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ON ISOMETRIES OF NON-ABELIAN LATTICE ORDERED GROUPS

JÁN JAKUBÍK

K. L. Swamy [5] defined an isometry of an abelian lattice ordered group $G$ to be a one-to-one mapping $f$ of $G$ onto $G$ such that the relation

$$(1) \quad |x - y| = |f(x) - f(y)|$$

for each $x, y \in G$ is valid. Cf. also Swamy [6].

In [3] the isometry of a lattice ordered group $G$ (that need not be abelian) has been defined as a one-to-one mapping of $G$ onto $G$ fulfilling the relation (1) and the relation

$$(2) \quad f([x \land y, x \lor y]) = [f(x) \land f(y), f(x) \lor f(y)]$$

for each $x, y \in G$.

It has been shown in [3] that if $G$ is abelian and if $f$ is a one-to-one mapping of $G$ onto $G$, then (1) implies (2). Thus in the case of abelian lattice ordered groups the above definitions of isometry are equivalent.

An isometry $f$ is said to be an 0-isometry if $f(0) = 0$. Each isometry can be represented as a composition of an 0-isometry and a translation. In [3] it has been shown that there exist a one-to-one correspondence between 0-isometries of $G$ and direct factors of $G$.

In this note it will be shown that for each lattice ordered group $G$ and each one-to-one mapping $f$ of $G$ onto $G$ the implication (1) $\Rightarrow$ (2) holds. Hence the condition (2) can be cancelled in the definition of isometry of a (non-abelian) lattice ordered group.

For the terminology and denotations, cf. Conrad [1] and Fuchs [2]. Let $G$ be a lattice ordered group. Let $f$ be a one-to-one mapping of $G$ onto $G$ fulfilling (1).

1. Lemma. Let $a, b \in G$, $a \leq b$, $x \in G$. Then the following conditions are equivalent:

(i) $x \in [a, b]$;

(ii) $|a - b| = |b - x| + |x - a|$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Suppose that (ii) is valid. Denote $a_1 = a \land x$, $b_1 = b \land x$, $a_2 = a \lor x$, $b_2 = b \lor x$, $r = a \lor b_1 = b \land a_2$. Then

$$|a - b| = |b - a| = b - a = (b - r) + (r - a) =$$

$$= (b_2 - a_2) + (b_1 - a_1),$$
\[ |b - x| = b_2 - b_1 = (b_2 - a_2) + (a_2 - b_1), \]
\[ |x - a| = a_2 - a_1 = (a_2 - b_1) + (b_1 - a_1). \]

From this and from (ii) we obtain \( a_2 - b_1 = 0 \), whence \( x = r \), and thus \( x \in [a, b] \).

2. Lemma. Let \( a, b \in G \), \( a \leq b \), \( u = f^{-1}(f(a) \land f(b)) \), \( v = f^{-1}(f(a) \lor f(b)) \). Then \( u \land v = a \), \( u \lor v = b \).

Proof. We have \( |f(a) - f(b)| = f(b) - f(u) + f(a) - f(u) \), hence
\[ |f(a) - f(b)| = |f(b) - f(u)| + |f(u) - f(a)| \]
and thus \( |a - b| = |b - u| + |u - a| \). In view of Lemma 1 we get \( u \in [a, b] \). Similarly we obtain \( v \in [a, b] \). Then \( b - a = |b - a| = |f(b) - f(a)| = |f(v) - f(u)| = |v - u| = v \land u - v \lor u \). On the other hand, from \( a \leq u \lor v \leq u \lor v \leq b \) we obtain
\[ b - a = (b - u \lor v) + (u \lor v - u \land v) + (u \land v - a), \]
thus \( a = u \land v \), \( b = u \lor v \).

The relation (1) implies \( |x - y| = |f^{-1}(x) - f^{-1}(y)| \) (i.e., the mapping \( f^{-1} \) fulfils (1) as well). From this and from Lemma 1 we obtain immediately:

3. Lemma. Let \( a, b \in G \). Suppose that \( a \leq b \) and \( f(a) \leq f(b) \). Then \( f([a, b]) = [f(a), f(b)] \).

4. Lemma. Let \( a, b \in G \). Suppose that \( a \leq b \) and \( f(a) \leq f(b) \). Then \( f([a, b]) = [f(b), f(a)] \).

Proof. Since \( f^{-1} \) fulfils (1) it suffices to verify that \( f([a, b]) \subseteq [f(b), f(a)] \) is valid. Let \( x \in [a, b] \). According to Lemma 2 there are \( u, v \in G \) such that
\[ u \land v = a, \quad u \lor v = x, \]
\[ f(a) \land f(x) = f(u), \quad f(a) \lor f(x) = f(v). \]
Since \( v \in [a, b] \), we have \( |b - a| = |b - v| + |v - a| \), hence
\[ |f(b) - f(a)| = |f(b) - f(v)| + |f(v) - f(a)| \]
and thus
\[ f(a) - f(b) = f(v) - f(b) + f(v) - f(a), \]
\[ f(a) - f(b) = (f(v) - f(a)) + (f(a) - f(b)) + (f(v) - f(a)). \]
If \( f(v) \neq f(a) \), then \( f(v) - f(a) > 0 \) and hence
\[ f(a) - f(b) < (f(v) - f(a)) + (f(a) - f(b)) + (f(v) - f(a)), \]
which is a contradiction. Therefore \( f(v) = f(a) \). This implies \( f(x) = f(u) \) and thus \( f(x) \leq f(a) \).

The proof of the relation \( f(b) \leq f(x) \) is analogous.

By summarizing, we obtain:
5. **Lemma.** Let \( a, b \in G, a \leq b \). Suppose that \( f(a) \) and \( f(b) \) are comparable. Then \( f([a, b]) = [f(a) \wedge f(b), f(a) \vee f(b)] \).

Let \( M_1 \) be the set of all intervals \([p, q] \subseteq G\) with \( f(p) \leq f(q) \). Further let \( M_2 \) be the set of all intervals \([p_1, q_1] \subseteq G\) with \( f(q_1) \leq f(p_1) \). From Lemma 2 we obtain:

6. **Corollary.** Let \( a, b \in G, a \leq b \). There are elements \( u, v \in [a, b] \) such that \([a, v], [u, b] \in M_1, [u, a], [v, b] \in M_2\).

7. **Lemma.** Let \( a, b, x \in G, a \leq b, x \in [a, b] \). Let \( u, v \) be as in Lemma 2. Denote \( x \wedge v = a, x \vee u = b, x \wedge u = a_2, x \vee u = b_2 \). Then \([a_2, b_1] \in M_1\) and \([a_1, b_2] \in M_2\).

Proof. According to Corollary 6 there exists \( y \in [a_2, b_1] \) such that \([a_2, y] \in M_2 \) and \([y, b_1] \in M_1\). Then for \( y_1 = v \wedge y \) we have \( a \leq y_1 \leq v \), hence according to Lemma 5, \([a, y_1] \in M_1\). On the other hand, \( y_1 \in [a, y] \in M_2\), thus (again by Lemma 5) we obtain \([a, y_1] \in M_2\). Therefore \( a = y_1 \) and hence \( y = a_2 \), implying \([a_2, b_1] \in M_1\). Similarly, according to Corollary 6 there is \( z \in [a_1, b_2] \) with \([a_1, z] \in M_1, [z, b_2] \in M_2\). Put \( z \wedge u = z_1 \). Then \([a, z] \in M_1, [a, z_1] \subseteq [a, z], \) whence \([a, z_1] \subseteq M_1\). At the same time, \([a, z_1] \subseteq [a, u] \subseteq M_2\), thus \([a, z_1] \in M_2\). Hence \( z_1 = a \) and so \( z = a_1 \). Therefore \([a_1, b_2] \in M_2\).

8. **Lemma.** Let \( a, b \in G, a \leq b, x \in [a, b] \). Let \( u, v \) be as in Lemma 2. Then \( f(x) \in [f(u), f(v)] \).

Proof. Let \( a_1, a_2 \) be as in Lemma 7. According to Lemma 7 and Lemma 5 we have

\[
f(u) \leq f(a_2) \leq f(x) \leq f(a_1) \leq f(v).
\]

9. **Lemma.** Let \( a_1, b_1 \in G, a_1 \wedge b_1 = a, a_1 \vee b_1 = b \). Let \( u, v \) be as in Lemma 2. Then \( f(a_1) \wedge f(b_1) = f(u), f(a_1) \vee f(b_1) = f(v) \).

Proof. Put \( f(a_1) \wedge f(b_1) = u_1, f(a_1) \vee f(b_1) = v_1 \). According to Lemma 8 we have \( u_1, v_1 \in [f(u), f(v)] \). Assume that either \( f(u) < u_1 \) or \( v_1 < f(v) \). Then

\[
|f(b_1) - f(a_1)| = |b_1 - a_1| = |b - a| = |f(b) - f(a)| = f(v) - f(u) =
= (f(v) - v_1) + (v_1 - u_1) + (u_1 - f(u)) > v_1 - u_1 = |f(b_1) - f(a_1)|,
\]

which is a contradiction. Hence \( f(a_1) \wedge f(b_1) = f(u) \) and \( f(a_1) \vee f(b_1) = f(v) \).

10. **Lemma.** Let \( a_1, b_1 \in G \). Then \( f([a_1 \wedge b_1, a_1 \vee b_1]) \subseteq [f(a_1) \wedge f(b_1), f(a_1) \vee f(b_1)] \).

This is an immediate consequence of Lemma 8 and Lemma 9.

Since \( f^{-1} \) fulfills (1), by using Lemma 10 for the mapping \( f^{-1} \) we obtain:

11. **Corollary.** Let \( a_1, b_1 \in G \). Then \( f^{-1}([f(a_1) \wedge f(b_1), f(a_1) \vee f(b_1)]) \subseteq [a_1 \wedge b_1, a_1 \vee b_1] \).

By summarizing, Lemma 10 and Corollary 11 yield:

12. **Proposition.** Let \( G \) be a lattice ordered group. Let \( f \) be a one-to-one mapping of \( G \) onto \( G \) fulfilling the condition (1). Then \( G \) fulfills the condition (2) as well.
For which types of lattice ordered groups are the conditions (1) and (2) equivalent? A partial answer to this question is given by the following proposition:

13. Proposition. Let \( G \neq \{0\} \) be a complete and completely distributive lattice ordered group. Then the following conditions are equivalent:

(i) If \( f \) is a one-to-one mapping of \( G \) onto \( G \), then (1) \( \iff \) (2).

(ii) \( G \) is isomorphic with the additive group of all integers (with the natural linear order).

Proof. In view of Proposition 12, the relation (1) \( \iff \) (2) in (i) can be replaced by (2) \( \Rightarrow \) (1). The proof of the implication (ii) \( \Rightarrow \) (i) is easy. Assume that (i) is valid. Since \( G \) is complete, it can be expressed as \( G = A \times B \), where \( A \) is a singular lattice ordered group and \( B \) is a vector lattice (cf. e.g., Conrad [1]). For \( g \in G \) we denote by \( g(A) \) and \( g(B) \) the component of \( g \) in \( A \) or \( B \), respectively. For each \( g \in G \) we put

\[
f(g) = g(A) + \frac{1}{2} g(B).
\]

Then \( f \) is a one-to-one mapping of \( G \) onto \( G \) fulfilling the condition (2). If \( B \neq \{0\} \), then \( f \) would fail to fulfill the condition (1); thus \( B = \{0\} \) and so \( G = A \).

Let \( 0 < s \) be a singular element of \( G \). Then \([0, s]\) is a Boolean algebra. Since \( G \) is complete and completely distributive, the Boolean algebra \([0, s]\) is complete and completely distributive. Hence \([0, s]\) is atomic. Therefore for each \( 0 < g \in G \) there exists \( a_i \in G \) such that \( a_i \leq g \) and \( a_i \) covers \( 0 \) in \( G \). Let \( \{a_i\}_{i \in I} \) be the set of all elements of \( G \) covering \( 0 \). For each \( i \in I \) let \( A_i = \{a_i\}^{\infty} \) be the polar of \( G \) generated by \( a_i \) (cf. Šik [7]). Then \( A_i \) is linearly ordered; since it is complete and contains an element covering \( 0 \), \( A_i \) is isomorphic with the additive group \( A_0 \) of all integers with the natural linear order. Since \( G \) is archimedean, \( A_i \) fails to be bounded and hence (cf. [4]) \( A_i \) is a direct factor of \( G \).

Assume that \( \text{card } I > 1 \). Choose \( i, j \in I, i \neq j \). Then \( G \) can be written as \( G = A_i \times A_j \times C \). For \( g \in G \) let \( g(A_k) \; (k \in \{i, j\}) \) and \( g(C) \) be the corresponding components of \( g \). Let \( \varphi_k \; (k \in \{i, j\}) \) be the isomorphism of \( A_k \) onto \( A_0 \). For each \( g \in G \) we set

\[
f_i(g) = \varphi_i^{-1}(\varphi_i(g(A_i))) + \varphi_j^{-1}(\varphi_j(g(A_j))) + g(C).
\]

Then \( f_i \) is a one-to-one mapping of \( G \) onto \( G \) fulfilling (2) that fails to fulfill the condition (1), which is a contradiction. Thus \( I \) is a one-element set, say \( I = \{i\} \). From this it follows that \( G = A_i \), completing the proof.

REFERENCES

ОБ ИЗОМЕТРИЯХ НЕАБЕЛЕВЫХ РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУПП

Я. Якубик

Резюме

Пусть $G$-решеточно упорядоченная группа. Предположим, что $f$ будет одно-однозначное отображение множества $G$ на $G$ такое, что $|f(x) - f(y)| = |x - y|$ для всех $x, y \in G$. (Отображения $f$ с этим свойством исследовал К. Л. Свами для случая абелевых решеточно упорядоченных групп.) В этой заметке доказано, что имеет место $f([x \land y, x \lor y]) = [f(x) \land f(y), f(x) \lor f(y)]$ для всех $x, y \in G$. 

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