad \left. - a\left(pb(x^*)^p - qc(x^*)^q\right)\right]
\end{aligned}$$

and

$$pb(x^*)^p - qc(x^*)^q \geq p\left[b(x^*)^p - c(x^*)^q\right] = p(1-a) > 0\,.$$

Hence

$$\begin{aligned}
\frac{dg}{du} &\leq \frac{1}{a + b(x^*)^p\,\mathrm{e}^{pu} - c(x^*)^q\,\mathrm{e}^{qu}} \left[\left(pb(x^*)^p - qc(x^*)^q\right)\left(1 - a - b(x^*)^p + c(x^*)^q\right)\right] \\
&= 0 \qquad \text{for} \quad u < 0\,.
\end{aligned}$$

This together with $g(0) = 0$ implies that $g(u) > 0$ for $u < 0$, that is

$$f(u) \geq \left[pb(x^*)^p - qc(x^*)^q\right]u \qquad \text{for} \quad u < 0\,.$$

We also have

$$\frac{df(0)}{du} = pb(x^*)^p - qc(x^*)^q$$

and so

$$\lim_{u\to 0} \frac{f(u)}{\left[pb(x^*)^p - qc(x^*)^q\right]u} = 1 .$$

Hence, by Lemma 1, every solution of (9) oscillates if and only if (8) holds. The proof is complete. □

3. Global Attractivity of (1)

In this section, we investigate the global attractivity of the positive equilibrium x^* of (1).

THEOREM 2. *Assume that* (2) *holds. Then every positive solution of* (1) *nonoscillatory about* x^* *tends to* x^* *as* $n \to \infty$.

Proof. Assume that $x_n > x^*$ for n sufficiently large. The proof when $x_n < x^*$ for n sufficiently large is similar and will be omitted. Set

$$x_n = x^* \,\mathrm{e}^{y_n} .$$

Then $y_n > 0$ for n sufficiently large and

$$y_{n+1} - y_n + \ln\left[a + b(x^*)^p \,\mathrm{e}^{py_{n-k}} - c(x^*)^q \,\mathrm{e}^{qy_{n-k}}\right] = 0 . \tag{11}$$

Thus for n sufficiently large

$$y_{n+1} - y_n \le -\ln\left[a + \left(b(x^*)^p - c(x^*)^q\right) \mathrm{e}^{qy_{n-k}}\right] \le 0 ,$$

and so $\lim\limits_{n\to\infty} y_n = \mu \in [0,\infty)$, say, exists.

We claim that $\mu = 0$. Otherwise, $\mu > 0$. Take

$$0 < \varepsilon < \frac{p-q}{p+q}\mu . \tag{12}$$

Then there exists $N_0 > 0$ such that for $n \ge N_0$,

$$\mu - \varepsilon < y_{n-k} < \mu + \varepsilon . \tag{13}$$

First, assume that $c \le 0$. From (11) and (13), it follows that

$$y_{n+1} - y_n + \ln\left[a + \left(b(x^*)^p - c(x^*)^q\right) \mathrm{e}^{q(\mu-\varepsilon)}\right] \le 0 \qquad \text{for } n \ge N_0$$

and by summing this inequality from N_0 to ∞ we get a contradiction.

Next, assume that $c > 0$. Then (11) and (13) yield

$$y_{n+1} - y_n + \ln\left[a + b(x^*)^p \,\mathrm{e}^{p(\mu-\varepsilon)} - c(x^*)^q \,\mathrm{e}^{q(\mu+\varepsilon)}\right] \le 0 . \tag{14}$$

In view of (12), we have

$$\ln\left[a + b(x^*)^p \,\mathrm{e}^{p(\mu-\varepsilon)} - c(x^*)^q \,\mathrm{e}^{q(\mu+\varepsilon)}\right] \ge \ln\left[a + \left(b(x^*)^p - c(x^*)^q\right) \mathrm{e}^{q(\mu+\varepsilon)}\right]$$

and so (14) yields

$$y_{n+1} - y_n + \ln\left[a + \left(b(x^*)^p - c(x^*)^q\right) \mathrm{e}^{q(\mu+\varepsilon)}\right] \le 0 \qquad \text{for} \quad n \ge N_0 .$$

By summing this inequality from N_0 to ∞ we get a contradiction. The proof is complete. □

THEOREM 3. *Assume that* (2) *holds. Set*

$$M_0 = \begin{cases} \left(\frac{1}{a}\right)^{k+1} & \text{if } c \le 0\,, \\ \frac{1}{\left(a+b\left(\frac{cq}{bp}\right)^p - c\left(\frac{cq}{bp}\right)^q\right)^{k+1}} & \text{if } c > 0\,. \end{cases}$$

Suppose that

$$\frac{(k+1)}{\ln M_0} \ln\left[a + b(x^* M_0)^p - c(x^* M_0)^q\right] < 1\,. \tag{15}$$

Then every positive solution of (1) *oscillatory about* x^* *tends to* x^* *as* $n \to \infty$.

P r o o f. Assume that $\{x_n\}_{n \ge -k}$ is an solution of (1) oscillatory about x^*. We will prove that $\lim_{n\to\infty} x_n = x^*$.

Let $\{n_i\}$ be an increasing sequence of positive integers such that $n_i \to \infty$ as $i \to \infty$ satisfying

$$x_{n_i} < x^* \quad \text{and} \quad x_{n_i+1} \ge x^* \qquad \text{for} \quad i = 1, 2, \dots,$$

and for each $i = 1, 2, \dots$, some of the terms x_j with $n_i < j \le n_{i+1}$ are greater than x^* and some are less than x^*. For each $i = 1, 2, \dots$, let m_i and M_i be the integers in the interval $[n_i, n_{i+1}]$ such that

$$x_{m_i+1} = \min\{x_j :\ n_i < j \le n_{i+1}\}\,,$$
$$x_{M_i+1} = \max\{x_j :\ n_i < j \le n_{i+1}\}\,.$$

Then for each $i = 1, 2, \dots$,

$$x_{m_i+1} < x^* \qquad \text{and} \qquad \Delta x_{m_i} \le 0$$

while

$$x_{M_i+1} > x^* \qquad \text{and} \qquad \Delta x_{M_i} \ge 0\,.$$

By (1), we have

$$0 \ge \Delta x_{m_i} = \frac{x_{m_i}\left[1 - (a + b x^p_{m_i-k} - c x^q_{m_i-k})\right]}{a + b x^p_{m_i-k} - c x^q_{m_i-k}}\,,$$

which indicates that $a + b x^p_{m_i-k} - c x^q_{m_i-k} \ge 1$, that is $x_{m_i-k} \ge x^*$. Therefore, there exists an integer $\overline{m_i}$ satisfying $\max\{n_i, m_i - k\} \le \overline{m_i} < m_i + 1$ and

$$x_{\overline{m_i}} \ge x^*\,, \qquad \text{and} \qquad x_j < x^* \quad \text{for } j = \overline{m_i} + 1, \dots, m_i + 1\,. \tag{16}$$

Similarly, there exists an integer $\overline{M_i}$ satisfying $\max\{n_i, M_i - k\} \le \overline{M_i} < M_i + 1$ and

$$x_{\overline{M_i}} \le x^*\,, \qquad \text{and} \qquad x_j > x^* \quad \text{for } j = \overline{M_i} + 1, \dots, M_i + 1\,. \tag{17}$$

Now we show that $\{x_n\}$ is bounded from above and bounded from below away from zero. In fact, since $x_n > 0$ for $n \geq 0$, it follows by (1) that

$$\frac{x_{n+1}}{x_n} = \frac{1}{a + bx_{n-k}^p - cx_{n-k}^q}. \tag{18}$$

First, assume $c \leq 0$. Then for $n \geq 0$, we have

$$\frac{x_{n+1}}{x_n} \leq \frac{1}{a} \qquad \text{for} \quad n \geq 0.$$

Hence, by multiplying this ineqality from $\overline{M_i}$ to M_i we have

$$\frac{x_{M_i+1}}{x_{\overline{M_i}}} \leq \left(\frac{1}{a}\right)^{M_i - \overline{M_i} + 1},$$

and so

$$x_{M_i+1} \leq x^* \left(\frac{1}{a}\right)^{k+1} = x^* M_0,$$

which clearly implies that

$$x_n \leq x^* M_0 \qquad \text{for} \quad n \geq 0.$$

By using this fact in (18), we find that for $n \geq 0$

$$\frac{x_{n+1}}{x_n} \geq \frac{1}{a + b(x^* M_0)^p - c(x^* M_0)^q}$$

and so

$$\frac{x_{m_i+1}}{x_{\overline{m_i}}} \geq \frac{1}{\left(a + b(x^* M_0)^p - c(x^* M_0)^q\right)^{k+1}} = M_1,$$

which implies that

$$x_n \geq x^* M_1 \qquad \text{for} \qquad n \geq 0.$$

Next, assume that $c > 0$. Then, in view of Lemma 2, we see from (18) that for $n \geq 0$,

$$\frac{x_{n+1}}{x_n} \leq \frac{1}{a + b\left(\frac{cq}{bp}\right)^p - c\left(\frac{cq}{bp}\right)^q}.$$

Hence, we have

$$\frac{x_{M_i+1}}{x_{\overline{M_i}}} \leq \frac{1}{\left(a + b\left(\frac{cq}{bp}\right)^p - c\left(\frac{cq}{bp}\right)^q\right)^{k+1}} = M_0$$

and so

$$x_{M_i+1} \leq x^* M_0,$$

which implies that

$$x_n \le x^* M_0 \qquad \text{for} \quad n \ge 0 .$$

Similarly, we have

$$x_n \ge x^* M_1 \qquad \text{for} \quad n \ge 0 .$$

Therefore, we have

$$M_1 x^* \le x_n \le x^* M_0 \qquad \text{for} \quad n \ge 0 .$$

Now set

$$g(u) = \begin{cases} \frac{1}{u} \ln\left[a + b(x^*)^p \, \mathrm{e}^{pu} - c(x^*)^q \, \mathrm{e}^{qu}\right] & \text{for } u \ne 0 , \\ pb(x^*)^p - qc(x^*)^q & \text{for } u = 0 . \end{cases}$$

Observe that the transformation

$$x_n = x^* \, \mathrm{e}^{y_n}$$

transforms (1) into

$$y_{n+1} - y_n = -g(y_{n-k}) y_{n-k} . \tag{19}$$

Clearly, to show that $\lim_{n\to\infty} x_n = x^*$, it suffices to show that

$$\lim_{n\to\infty} y_n = 0 . \tag{20}$$

To this end, observe that

$$\ln M_1 \le y_n \le \ln M_0 \qquad \text{for} \quad n \ge 0 . \tag{21}$$

First we show that there is a $\delta > 0$ such that

$$\delta \le g(y_n) \le g(\ln M_0) \qquad \text{for} \quad n \ge 0 . \tag{22}$$

Observe that

$$f(u) = \begin{cases} \frac{\mathrm{e}^u - 1}{u} & \text{for } u \ne 0 , \\ 1 & \text{for } u = 0 \end{cases}$$

is increasing, $f > 0$, $p > q$, and $pb(x^*)^p > qc(x^*)^q$. Thus for $u < 0$,

$$\begin{aligned} g(u) &= \frac{1}{u} \ln\left[1 + b(x^*)^p(\mathrm{e}^{pu} - 1) - c(x^*)^q(\mathrm{e}^{qu} - 1)\right] \\ &\le pb(x^*)^p \frac{\mathrm{e}^{pu} - 1}{pu} - qc(x^*)^q \frac{\mathrm{e}^{qu} - 1}{qu} \\ &\le \left(pb(x^*)^p - qc(x^*)^q\right) f(qu) \\ &\le pb(x^*)^p - qc(x^*)^q \\ &= g(0) \end{aligned} \tag{23}$$

and

$$g(u) = \frac{1}{u} \ln\left[a + b(x^* \, \mathrm{e}^u)^p - c(x^* \, \mathrm{e}^u)^q\right] > 0 . \tag{24}$$

Also, as g is increasing for $u \geq 0$, it follows that

$$g(0) \leq g(u) \leq g\big(\ln(M_0)\big) \qquad \text{for} \quad 0 \leq u \leq \ln M_0 . \tag{25}$$

Therefore, by using (21), (23), (24) and (25) and because g is continuous, we see that (22) holds.

Next, define the nonnegative function

$$V(y_n) = \Big[y_n - \sum_{i=n-k}^{n} g(y_i) y_i\Big]^2 + \sum_{i=n-k}^{n} \Big[g(y_{i+k+1}) \sum_{j=i}^{n} g(y_j) y_j^2\Big]$$

for $n \geq N_0$. Calculating the difference of V along the solutions of (19) and using the fact that $2y_i y_{n+1} \leq y_i^2 + y_{n+1}^2$, we see that

$$\begin{aligned}
&V(y_{n+1}) - V(y_n) \\
&= \Big[y_{n+1} - \sum_{i=n-k+1}^{n+1} g(y_i) y_i\Big]^2 - \Big[y_n - \sum_{i=n-k}^{n} g(y_i) y_i\Big]^2 \\
&\quad + \sum_{i=n-k+1}^{n+1} \Big(g(y_{i+k+1}) \sum_{j=i}^{n+1} g(y_j) y_j^2\Big) - \sum_{i=n-k}^{n} \Big(g(y_{i+k+1}) \sum_{j=i}^{n} g(y_j) y_j^2\Big) \\
&= -\, g(y_{n+1}) y_{n+1} \Big[2y_{n+1} + g(y_{n+1}) y_{n+1} - 2 \sum_{i=n-k+1}^{n+1} g(y_i) y_i\Big] \\
&\quad + g(y_{n+1}) y_{n+1}^2 \sum_{i=n-k+1}^{n+1} g(y_{i+k+1}) - g(y_{n+1}) \sum_{i=n-k}^{n} g(y_i) y_i^2 \\
&= -\, 2g(y_{n+1}) y_{n+1}^2 + 2g(y_{n+1}) y_{n+1} \sum_{i=n-k+1}^{n+1} g(y_i) y_i \\
&\quad - g(y_{n+1}) \sum_{i=n-k+1}^{n+1} g(y_i) y_i^2 - g^2(y_{n+1}) y_{n+1}^2 \\
&\quad + g(y_{n+1}) y_{n+1}^2 \sum_{i=n-k+1}^{n+1} g(y_{i+k+1}) + g^2(y_{n+1}) y_{n+1}^2 - g(y_{n+1}) g(y_{n-k}) y_{n-k}^2 \\
&\leq -\, g(y_{n+1}) y_{n+1}^2 \Big[2 - \sum_{i=n-k+1}^{n+1} g(y_i) - \sum_{i=n-k+1}^{n+1} g(y_{i+k+1})\Big] .
\end{aligned}$$

This, in view of (22), yields

$$V(y_{n+1}) - V(y_n) \leq -2\big[1 - g(\ln M_0)(k+1)\big] g(y_{n+1}) y_{n+1}^2 .$$

By summing both side of this inequality we see that for $n \geq N_0$,

$$V(y_{n+1}) + 2\big[1 - g(\ln M_0)(k+1)\big] \sum_{i=N_0+1}^{n+1} g(y_i)y_i^2 \leq V(y_{N_0}) .$$

Hence,

$$\sum_{n=1}^{\infty} g(y_n)y_n^2 < \infty ,$$

which, in view of (22), implies that

$$\sum_{n=1}^{\infty} y_n^2 < \infty . \tag{26}$$

Clearly, this fact implies that (20) holds. The proof is complete. □

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REFERENCES

[1] KOCIC, V. L.—LADAS, G.: *Global Behavior of Nonlinear Difference Equations of Higher Order with Application*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[2] CHEN, MING-PO—YU, J. S.: *Oscillation and global attractivity in a delay logistic difference equation*, J Differ. Equations Appl. **1** (1995), 227–237.

[3] SO, J. W-H—YU, J. S.: *On the stability and uniform persistence of a discrete model of Nicholson's blowflies*, J. Math. Anal. Appl **193** (1995), 233–244.

[4] LIU, BING—YAN, JURANG: *Oscillatory and asymptotic behavior of second order nonlinear difference equations*, Proc. Edinburgh Math. Soc. (2) **39** (1996), 523–533.

[5] ZHOU, XIAOLIANG—YAN, JURANG: *Oscillatory and asymptotic properties of higher order nonlinear difference equations*, Nonlinear Anal. **31** (1998), 493–502.

[6] HE, XUEZHONG: *Oscillatory and asymptotic behaviour of second order nonlinear difference equations*, J. Math. Anal. Appl. **175** (1993), 482–498.

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