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PERIODIC SOLUTIONS IN SYSTEMS AT RESONANCES WITH SMALL RELAY HYSTERESIS

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ABSTRACT. We study the existence of periodic solutions for certain systems of constant ordinary differential equations at resonances with relay hysteresis.

1. Introduction

In this paper, we deal with relay hysteresis [5]. So there is given a pair of real numbers $\alpha < \beta$ (thresholds) and a pair of real-valued continuous functions $h_o \in C([\alpha,\infty),\mathbb{R})$, $h_c \in C((-\infty,\beta],\mathbb{R})$ such that $h_o(u) \geq h_c(u)$ $\forall u \in [\alpha,\beta]$. Moreover, we suppose that h_o, h_c are bounded on $[\alpha,\infty)$, $(-\infty,\beta]$, respectively.

For a given continuous input $u(t)$, $t \geq t_0$, one defines the output $v(t) = f(u)(t)$ of the relay hysteresis operator as follows

$$f(u)(t) = \begin{cases} h_o\bigl(u(t)\bigr) & \text{if } u(t) \geq \beta\,, \\ h_c\bigl(u(t)\bigr) & \text{if } u(t) \leq \alpha\,, \\ h_o\bigl(u(t)\bigr) & \text{if } u(t) \in (\alpha,\beta) \text{ and } u\bigl(\tau(t)\bigr) = \beta\,, \\ h_c\bigl(u(t)\bigr) & \text{if } u(t) \in (\alpha,\beta) \text{ and } u\bigl(\tau(t)\bigr) = \alpha\,, \end{cases}$$

where $\tau(t) = \sup\{s:\ s \in [t_0,t],\ u(s) = \alpha \text{ or } u(s) = \beta\}$. If $\tau(t)$ does not exist (i.e. $u(\sigma) \in (\alpha,\beta)$ for $\sigma \in [t_0,t]$), then $f(u)(\sigma)$ is undefined and we have initially to set the relay open or closed when $u(t_0) \in (\alpha,\beta)$. Of course, when either $h_o(\beta) > h_c(\beta)$ or $h_o(\alpha) > h_c(\alpha)$ then $f(u)$ is generally discontinuous.

Electrical engineers are interested in the periodic behaviour of circuits with hysteresis. A circuit with a relay hysteresis could be modelled by

$$L_m y = f(y)\,,$$

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where L_m is an mth-order differential operator.

In this paper, in order to deal with much more general equations, we are interested in the periodic oscillations of systems given by

$$\dot{x} = Ax + \mu f(x_1)b, \tag{1.1}$$

where A is a constant $n \times n$ matrix, x_1 is the first component of $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ is a constant vector and $\mu \in \mathbb{R}$ is a small parameter. Similar systems are studied in [1] and [5]–[7].

In contrast to these papers, we assume that (1.1) is at resonance, i.e. $\dot{x} = Ax$ has a nonzero periodic solution. The aim of this paper is to find conditions ensuring the existence of periodic oscillations of (1.1) for $\mu \neq 0$ small. Since (1.1) is generally discontinuous, we consider this as a differential inclusion. The method used in this paper is a combination of [3] and [6], i.e. we apply to (1.1) a Lyapunov-Schmidt decomposition procedure together with topological degree theory for multivalued mappings [2]. Periodically forced problems of (1.1) are also investigated. We end the paper with examples of unforced and forced third-order ordinary differential equations with a small relay hysteresis.

2. The existence of periodic solutions

We suppose that the following condition holds

i) $W = \{x \in \mathbb{R}^n : x = \mathrm{e}^A x\} \neq \{0\}$ and there is an $x_0 \in W$ such that $Ax_0 \neq 0$.

By [4] we have

$$W^* = \{x \in \mathbb{R}^n : x = \mathrm{e}^{-A^*} x\} \neq \{0\}, \qquad \dim W^* = \dim W = d > 1.$$

Moreover, the linear equation

$$\dot{x} = Ax + h(t), \qquad h \in L_2 = L_2([0,1], \mathbb{R}^n)$$

has a solution $x \in W^{1,\infty} = W^{1,\infty}([0,1], \mathbb{R}^n)$ satisfying $x(0) = x(1)$ if and only if

$$\forall\, w \in W^* \qquad \int_0^1 \langle h(s), \mathrm{e}^{-A^* s} w \rangle \, \mathrm{d}s = 0.$$

Here $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$. The norm on $W^{1,\infty}$ is denoted by $\| \cdot \|$.

Let $x = \mathcal{K}h$ be the unique such solution satisfying

$$\forall\, z \in W \qquad \int_0^1 \langle x(s), \mathrm{e}^{As} z \rangle \, \mathrm{d}s = 0.$$

We put

$$X = \left\{ x \in W^{1,\infty} : \int_0^1 \langle x(s), e^{As} z\rangle \, ds = 0 \quad \forall z \in W \right\}.$$

Let

$$\Pi \colon L_2 \to \left\{ h \in L_2 : \int_0^1 \langle h(s), e^{-A^* s} w\rangle \, ds = 0 \quad \forall w \in W^* \right\}$$

be the orthogonal projection. Of course, $\mathcal{K}\colon \operatorname{im}\Pi \to X$ is linear and bounded.

By taking a basis $\{w_1, \ldots, w_d\}$ of W, we put $\gamma_i(t) = e^{At} w_i$, $i = 1, \ldots, d$. Let

$$\gamma(\theta, t) = \sum_{i=1}^{d-1} \theta_i \gamma_i(t), \qquad \theta_i \in \mathbb{R}.$$

By i), we obtain that $d \geq 2$ and $\{w_1, \ldots, w_d\}$ can be chosen such that any solution of $\dot{x} = Ax$, $x(0) \in W$ has the form $\gamma(\theta, t+\omega)$, $\omega \in \mathbb{R}$, $\theta \in \mathbb{R}^{d-1}$. From now on, $\{w_1, \ldots, w_d\}$ will be such a basis. Let $\gamma_1(\theta, t)$ be the first component of $\gamma(\theta, t)$. We need the following conditions to hold:

ii) There is an open bounded subset $\emptyset \neq \mathcal{O} \subset \mathbb{R}^{d-1}$ such that $\forall \theta \in \mathcal{O}$ and $\forall t_0 \in \mathbb{R}$

$$\gamma_1(\theta, t_0) = \alpha, \beta \implies \dot{\gamma}_1(\theta, t_0) \neq 0.$$

iii) $\forall \theta \in \mathcal{O} \quad \min_{t\in\mathbb{R}} \gamma_1(\theta, t) < \alpha$, $\max_{t\in\mathbb{R}} \gamma_1(\theta, t) > \beta$.

Now in (1.1) we make the following change of variables

$$x\big((1+\mu\omega)t\big) = \mu z(t) + \gamma(\theta, t), \qquad \omega \in \mathbb{R}.$$

The conditions ii) and iii) imply that if $z \in X$ satisfies $\|z\| \leq K$ and μ is sufficiently small, then $\mu z_1(t) + \gamma_1(\theta, t)$ crosses α and β strictly monotonically for arbitrary $\theta \in \mathcal{O}$.

We rewrite (1.1) as a differential inclusion of the form

$$\dot{x} - Ax \in \mu F(x_1) b, \tag{2.1}$$

where F is a multivalued mapping defined as follows

$$F(u)(t) = \begin{cases} f(u)(t) & \text{if } u(t) \neq \alpha, \beta, \\ h_c(\alpha) & \text{if } u(t) = \alpha,\ u(\tau(s)) = \alpha, \text{ for any } s < t \text{ near } t, \\ h_o(\beta) & \text{if } u(t) = \beta,\ u(\tau(s)) = \beta, \text{ for any } s < t \text{ near } t, \\ [h_c(\alpha), h_o(\alpha)] & \text{if } u(t) = \alpha,\ u(\tau(s)) = \beta, \text{ for any } s < t \text{ near } t, \\ [h_c(\beta), h_o(\beta)] & \text{if } u(t) = \beta,\ u(\tau(s)) = \alpha, \text{ for any } s < t \text{ near } t. \end{cases}$$

ii) and iii) imply that if $u(t) = \mu z_1(t) + \gamma_1(\theta, t)$ with $z \in X$ bounded and μ sufficiently small, then $F(u)$ is well-defined. By a solution of a differential inclusion in this paper we mean a function which is absolute continuous and which satisfies that differential inclusion almost everywhere.

Hence (2.1) has the form

$$\dot{z}(t) - Az(t) \in (1+\mu\omega)F\big(\mu z_1 + \gamma_1(\theta,\cdot)\big)(t)b + \omega A\big(\mu z(t) + \gamma(\theta,t)\big)\,. \tag{2.2}$$

By taking the mapping

$$\begin{aligned} &G(z,\omega,\theta,\mu,\lambda) = \\ &= \big\{h \in L_2 : \text{ satisfying the relation} \\ &\qquad h(t) \in (1+\lambda\mu\omega)F\big(\lambda\mu z_1 + \gamma_1(\theta,\cdot)\big)(t)b + \omega A\big(\lambda\mu z(t) + \gamma(\theta,t)\big) \\ &\qquad\qquad \text{a.e. on } [0,1]\big\}\,, \end{aligned}$$

(2.2) has the form

$$\dot{z} - Az \in G(z,\omega,\theta,\mu,1)\,. \tag{2.3}$$

Using Π and $\mathcal{K}$, we rewrite (2.3) as follows

$$\begin{cases} 0 \in H(z,\omega,\theta,\mu,1) \\ H(z,\omega,\theta,\mu,\lambda) = \big\{(z - \lambda\mathcal{K}\Pi h, \mathcal{L}h) : h \in G(z,\omega,\theta,\mu,\lambda)\big\}\,, \end{cases} \tag{2.4}$$

where $\mathcal{L}\colon L_2 \to \mathbb{R}^d$ is defined by

$$\mathcal{L}h = \left(\int_0^1 \langle h(s), \mathrm{e}^{-A^*s}\,\tilde{w}_1\rangle\,\mathrm{d}s, \dots, \int_0^1 \langle h(s), \mathrm{e}^{-A^*s}\,\tilde{w}_d\rangle\,\mathrm{d}s\right)$$

for a basis $\{\tilde{w}_1,\dots,\tilde{w}_d\}$ of W^*.

Since f is bounded in (1.1), for arbitrary $\Gamma > 0$ there exist $\mu_0 > 0$ and $K > 0$ such that

$$\begin{gathered} \|\mathcal{K}\Pi h\| \le K \qquad \text{for arbitrary} \quad h \in G(z,\omega,\theta,\mu,\lambda)\,, \\ \|z\| \le K+1\,, \qquad |\omega| \le \Gamma\,, \qquad \theta \in \mathcal{O}\,, \qquad |\mu| \le \mu_0\,, \qquad \lambda \in [0,1]\,. \end{gathered}$$

Moreover, if μ_0 is sufficiently small then by ii) and iii), the mapping

$$H\colon \Omega \times [-\mu_0,\mu_0] \times [0,1] \to 2^{X\times\mathbb{R}^d} \tag{2.5}$$

is well-defined and singlevalued, where

$$\Omega = \big\{(z,\omega,\theta) \in X \times \mathbb{R}^d : \|z\| < K+1\,,\ (\omega,\theta) \in \mathcal{B}\big\}$$

and $\mathcal{B}$ is an open bounded non-empty subset satisfying $\overline{\mathcal{B}} \subset \mathbb{R} \times \mathcal{O}$.

The arguments of [6; pp. 677–678] imply that $H\colon \Omega \times [-\mu_0,\mu_0] \times [0,1] \to X \times \mathbb{R}^d$ is continuous and also compact. Similarly, the mapping given by

$$\begin{gathered} M\colon \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d\,, \qquad M(\omega,\theta) = \mathcal{L}h \\ h(t) = F\big(\gamma_1(\theta,\cdot)\big)(t)b + \omega A\gamma(\theta,t) \qquad \text{a.e. on} \quad [0,1] \end{gathered} \tag{2.6}$$

is continuous.

THEOREM 2.1. *Assume that* i) – iii) *hold. If there is a non-empty open bounded set* $\mathcal{B}$ *such that* $\overline{\mathcal{B}} \subset \mathbb{R} \times \mathcal{O}$ *and*

(i) $0 \notin M(\partial\mathcal{B})$,
(ii) $\deg(M, \mathcal{B}, 0) \neq 0$,

where deg *is the Brouwer degree and* M *is given by* (2.6), *then there are constants* $K_1 > 0$ *and* $\mu_0 > 0$ *such that for arbitrary* $|\mu| < \mu_0$, *there exist* $(\omega_\mu, \theta_\mu) \in \mathcal{B}$ *and a* $(1+\mu\omega_\mu)$*-periodic solution* x_μ *of* (1.1) *satisfying*

$$\sup_{t\in\mathbb{R}} \big|x_\mu(t) - \gamma(\theta_\mu,\, t/(1+\mu\omega_\mu))\big| \le K_1|\mu|\,.$$

Proof. First we show

$$0 \notin H\big(\partial\Omega \times [-\mu_0, \mu_0] \times [0,1]\big)$$

for arbitrary $\mu_0 > 0$ sufficiently small. Assume the contrary. Then there exist

$$[0,1] \ni \lambda_i \to \lambda_0\,, \quad \|z_i\| \le K+1\,, \quad \mu_i \to 0\,, \qquad i \in \mathbb{N}$$
$$\partial\mathcal{B} \ni (\omega_i, \theta_i) \to (\omega_0, \theta_0) \in \partial\mathcal{B}\,, \qquad h_i \in G(z_i, \omega_i, \theta_i, \mu_i, \lambda_i)$$

such that

$$\mathcal{L}h_i = 0\,.$$

We can assume that $z_i \to z$ in $C([0,1], \mathbb{R}^n)$ and h_i tends weakly to some $h_0 \in L^2$. Then by applying the standard arguments (see the proof of [2; Remarks 5.5.1]), we obtain

$$h \in G(z, \omega_0, \theta_0, 0, \lambda_0) \qquad \text{and} \qquad \mathcal{L}h_0 = 0\,,$$

i.e. $0 = M(\omega_0, \theta_0)$ for some $(\omega_0, \theta_0) \in \partial\mathcal{B}$. This contradicts (i) of this theorem.

Consequently, we compute for μ sufficiently small

$$\begin{aligned}\deg\big(H(\cdot\,,\cdot\,,\cdot\,,\mu,1), \Omega, 0\big) &= \deg\big(H(\cdot\,,\cdot\,,\cdot\,,\mu,0), \Omega, 0\big)\\ &= \deg(M, \mathcal{B}, 0) \neq 0\,.\end{aligned}$$

Thus, (2.4) has a solution $(z, \omega, \theta) \in \Omega$ for arbitrary sufficiently small μ. The proof is finished. □

Now we return to the differential equation

$$\begin{gathered} L_m y = \sum_{i=0}^{m} a_i y^{(i)} = \mu f(y)\,, \\ a_i \in \mathbb{R}, \qquad a_m = 1\,, \qquad y^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} y\,. \end{gathered} \tag{2.7}$$

Of course, (2.7) can be rewritten in the form of (1.1). We put

$$L_m^* y = \sum_{i=0}^{m} (-1)^i a_i y^{(i)} .$$

Let $\phi_1, \dots, \phi_d$, respectively $\psi_1, \dots, \psi_d$, be a basis of the space of all 1-periodic solutions of $L_m y = 0$, respectively $L_m^* y = 0$. We suppose that i)–iii) hold for (2.7) and also ϕ_d is non-constant. A tedious computation shows that the mapping (2.6) for (2.7) of the form $M\colon \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d$ is given by

$$\begin{aligned} M(\omega, \theta) &= \left(\int_0^1 h(s)\psi_1(s)\, ds, \dots, \int_0^1 h(s)\psi_d(s)\, ds \right), \\ h(t) &= F\big(\eta(\theta, \cdot)\big)(t) + \omega \sum_{i=1}^{m} i a_i \eta^{(i)}(\theta, t) \qquad \text{a.e. on} \quad [0,1], \end{aligned} \tag{2.8}$$

where $\eta(\theta, t) = \sum_{i=1}^{d-1} \theta_i \phi_i(t)$. Theorem 2.1 implies the following result.

THEOREM 2.2. *Assume that ϕ_d is non-constant and that the following conditions hold:*

a) *There is an open bounded subset $\emptyset \neq \mathcal{O} \subset \mathbb{R}^{d-1}$ such that $\forall\, \theta \in \mathcal{O}$ and $\forall\, t_0 \in \mathbb{R}$*

$$\eta(\theta, t_0) = \alpha,\ \beta \implies \dot{\eta}(\theta, t_0) \neq 0 .$$

b) $\forall\, \theta \in \mathcal{O} \qquad \min_{t \in \mathbb{R}} \eta(\theta, t) < \alpha,\ \max_{t \in \mathbb{R}} \eta(\theta, t) > \beta .$

If there is a non-empty open bounded set $\mathcal{B}$ such that $\overline{\mathcal{B}} \subset \mathbb{R} \times \mathcal{O}$ and

(i) $0 \notin M(\partial \mathcal{B})$,

(ii) $\deg(M, \mathcal{B}, 0) \neq 0$,

where M is given by (2.8), then there exist constants $K_1 > 0$ and $\mu_0 > 0$ such that for arbitrary $|\mu| < \mu_0$, there exist $(\omega_\mu, \theta_\mu) \in \mathcal{B}$ and an $(1 + \mu\omega_\mu)$-periodic solution y_μ of (2.7) satisfying

$$\sup_{t \in \mathbb{R}} \big| y_\mu(t) - \eta\big(\theta_\mu, t/(1 + \mu\omega_\mu)\big) \big| \leq K_1 |\mu| .$$

The results of [3] can be modified to give existence results of subharmonic solutions of nonautonomous periodic versions of (1.1) expressed in the following theorems.

THEOREM 2.3. *Consider*

$$\dot{x} = Ax + \mu\big(f(x_1)b + q(t)\big) , \tag{2.9}$$

where $q \in C(\mathbb{R}, \mathbb{R}^n)$ *is* 1*-periodic and* A, f, b *are given in* (1.1). *Assume that* i)–iii) *hold. If there is a non-empty open bounded set* $\mathcal{B}$ *such that* $\overline{\mathcal{B}} \subset \mathbb{R} \times \mathcal{O}$ *and*

(i) $0 \notin M(\partial\mathcal{B})$,
(ii) $\deg(M, \mathcal{B}, 0) \neq 0$,

where M *is given by*

$$\begin{aligned} &M: \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d, \qquad M(\omega, \theta) = \mathcal{L}h, \\ &h(t) = F\big(\gamma_1(\theta, \cdot)\big)(t)b + q(t+\omega) \qquad a.e. \text{ on } \quad [0,1], \end{aligned} \tag{2.10}$$

then there exist constants $K_1 > 0$ *and* $\mu_0 > 0$ *such that for arbitrary* $|\mu| < \mu_0$, *there are* $(\omega_\mu, \theta_\mu) \in \mathcal{B}$ *and a* 1*-periodic solution* x_μ *of* (2.9) *satisfying*

$$\sup_{t \in \mathbb{R}} |x_\mu(t) - \gamma(\theta_\mu, t - \omega_\mu)| \le K_1 |\mu|.$$

THEOREM 2.4. *Consider*

$$L_m y = \mu\big(f(y) + q(t)\big), \tag{2.11}$$

where L_m, f *are given in* (2.7) *and* $q \in C(\mathbb{R}, \mathbb{R})$ *is* 1*-periodic. Assume that* ϕ_d *is non-constant, and* a) *and* b) *of Theorem 2.2 hold. If there is a non-empty open bounded set* $\mathcal{B}$ *such that* $\overline{\mathcal{B}} \subset \mathbb{R} \times \mathcal{O}$ *and*

(i) $0 \notin M(\partial\mathcal{B})$,
(ii) $\deg(M, \mathcal{B}, 0) \neq 0$,

where $M: \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d$ *is given by*

$$\begin{aligned} &M(\omega, \theta) = \left(\int_0^1 h(s)\psi_1(s)\, \mathrm{d}s, \ldots, \int_0^1 h(s)\psi_d(s)\, \mathrm{d}s \right), \\ &h(t) = F\big(\eta(\theta, \cdot)\big)(t) + q(t+\omega) \qquad a.e. \text{ on } \quad [0,1], \end{aligned} \tag{2.12}$$

then there exist constants $K_1 > 0$ *and* $\mu_0 > 0$ *such that for arbitrary* $|\mu| < \mu_0$, *there are* $(\omega_\mu, \theta_\mu) \in \mathcal{B}$ *and a* 1*-periodic solution* y_μ *of* (2.11) *satisfying*

$$\sup_{t \in \mathbb{R}} |y_\mu(t) - \eta(\theta_\mu, t - \omega_\mu)| \le K_1 |\mu|.$$

Remark 2.5. The boundedness of h_o and h_c on $[\alpha, \infty)$, respectively $(-\infty, \beta]$, is not essential.

Remark 2.6. The smallness of μ_0 in Theorems 2.1–2.4 can be estimated.

3. Examples

Let us consider the problem

$$\dddot{y} + \ddot{y} + \dot{y} + y = \mu f(y)\,, \tag{3.1}$$

where f is of the form

$$\alpha = -\delta\,, \qquad \beta = \delta\,, \qquad \delta > 0\,, \qquad h_o = g + p\,, \qquad h_c = g - p$$

with $p > 0$ constant and $g \in C(\mathbb{R}, \mathbb{R})$. We apply Theorem 2.2. Now we have

$$\phi_1(t) = \psi_1(t) = \sin t\,, \qquad \phi_2(t) = \psi_2(t) = \cos t\,, \qquad \eta(\theta, t) = \theta \sin t\,.$$

By taking $\mathcal{O} = (\delta, \infty)$, the conditions a) and b) of Theorem 2.2 are satisfied. Let $t_0 = \arcsin \frac{\delta}{\theta}$ for $\theta \in \mathcal{O}$. We compute (2.8) for this case

$$M(\omega, \theta) = \big(M_1(\omega, \theta), M_2(\omega, \theta)\big)\,, \tag{3.2}$$

where

$$\begin{aligned}
M_1(\omega, \theta) &= \int_0^{2\pi} \omega(\theta \cos t - 2\theta \sin t - 3\theta \cos t) \sin t \; dt + \int_{t_0}^{t_0+\pi} \big(g(\theta \sin t) + p\big) \sin t \; dt \\
&\qquad + \int_{t_0+\pi}^{t_0+2\pi} \big(g(\theta \sin t) - p\big) \sin t \; dt \\
&= -2\pi\theta\omega + \int_0^{2\pi} g(\theta \sin t) \sin t \; dt + 4p \cos t_0 \\
&= -2\pi\theta\omega + 4p\sqrt{1 - \frac{\delta^2}{\theta^2}} + \int_0^{2\pi} g(\theta \sin t) \sin t \; dt\,,
\end{aligned}$$

$$\begin{aligned}
M_2(\omega, \theta) &= \int_0^{2\pi} \omega(\theta \cos t - 2\theta \sin t - 3\theta \cos t) \cos t \; dt + \int_{t_0}^{t_0+\pi} \big(g(\theta \sin t) + p\big) \cos t \; dt \\
&\qquad + \int_{t_0+\pi}^{t_0+2\pi} \big(g(\theta \sin t) - p\big) \cos t \; dt \\
&= -2\pi\theta\omega + \int_0^{2\pi} g(\theta \sin t) \cos t \; dt - 4p \sin t_0 \\
&= -2\pi\theta\omega - 4\frac{\delta p}{\theta}\,.
\end{aligned}$$

We have the following result.

THEOREM 3.1. *If there exist numbers $\delta < a_1 < a_2$ such that the numbers*

$$4p\left(\frac{\delta}{a_1} + \sqrt{1 - \frac{\delta^2}{a_1^2}}\right) + \int_0^{2\pi} g(a_1 \sin t) \sin t \, dt\,,$$

$$4p\left(\frac{\delta}{a_2} + \sqrt{1 - \frac{\delta^2}{a_2^2}}\right) + \int_0^{2\pi} g(a_2 \sin t) \sin t \, dt$$

have opposite signs, then there is a constant $K > 0$ such that for arbitrary sufficiently small μ there exist $\theta_\mu \in (a_1, a_2)$, $\omega_\mu \in (3D, D)$, $D = -\frac{\delta p}{2\pi}\left(\frac{1}{a_2^2} + \frac{1}{a_1^2}\right)$ and a $2\pi(1+\mu\omega_\mu)$-periodic solution y_μ of (3.1) satisfying

$$\sup_{t\in\mathbb{R}} \left| y_\mu(t) - \theta_\mu \sin \frac{t}{1+\mu\omega_\mu} \right| \le K|\mu|\,.$$

P r o o f . It is sufficient to verify (i) and (ii) of Theorem 2.2 when M is given by (3.2) and $\mathcal{B} = (3D, D) \times (a_1, a_2)$.

We put (3.2) in the homotopy

$$M(\omega,\theta,\lambda) = \big(M_1(\omega,\theta,\lambda), M_2(\omega,\theta,\lambda)\big)\,, \qquad \lambda \in [0,1]\,,$$

where

$$M_1(\omega,\theta,\lambda) = -2\pi\theta\big(\omega - 2(1-\lambda)D\big) + 4p\sqrt{1 - \frac{\delta^2}{\theta^2}} + \int_0^{2\pi} g(\theta \sin t) \sin t \, dt + 4\frac{\delta p}{\theta} - \lambda 4 \frac{\delta p}{\theta}\,,$$

$$M_2(\omega,\theta,\lambda) = -2\pi\theta\big(\omega - 2(1-\lambda)D\big) - \lambda 4\frac{\delta p}{\theta}\,.$$

It is clear that

$$\forall\, \lambda \in [0,1] \quad M(\partial\mathcal{B},\lambda) \neq 0\,.$$

Consequently, we obtain

$$\deg\big(M(\cdot\,,\cdot\,,1),\mathcal{B},0\big) = -\deg\big(M_1(2D,\cdot\,,0),(a_1,a_2),0\big) \neq 0\,.$$

The proof is finished by using Theorem 2.2. □

Let us take $g(x) = c_1 x + c_2$ with $c_{1,2}$ constant. We compute

$$4p\left(\frac{\delta}{\theta} + \sqrt{1 - \frac{\delta^2}{a\theta^2}}\right) + \int_0^{2\pi} (c_1 \theta \sin t + c_2) \sin t \, dt = 4p\left(\frac{\delta}{\theta} + \sqrt{1 - \frac{\delta^2}{\theta^2}}\right) + c_1\theta\pi\,.$$

COROLLARY 3.2. *If $g(x) = c_1 x + c_2$ in (3.1) with constant $c_{1,2}$ such that $c_1 < 0$ and $4p > -c_1\delta\pi$, then the conclusion of Theorem 3.1 holds.*

Proof. In Theorem 3.1, it is enough to take $a_1 > \delta$ near to δ and $a_2 > a_1$ sufficiently large. □

Now we consider a forced problem of (3.1)

$$\dddot{y} + \ddot{y} + \dot{y} + y = \mu\big(f(y) + \sin t\big)\,, \tag{3.3}$$

where f is given in (3.1). According to Theorem 2.4 and the computations for (3.2), the mapping (2.12) for (3.3) has the form

$$M(\omega,\theta) = \big(M_1(\omega,\theta), M_2(\omega,\theta)\big)\,, \tag{3.4}$$

where

$$\begin{aligned} M_1(\omega,\theta) &= 4p\sqrt{1-\frac{\delta^2}{\theta^2}} + \int_0^{2\pi} g(\theta\sin t)\sin t\ dt + \int_0^{2\pi} \sin(t+\omega)\sin t\ dt \\ &= 4p\sqrt{1-\frac{\delta^2}{\theta^2}} + \int_0^{2\pi} g(\theta\sin t)\sin t\ dt + \pi\cos\omega\,, \end{aligned}$$

$$\begin{aligned} M_2(\omega,\theta) &= -4\frac{\delta p}{\theta} + \int_0^{2\pi} \sin(t+\omega)\cos t\ dt \\ &= -4\frac{\delta p}{\theta} + \pi\sin\omega\,. \end{aligned}$$

Assume that $4p = \pi$ and $\pi/2 < \omega < \pi$. Then the equations $M_1 = 0$, $M_2 = 0$ are equivalent to

$$\int_0^{2\pi} g\Big(\frac{\sin t}{\sin\omega}\delta\Big)\sin t\ dt = 0\,.$$

Theorem 2.4 implies the following result.

THEOREM 3.3. *Assume that $4p = \pi$ and $g \in C^1(\mathbb{R},\mathbb{R})$. If the function*

$$\rho \mapsto \int_0^{2\pi} g(\delta\rho\sin t)\sin t\ dt$$

has a simple root $\rho_0 > 1$, then by putting $1/\rho_0 = \sin\omega_0$, $\pi/2 < \omega_0 < \pi$, there is a constant $K > 0$ such that for any μ sufficiently small there are (ω_μ, θ_μ) near to $(\omega_0, \delta\rho_0)$ and a 2π-periodic solution y_μ of (3.3) satisfying

$$\sup_{t\in\mathbb{R}} |y_\mu(t) - \theta_\mu\sin(t-\omega_\mu)| \le K|\mu|\,.$$

P r o o f. If $\rho_0 > 1$ is a simple root of $\rho \mapsto \int_0^{2\pi} g(\delta\rho\sin t)\sin t\, dt$, then $\theta_0 = \delta\rho_0$, $1/\rho_0 = \sin\omega_0$, $\pi/2 < \omega_0 < \pi$ is a simple zero of $M = 0$ given by (3.4), i.e. $M(\omega_0,\theta_0) = 0$ and $DM(\omega_0,\theta_0)$ is invertible. The proof is finished by Theorem 2.4 when $\mathcal{B}$ is taken as a small open neighbourhood of (ω_0,θ_0). □

Let us take $g(x) = c_1x^3 + c_2x$ with $c_{1,2}$ constant. Then

$$\int_0^{2\pi} g(\delta\rho\sin t)\sin t\, dt = \frac{3}{4}\pi c_1\delta^3\rho^3 + \pi\delta c_2\rho\,.$$

Theorem 3.3 gives the next result.

COROLLARY 3.4. *Assume that* $4p = \pi$. *If* $g(x) = c_1x^3 + c_2x$ *in* (3.1) *with constant* $c_{1,2}$ *such that* $c_1c_2 < \frac{-3}{4}c_1^2\delta^2$, *then the conclusion of Theorem* 3.3 *holds.*

P r o o f. The assumption $c_1c_2 < \frac{-3}{4}c_1^2\delta^2$ implies the existence of a simple root $\rho_0 > 1$ of the equation

$$\frac{3}{4}\pi c_1\delta^3\rho^3 + \pi\delta c_2\rho = 0\,.$$

□

Now we assume that $g(x) = c_1x$ with constant $c_1 > 0$ in (3.3). Then (3.4) has the form

$$M_1(\omega,\theta) = 4p\sqrt{1-\frac{\delta^2}{\theta^2}} + \pi\cos\omega + c_1\theta\pi\,,$$
$$M_2(\omega,\theta) = -4\frac{\delta p}{\theta} + \pi\sin\omega\,.$$

By assuming $\pi > 4p$, the equation $M(\omega,\theta) = 0$ with $\theta > \delta$ and $\pi/2 < \omega < \pi$ is equivalent to

$$4p\sqrt{1-\frac{\delta^2}{\theta^2}} - \pi\sqrt{1-\frac{16\delta^2p^2}{\theta^2\pi^2}} + c_1\theta\pi = 0\,,$$

i.e.

$$8\pi c_1p\sqrt{\theta^2-\delta^2} + c_1^2\theta^2\pi^2 = \pi^2 - 16p^2\,. \tag{3.5}$$

If $\pi^2 - 16p^2 > c_1^2\delta^2\pi^2$, then (3.5) has a unique simple root

$$\theta_0 = \sqrt{\left(\frac{-4p + \pi\sqrt{1-\delta^2c_1^2}}{c_1\pi}\right)^2 + \delta^2}\,. \tag{3.6}$$

Like for Corollary 3.4, we obtain

THEOREM 3.5. *Assume that* $g(x) = c_1 x$ *with constant* $c_1 > 0$ *such that* $\pi^2 - 16p^2 > c_1^2\delta^2\pi^2$. *Then there exists a constant* $K > 0$ *such that for arbitrary sufficiently small* μ *there exist* (ω_μ, θ_μ) *near to* (ω_0, θ_0) *given by* (3.6) *and* $\pi/2 < \omega_0 < \pi$, $\sin\omega_0 = \frac{4\delta p}{\pi\theta_0}$, *and a* 2π*-periodic solution* y_μ *of* (3.3) *satisfying*

$$\sup_{t\in\mathbb{R}} |y_\mu(t) - \theta_\mu \sin(t - \omega_\mu)| \le K|\mu| .$$

Similarly we have

THEOREM 3.6. *Assume that* $g(x) = c_1 x$ *with constant* $c_1 < 0$ *such that* $16p^2(1 - c_1^2\delta^2) > \pi^2$. *Then there exists a constant* $K > 0$ *such that for arbitrary sufficiently small* μ *there exist* (ω_μ, θ_μ) *near* (ω_0, θ_0) *given by*

$$\theta_0 = \frac{1}{\pi}\sqrt{\left(\frac{\pi - 4p\sqrt{1-\delta^2c_1^2}}{c_1}\right)^2 + 16\delta^2p^2}\,,$$

$$\sin\omega_0 = \frac{4\delta p}{\pi\theta_0}\,, \qquad \pi/2 < \omega_0 < \pi\,,$$

and a 2π*-periodic solution* y_μ *of* (3.3) *satisfying*

$$\sup_{t\in\mathbb{R}} |y_\mu(t) - \theta_\mu \sin(t - \omega_\mu)| \le K|\mu| .$$

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