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INVARIANT MEASURES ON LOCALLY COMPACT SPACES AND A TOPOLOGICAL CHARACTERIZATION OF UNIMODULAR LIE GROUPS

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The present paper considers the following problem. There is given a locally compact topological space X and a system $\mathscr{F}$ of homeomorphisms of X. Under what conditions on the system $\mathscr{F}$ does there exist a nonzero $\mathscr{F}$-invariant Borel measure on X? This problem was fully solved for compact spaces in Roberts' work [7].

For locally compact spaces the existence of an invariant measure is known in two cases. The first requires that the topology on X is induced by a uniformity, which has an $\mathscr{F}$-invariant base (see [2, p. 123] and [8]), the last work is based on weaker assumptions, but the idea of the construction is the same.

The second case requires X to be a homogeneous space of some locally compact group and a necessary and sufficient condition for the existence of an invariant measure is a classical result of H. Weyl (see [1, p. 172]).

We may assume that $\mathscr{F}$ is a group under the composition of mappings (see Proposition 2) and $\mathscr{F}$ is denoted by $\mathscr{G}$. This paper considers only groups $\mathscr{G}$ of homeomorphisms such that the orbit $\mathscr{G}(x)$ is a dense subset of X for every point $x \in X$. For such groups we define two set functions $(B:A)$ and $[B:A]$. The first denotes the minimal number of $\mathscr{G}$-images of the set A covering the set B. The second is the maximal number of pairwise disjoint $\mathscr{G}$-images of the set A which are contained in the set B. The main result of this paper, Theorem 12, states that the existence of $\alpha \in (0, 1)$ such that the inequalities $\alpha(U:A) \leq [U:A] \leq (U:A)$ are satisfied for sufficiently many Borel sets guarantees the existence and the uniqueness of a nonzero $\mathscr{G}$-invariant measure.

The right inequality, denoted by (N), is a necessary condition for the existence of a $\mathscr{G}$-invariant measure. The left inequality is denoted by (C) and it has a complementary character with respect to the condition (N). Proposition 11 and Theorem 16 give nontrivial examples of groups of homeomorphisms which satisfy the condition (C).

The paper ends with a topological characterization of unimodular Lie groups (Theorem 19).

1. Definitions and preliminary results

For a locally compact space X the symbol $\mathscr{B}(X)$ denotes the minimal σ-ring containing all compact subsets of X. The members of $\mathscr{B}(X)$ are called Borel sets in X.

A set A is called bounded if its closure $\bar{A}$ is a compact subset of X.

A Borel measure m on the locally compact space X is a set function m: $\mathscr{B}(X) \to \langle 0, \infty \rangle$ such that

$$m(\emptyset) = 0,$$

$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ for every sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint Borel sets, $m(K) < \infty$ for every compact set $K \subset X$.

A Borel measure m on X is called regular if for every Borel set A

$$\begin{aligned} m(A) &= \sup\{m(K) : K \subset A,\ K \textit{ is compact}\} = \\ &= \inf\{m(U) : A \subset U,\ U \textit{ is open Borel}\}. \end{aligned}$$

A mapping F: $X \to X$ is measurable if $F^{-1}(A)$ is a Borel set for every Borel set $A \subset X$.

If the mapping F: $X \to X$ is measurable and m is a Borel measure on X, then m is called F-invariant if $m(A) = m(F^{-1}(A))$ for every Borel set $A \in \mathscr{B}(X)$.

If $\mathscr{F}$ is a system of measurable mappings of the space X, then a Borel measure is called $\mathscr{F}$-invariant if it is F-invariant for every $F \in \mathscr{F}$.

A system $\mathscr{G}$ of homeomorphisms of a locally space X is called a group of homeomorphisms if it is a group under the composition of mappings.

Proposition 1. *Let X be a locally compact space and F: $X \to X$ be a homeomorphism. Then F is measurable and a Borel measure m is F-invariant if and only if $m(A) = m(F(A))$ for every $A \in \mathscr{B}(X)$.*

Proposition 2. *Let X be a locally compact space, $\mathscr{F}$ be a system of homeomorphisms of X and $\mathscr{G}(\mathscr{F})$ be the group of homeomorphisms which is generated by the system $\mathscr{F}$. Every Borel measure m is $\mathscr{F}$-invariant if and only if it is $\mathscr{G}(\mathscr{F})$-invariant.*

Definition 3. *Let X be a locally compact space and $\mathscr{G}$ be a group of homeomorphisms of X. The group $\mathscr{G}$ is called transitive on X if for every $x, y \in X$ there exists $F \in \mathscr{G}$ such that $F(x) = y$. $\mathscr{G}$ is called minimal if for every $x \in X$ the set $\mathscr{G}(x) = \{y$: $\exists F \in \mathscr{G}$: $F(x) = y\}$ is dense in X.*

Proposition 4. *Every transitive group is minimal. The group $\mathscr{G}$ is minimal if and only if for every nonempty open subset $U \subset X$ the system $\{F(U) : F \in \mathscr{G}\}$ covers X.*

Definition 5. *Let X be a locally compact space and $\mathscr{G}$ be a group of homeomorphisms of X and m be a nonzero Borel $\mathscr{G}$-invariant measure on X. We say that $\mathscr{G}$*

is metrically transitive with respect to m if for every pair of Borel sets A, B such that $m(A)\cdot m(B) > 0$ *there exists* $F\in\mathcal{G}$ *such that* $m(A\cap F(B)) > 0$.

Definition 6. *Let X be a locally compact space and* $\mathcal{G}$ *be a minimal group of homeomorphisms of X. For any bounded set* $B\subset X$ *and any bounded Borel set A with nonempty interior put*:

$$(B:A) = \min\{n\in N\colon \exists T_1,\dots,T_n\in\mathcal{G} : B\subset \bigcup_{i=1}^{n} T_i(A)\}$$

$$[B:A] = \sup\{n\in N\colon \exists T_1,\dots,T_n\in\mathcal{G} : \bigcup_{i=1}^{n} T_i(A)\subset B,$$
$$T_i(A)\cap T_j(A)=\emptyset \quad \textit{for} \quad i\neq j\}.$$

We say that $\mathcal{G}$ *satisfies the condition (N) if the inequality* $[B:A]\le (B:A)$ *holds for any bounded set* $B\subset X$ *and any bounded Borel set A with the nonempty interior.*

2. Construction of an invariant measure

In this section we shall prove the necessity of the condition (N) for the existence of a $\mathcal{G}$-invariant measure and then we shall give a construction of an invariant measure under a complementary condition (C).

Theorem 7. *Let X be a locally compact space and* $\mathcal{G}$ *be a minimal group of homeomorphisms of X. Then*:

(i) $0\le (B:A) < \infty,\ 0\le [B:A]$

(ii) $(\emptyset:A) = [\emptyset:A] = 0$

(iii) $B_1\subset B_2 \Rightarrow (B_1:A)\le (B_2:A),\ [B_1:A]\le [B_2:A]$

(iv) $((B_1\cup B_2):A)\le (B_1:A)+(B_2:A)$

(v) $B_1\cap B_2=\emptyset \Rightarrow [(B_1\cup B_2):A]\ge [B_1:A]+[B_2:A]$

(vi) $[F(B):A]=[B:A],\ (F(B):A)=(B:A)$

(vii) $(B:A)\le (B:C)\cdot(C:A)$,

where B, B_1, B_2 *are arbitrary bounded sets, A, C are arbitrary bounded Borel sets with nonempty interiors and F is an arbitrary homeomorphism from* $\mathcal{G}$.

(viii) *The condition (N) is necessary for the existence of a nonzero* $\mathcal{G}$*-invariant Borel measure.*

(ix) *If* $\mathcal{G}'$ *is another group of homeomorphisms of X which contains* $\mathcal{G}$ *and the numbers* $(B:A)'$, $[B:A]'$ *are determined by* $\mathcal{G}'$, *then the inequalities* $[B:A]'\ge [B:A]$, $(B:A)'\le (B:A)$ *hold for any bounded set B and any bounded Borel set A with the nonempty interior.*

Proof. We shall prove only the statement (viii), because the others follow immediately from the last definition. Suppose we have a locally compact space

X, a minimal group $\mathscr{G}$ of homeomorphisms of X and a nonzero $\mathscr{G}$-invariant Borel measure m. We shall prove that $m(U) > 0$ for any nonempty open Borel set $U \subset X$.

Let $m(U) = 0$ for some nonempty open Borel set $U \subset X$. Take a compact set $K \subset X$. Since $\mathscr{G}$ is minimal the system $\{F(U): F \in \mathscr{G}\}$ covers X and K as well. As K is compact there exist $F_1, \ldots, F_n \in \mathscr{G}$ such that $K \subset \bigcup_{i=1}^{n} F_i(U)$. The properties of the measure m imply the following inequalities: $0 \leq m(K) \leq m\left(\bigcup_{i=1}^{n} F_i(U)\right) \leq$ $\leq \sum_{i=1}^{n} mF_i(U) = n \cdot m(U) = 0$, which means $m(K) = 0$ for the arbitrary compact set $K \subset X$.

If A is a bounded Borel set, then $m(A) = 0$ because $m(\bar{A}) = 0$. Since every Borel set $A \subset X$ is σ-bounded the measure m is zero, which is a contradiction.

Let B be any bounded subset of X and A be any bounded Borel set with the nonempty interior. Then $0 < m(A) < \infty$ because $m(\operatorname{int} A) > 0$ and $m(\bar{A}) < \infty$.

Let $F_1, \ldots, F_n$, $T_1, \ldots, T_k$ be homeomorphisms from $\mathscr{G}$ such that $\bigcup_{i=1}^{n} F_i(A) \subset$ $\subset B \subset \bigcup_{i=1}^{k} T_i(A)$, and $F_i(A) \cap F_j(A) = \emptyset$ for $i \neq j$. Then $n \cdot m(A) = \sum_{i=1}^{n} m(F_i(A)) =$ $= m\left(\bigcup_{i=1}^{n} F_i(A)\right) \leq m(B) \leq m\left(\bigcup_{i=1}^{k} T_i(A)\right)$.

It means that $k \geq n$. This inequality implies the inequality $(B:A) \geq [B:A]$. The proof is complete.

Suppose we have a locally compact space X and a minimal group $\mathscr{G}$ of homeomorphisms on X, which satisfies the condition (N). There is a question whether there exists a nonzero $\mathscr{G}$-invariant Borel measure. We can construct a $\mathscr{G}$-invariant Borel measure but it is not clear if this measure is nonzero. We are going to formulate the condition (C), which guarantees the nontriviality of the measure.

Definition 8. *Let X be a nonempty set and $\mathscr{B}$ be a nonempty system of nonempty subsets of X. $\mathscr{B}$ is called a filter base if $\forall A \in \mathscr{B}\ \forall B \in \mathscr{B}\ \exists C \in \mathscr{B}$: $C \subset A \cap B$.*

Definition 9. *Let X be a locally compact space, W be a nonempty open subset of X and $\mathscr{S}$ be a system of open subsets of X. $\mathscr{S}$ is called separating on W if for any compact set $K \subset W$ and any open subset $U \subset X$ such that $K \subset U$ there exists $U' \in \mathscr{S}$ such that $K \subset U' \subset U$.*

Definition 10. *Let X be a locally compact space and $\mathscr{G}$ be a minimal group of homeomorphisms of the space X. We say that $\mathscr{G}$ satisfies the condition (C) if there exist $\mathscr{B}$, $\mathscr{S}$, W and α such that:*

(i) *$\mathscr{B}$ is a filter base on X. All elements of $\mathscr{B}$ are bounded Borel sets with nonempty interiors.*

(ii) *W is a nonempty open subset of X. $\mathcal{S}$ is a system which consists of bounded open subsets of X. $\mathcal{S}$ is separating on W.*
(iii) *α is a real number from* $(0, 1)$.
(iv) $\forall U \in \mathcal{S}\ \exists A \in \mathcal{B}\ \forall A' \in \mathcal{B} : A' \subset A \Rightarrow [U : A'] \geq \alpha(U : A')$.

Comparing the conditions (N) and (C) we see that (C) is nearly opposite to (N). If the group $\mathcal{G}$ of homeomorphisms satisfies both conditions (N) and (C), it means roughly speaking that the numbers $(U : A)$ and $[U : A]$ are of the "same order" for sufficiently many U and A.

Proposition 11. *Let X be a locally compact spaced and $\mathcal{G}$, $\mathcal{G}'$ be minimal groups of homeomorphisms of X. If $\mathcal{G}$ satisfies the condition (C) and $\mathcal{G} \subset \mathcal{G}'$, then $\mathcal{G}'$ satisfies the condition (C) as well.*

Theorem 12. *Let X be a locally compact space and $\mathcal{G}$ be a minimal group of homeomorphisms which satisfies the condition (C) and the following variant of the condition (N): $[U : A] \leq (U : A)$ for every bounded open subset $U \subset X$ and every $A \in \mathcal{B}$, where $\mathcal{B}$ is the filter base from Definition* 10. *Then*:

(i) *There exists a nonzero regular Borel $\mathcal{G}$-invariant measure m.*
(ii) *Every regular Borel $\mathcal{G}$-invariant measure is a constant multiple of the measure m.*
(iii) *The group $\mathcal{G}$ is metrically transitive with respect to the measure m.*

Proof. Let W, $\mathcal{S}$, $\mathcal{B}$ and α be from Definition 10. There exists $U_0 \in \mathcal{S}$ such that $\emptyset \neq U_0 \subset \bar{U}_0 \subset W$. For any bounded open subset $U \subset X$ and any $A \in \mathcal{B}$ put:

$$\lambda_A(U) = \frac{[U : A]}{(U_0 : A)}, \qquad \Lambda_A(U) = \frac{(U : A)}{(U_0 : A)}.$$

We obtain two systems of set functions $\{\lambda_A\}_{A \in \mathcal{B}}$, $\{\Lambda_A\}_{A \in \mathcal{B}}$. The set functions λ_A and Λ_A have the following properties:

$$0 \leq \lambda_A(U) \leq \Lambda_A(U) \leq (U : U_0) \tag{1}$$

$$\forall U \in \mathcal{S}\ \exists A \in \mathcal{B}\ \forall A' \in \mathcal{B} : A' \subset A \Rightarrow \lambda_{A'}(U) \geq \alpha \cdot \Lambda_{A'}(U) \tag{2}$$

$$\lambda_A(\emptyset) = \Lambda_A(\emptyset) = 0 \tag{3}$$

$$U_1 \subset U_2 \Rightarrow \lambda_A(U_1) \leq \lambda_A(U_2),\ \Lambda_A(U_1) \leq \Lambda_A(U_2) \tag{4}$$

$$\Lambda_A(U_1 \cup U_2) \leq \Lambda_A(U_1) + \Lambda_A(U_2) \tag{5}$$

$$\emptyset = U_1 \cap U_2 \Rightarrow \lambda_A(U_1 \cup U_2) \geq \lambda_A(U_1) + \lambda_A(U_2) \tag{6}$$

$$\lambda_A(F(U)) = \lambda_A(U),\ \Lambda_A(U) = \Lambda_A(F(U)) \text{ for any } F \in \mathcal{G} \tag{7}$$

$$\Lambda_A(U_0) = 1. \tag{8}$$

From the systems $\{\lambda_A\}_{A\in\mathscr{B}}$, $\{\Lambda_A\}_{A\in\mathscr{B}}$ of set functions we can construct two "limit" set functions λ and Λ.

The construction of λ and Λ is the same as the construction of the Haar content in [3, pp. 245—248], the only difference is that we have two systems of set functions and we construct two set functions. The "limit" set functions λ and Λ have the properties:

(1′) $$0 \le \lambda(U) \le \Lambda(U) \le (U : U_0)$$

(2′) $$\forall U \in \mathscr{S}: \lambda(U) \ge \alpha \cdot \Lambda(U)$$

(3′) $$\lambda(\emptyset) = \Lambda(\emptyset) = 0$$

(4′) $$U_1 \subset U_2 \Rightarrow \lambda(U_1) \le \lambda(U_2),\ \Lambda(U_1) \le \Lambda(U_2)$$

(5′) $$\Lambda(U_1 \cup U_2) \le \Lambda(U_1) + \Lambda(U_2)$$

(6′) $$U_1 \cap U_2 = \emptyset \Rightarrow \lambda(U_1 \cup U_2) \ge \lambda(U_1) + \lambda(U_2)$$

(7′) $$\lambda(F(U)) = \lambda(U),\ \Lambda(F(U)) = \Lambda(U) \text{ for any } F \in \mathscr{G}$$

(8′) $$\Lambda(U_0) = 1.$$

For any compact subset $K \subset X$ put:

$$\lambda(K) = \inf\{\lambda(U): K \subset U,\ U \textit{ is a bounded open subset}\}$$
$$\Lambda(K) = \inf\{\Lambda(U): K \subset U,\ U \textit{ is a bounded open subset}\}.$$

If some compact set is also open, then we obtain the same values of $\lambda(K)$, $\Lambda(K)$ as we had before.

For any compact sets K, K_1, $K_2 \subset X$ we have:

(1″) $$0 \le \lambda(K) \le \Lambda(K) < \infty$$

(2″) $$K \subset W \Rightarrow \lambda(K) \ge \alpha \cdot \Lambda(K)$$

(3″) $$\lambda(\emptyset) = \Lambda(\emptyset) = 0$$

(4″) $$K_1 \subset K_2 \Rightarrow \lambda(K_1) \le \lambda(K_2),\ \Lambda(K_1) \le \Lambda(K_2)$$

(5″) $$\Lambda(K_1 \cup K_2) \le \Lambda(K_1) + \Lambda(K_2)$$

(6″) $$K_1 \cap K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) \ge \lambda(K_1) + \lambda(K_2)$$

(7″) $$\lambda(F(K)) = \lambda(K),\ \Lambda(F(K)) = \Lambda(K) \text{ for any } F \in \mathscr{G}$$

(8″) $$\Lambda(\bar{U}_0) \ge 1.$$

All properties (1″)—(8″) follow from (1′)-(8′) and the construction. The implication (2′) $\Rightarrow$ (2″) holds because the system $\mathscr{S}$ is separating on W and the set functions λ and Λ are monotone. The implication (6′) $\Rightarrow$ (6″) holds because in a locally compact space every pair of disjoint compact sets may be separated by a pair of disjoint bounded open subsets. The other properties are obvious.

For every compact set $K \subset X$ put:

$$\mu(K) = \inf\left\{ \sum_{i=1}^{n} \lambda(K_i)\colon \textit{all } K_i \textit{ are compact sets such that } K = \bigcup_{i=1}^{n} K_i \right\}.$$

For any compact sets K, L we have:

$$0 \le \mu(K) \le \lambda(K) < \infty \tag{9}$$

$$\mu(\emptyset) = 0 \tag{10}$$

$$K \subset L \Rightarrow \mu(K) \le \mu(L) \tag{11}$$

$$\mu(K \cup L) \le \mu(K) + \mu(L) \tag{12}$$

$$K \cap L = \emptyset \Rightarrow \mu(K \cup L) = \mu(K) + \mu(L) \tag{13}$$

$$K \subset W \Rightarrow \mu(K) \ge \alpha \Lambda(K),\ \mu(\bar{U}_0) \ge \alpha \tag{14}$$

$$\mu(F(K)) = \mu(K) \text{ for any } F \in \mathscr{G}. \tag{15}$$

All properties (9)—(15) follow from (1″)—(8″). The inequality $\mu(\bar{U}_0) \ge \alpha$ follows from the fact that $\bar{U}_0$ is a compact subset of W and $\Lambda(\bar{U}_0) \ge 1$. The properties (9)—(15) say that μ is a nontrivial $\mathscr{G}$-invariant content. If we put $m^*(U) = \sup\{m(K)\colon\ K \subset U,\ K \textit{ is compact}\}$ for every open Borel set and $m(A) = \inf\{m^*(U)\colon A \subset U, U \textit{ is open Borel}\}$ for every Borel set $A \subset X$, then we obtain a regular Borel $\mathscr{G}$-invariant measure on X which is nonzero because $m(\bar{U}_0) \ge \mu(\bar{U}_0) \ge \alpha$.

Now we are going to prove that for any $\mathscr{G}$-invariant regular Borel measure m' on X there exists a constant $C \ge 0$ such that $m'(A) = C \cdot m(A)$ for every Borel subset $A \subset X$. If m' is the zero measure, then C is equal to zero. If m' is nonzero, then $\infty > m'(U_0) > 0$ (see the proof of Theorem 7). We may assume that $m'(U_0) = 1$. From Definition 6 and the properties of measure m' it follows that $[B\colon A] \cdot m'(A) \le m'(B) \le (B\colon A) \cdot m'(A)$ for any bounded Borel subset $B \subset X$ and any $A \in \mathscr{B}$. Therefore

$$\lambda_A(U) = \frac{[U\colon A]}{(U_0\colon A)} \le \frac{m'(U)}{m'(A)} \cdot \frac{m'(A)}{m'(U_0)} = m'(U).$$

Since $U_0 \in \mathscr{S}$ there exists $A_0 \in \mathscr{B}$ such that $(U_0\colon A) \le \alpha^{-1}[U_0\colon A]$ for any $A \subset A_0$, $A \in \mathscr{B}$. For such A we have

$$\Lambda_A(U) = \frac{(U\colon A)}{(U_0\colon A)} \ge \alpha \cdot \frac{(U\colon A)}{[U_0\colon A]} \ge \alpha \cdot \frac{m'(U)}{m'(A)} \cdot \frac{m'(A)}{m'(U_0)} = \alpha \cdot m'(U).$$

Take $U \in \mathscr{S}$. There exists $A_1 \in \mathscr{B}$ such that $\lambda_A(U) \geq \alpha \cdot \Lambda_A(U)$ for any $A \in \mathscr{B}$, $A \subset A_1$. Since $\mathscr{B}$ is a filter base for any $U \in \mathscr{S}$ there exists $A_2 \in \mathscr{B}$ such that

$$m'(U) \geq \lambda_A(U) \geq \alpha \cdot \Lambda_A(U) \geq \alpha^2 m'(U) \text{ for any } A \in \mathscr{B} \;\; A \subset A_2.$$

For the "limit" set functions λ, Λ we have the following inequalities

$$m'(U) \geq \lambda(U) \geq \alpha \cdot \Lambda(U) \geq \alpha^2 \cdot m'(U) \text{ for any } U \in \mathscr{S}$$

Since m' is regular and $\mathscr{S}$ is separating on W we have: $m(K) \geq \lambda(K) \geq \geq \alpha \cdot \Lambda(K) \geq \alpha^2 \cdot m'(K)$ for any compact subset $K \subset W$. From the last inequality and the properties of μ it follows that $m'(K) \geq \mu(K) \geq \alpha^2 \cdot m(K)$ for any compact subset $K \subset W$.

The regularity of the measure m' gives $m'(U) \geq m^*(U) \geq \alpha^2 \cdot m'(U)$ for any open subset $U \subset W$ and $m'(A) \geq m(A) \geq \alpha^2 \cdot m'(A)$ for any Borel $A \subset W$. The last inequality holds in the case when $A = F(B)$, where $B \subset W$ is a Borel set and F is a homeomorphism from $\mathscr{G}$.

Let A be an arbitrary Borel set in X. Since every Borel set is σ bounded and $\mathscr{G}$ is a minimal group there exists a sequence $\{F_i\}_{i=1}^{\infty}$ of homeomorphisms from $\mathscr{G}$ such that $A \subset \bigcup_{i=1}^{\infty} F_i(U_0)$.

Put $A_1 = A \cap F_1(U_0)$ and

$$A_n = (A \cap F_n(U_0)) \setminus \left(\bigcup_{i=1}^{n-1} F_i(U_0) \right) \quad \text{for} \quad n \geq 2$$

We have a sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint Borel sets such that

$$A = \bigcup_{i=1}^{\infty} A_i, \; F_n^{-1}(A_n) \subset U_0 \subset W \quad \text{for all natural } n.$$

Since $m'(A_i) \geq m(A_i) \geq \alpha^2 \cdot m'(A_i)$, we have $m'(A) \geq m(A) \geq \alpha^2 \cdot m'(A)$. When the assumption $m'(U_0) = 1$ is not true, then the last inequality must be replaced by

$$C_1 m'(A) \geq m(A) \geq C_2 m'(A) \quad \text{because} \quad 0 < m'(U_0) < \infty.$$

Now we are going to prove part (iii) and then we shall finish the proof of uniqueness.

Let A and B be Borel sets such that $m(A) > 0$, $m(B) > 0$. If $m(A \cap T(B)) = 0$ for every $T \in \mathscr{G}$, then the set function m'' defined for Borel sets E by the formula:

$$m''(E) = \sup \left\{ m\left(E \cap \bigcup_{i=1}^{n} T_i(B) \right) : T_1, \ldots, T_n \in \mathscr{G} \right\}$$

is a regular Borel $\mathscr{G}$-invariant measure such that $m''(A) = 0$, $m''(B) = m(B) > 0$ and this contradicts the inequality $C_1 m''(A) \geq m(A) \geq C_2 m'(A)$.

Therefore $m(A \cap T(B)) > 0$ fore some $T \in \mathscr{G}$ and part (iii) is proved.

Let E be a Borel set such that $0 < m(E) < \infty$. Since $C_1 m'(A) \geq m(A) \geq$ $\geq C_2 m'(A)$ there exists a measurable function $\varphi \colon E \to \langle 0, \infty \rangle$ such that

$$m'(A) = \int_A \varphi(x)\, \mathrm{d}m(x) \text{ for any } A \subset E \text{ and } \varphi(x) \in \langle C_1^{-1}, C_2^{-1} \rangle$$

almost everywhere with respect to m.

Let the function Φ: $\langle 0, \infty \rangle \to \langle 0, \infty)$ be defined by the formula $\Phi(t) = m(\{x \in E \colon \varphi(x) < t\})$.

Obviously $\quad \Phi(t) = 0$ for $t \in (0, C_1^{-1}\rangle$,

$\Phi(t) = m(E)$ for $t \in (C_2^{-1}, \infty)$ and

Φ is a nondecreasing function.

We shall prove that $\Phi(\langle 0, \infty \rangle) = \{0, m(E)\}$.

Suppose that $0 \neq \Phi(t_0) \neq m(E)$ for some $t_0 \in \langle 0, \infty \rangle$.

Then $0 < \Phi(t_0) < m(E)$, $0 < t_0 < \infty$.

Put $A_0 = \{x \in E \colon \varphi(x) < t_0\}$, $B_0 = E \setminus A_0$.

Then $m(A_0) = \Phi(t_0) > 0$, $m(B_0) = m(E) - \Phi(t_0) > 0$.

There exists some $T \in \mathscr{G}$ for which $m(A_0 \cap T(B_0)) > 0$.

We have two inequalities:

$$m'(A_0 \cap T(B_0)) = \int_{A_0 \cap T(B_0)} \varphi(x)\, \mathrm{d}m(x) < t_0 m(A_0 \cap T(B_0))$$

and

$$m'(T^{-1}(A_0 \cap T(B_0)) = m'(T^{-1}(A_0) \cap B_0) = \int_{B_0 \cap T^{-1}(A_0)} \varphi(x)\, \mathrm{d}m(x) \geq$$

$$\geq t_0 \cdot m(T^{-1}(A_0) \cap B_0) = t_0 \cdot m(T^{-1}(A_0 \cap T(B_0)),$$

which contradicts the $\mathscr{G}$-invariantness of the measures m and m'.

Therefore $\Phi(\langle 0, \infty \rangle) = \{0, m(E)\}$.

It means $\varphi(x) = \sup\{t \in \langle 0, \infty \rangle \colon \Phi(t) = 0\} = \text{const}$ almost everywhere on E with respect to the measure m.

We have just proved that for any Borel measure $E \subset X$ such that $0 < m(E) < \infty$ there exists a constant $C > 0$ such that $m'(A) = C \cdot m(A)$ for any Borel set $A \subset E$. It is clear that a constant C must be the same for all Borel sets $E \subset X$ such that $0 < m(E) < \infty$. This proves the equality $m'(E) = C \cdot m(E)$ for the Borel sets $E \subset X$ with the property $0 < m(E) < \infty$. But the equality $m'(E) = C \cdot m(E)$ holds also in the case when $m(E) = 0$ or $m(E) = \infty$ because of the inequality $C_1 \cdot m'(E) \geq m(E) \geq C_2 \cdot m'(E)$.

The proof of Theorem 12 is complete.

3. Examples and applications

We are going to prove that the group of all translations of the Euclidean space R^n satisfies the condition (C). We begin the proof with a construction of a separating system on R^n.

For every natural number i let $\mathscr{R}_i$ denote the system of all closed hypercubes of the form $\prod_{j=1}^{n} \langle a_j 2^{-i}, (a_j + 1)2^{-i}\rangle$, where a_j are arbitrary integers.

Let $\mathscr{S}_i$ denote the system of all sets U of the form $U = \operatorname{int}\left(\bigcup_{s=1}^{m} K_s\right)$, where $K_s \in \mathscr{R}_i$ for all $s = 1, \ldots, m$ and int is the interior with respect to the standard topology. Let $\mathscr{S} = \bigcup_{i=1}^{\infty} \mathscr{S}_i$.

Proposition 13. *For any nonempty open subset W the system $\mathscr{S}_W = \{U: U \in \mathscr{S}, \bar{U} \subset W\}$ is separating on W.*

Proof. We are to prove that for any compact set $K \subset W$ and for any open set V which contains the set K there exists $U \in \mathscr{S}_W$ such that $K \subset U \subset V$.

Let K be an arbitrary compact set in W and V its open neighbourhood. If $V \cap W = R^n$, let $i = 1$. If $V \cap W \neq R^n$, we take a natural number i such that $2^{-i} < \delta(\sqrt{n})^{-1}$, where

$$\delta = \inf\{\|x - y\|: x \in K, y \in R^n \setminus (V \cap W)\}.$$

Let $\mathscr{M}$ be the system $\{K_a: K_a \in \mathscr{R}_i, K_a \cap K \neq \emptyset\}$. The system $\mathscr{M}$ is finite because K is compact, and

$$K \subset \operatorname{int}\left(\bigcup_{\mathscr{M}} K_a\right) \subset \bigcup_{\mathscr{M}} K_a = \overline{\bigcup_{\mathscr{M}} K_a} \subset V \cap W.$$

The first inclusion holds because every point $x \in R^n$ is contained in the interior of the union of all $K_a \in \mathscr{R}_i$ which contain the point x. The last statement holds particularly for every $x \in K$. But every hypercube $K_a \in \mathscr{R}_i$ which contains $x \in K$ belongs to $\mathscr{M}$. The equality $\bigcup_{\mathscr{M}} K_a = \overline{\bigcup_{\mathscr{M}} K_a}$ is true because all K_a are closed and $\mathscr{M}$ is a finite system. If $x \in \bigcup_{\mathscr{M}} K_a$, then $\|x - y\| \leq \sqrt{n}2^{-i} < \delta$ for some $y \in K$, which means that $x \in U \cap W$ because $\|x - y\| \geq \delta$ for every $x \in R^n \setminus (V \cap W)$ and every $y \in K$. The set $U = \operatorname{int}\left(\bigcup_{\mathscr{M}} K_a\right)$ has the required properties.

Proposition 14. *Let X be R^n with the standard topology, $\| \;\|$ be the Euclidean norm, $\mathscr{G}$ be the group of all translations and $B_r = \{x \in R^n: \|x\| \leq r\}$ for $r > 0$.*

Then the system $\{B_r\}_{r>0}$ *is a filter base and for any* $U \in \mathcal{S}_i$ *and* $r \in (0, 2^{-(i+1)})$ *we have*:

$$[U: B_r] \geq m\,2^{-n(i+2)} r^{-n}$$

$$(U: B_r) \leq m(\sqrt{n})^n 2^{-in} r^{-n} \quad \textit{and}$$

$$[U: B_r] \geq (4\sqrt{n})^{-n} (U: B_r).$$

$\left(m\right.$ depends on U and denotes the number of pairwise different cubes $K_s \in \mathcal{R}_i$, $s = 1, \ldots, m$, such that $\left. U = \operatorname{int}\left(\bigcup_{s=1}^{m} K_s\right).\right)$

Proof. Since $\bigcup_{j=1}^{m} \operatorname{int} K_j \subset \operatorname{int} \bigcup_{j=1}^{m} K_j$ we have

$$\left[\operatorname{int}\left(\bigcup_{j=1}^{m} K_j\right): B_r\right] \geq \left[\bigcup_{j=1}^{m} (\operatorname{int} K_j): B_r\right].$$

$\operatorname{int}(K_j)$ are pairwise disjoint because $K_j \in \mathcal{R}_i$ are pairwise different.

Obviously, every open cube $\operatorname{int} K_j$ is obtained by some translation of the open cube $K_0 = (0, 2^{-i})^n$. From the statements (v) and (vi) of Theorem 7 we have the inequality $\left[\left(\bigcup_{j=1}^{m} \operatorname{int} K_j\right): B_r\right] \geq m \cdot [K_0: B_r]$.

Let $0 < r < 2^{-(i+1)}$. Let k be that nonnegative integer for which $2^{-(i+k+2)} \leq r < 2^{-(i+k+1)}$. This k is unique.

Consider the system of open cubes $\left\{\prod_{s=1}^{n} (\alpha_s \cdot 2^{-(k+i)}, (\alpha_s + 1) \cdot 2^{-(k+i)})\right\}$, where α_s are arbitrary numbers from $\{0, \ldots, 2^k - 1\}$. Then we have 2^{kn} pairwise disjoint cubes with vertices of the length $2^{-(k+i)}$ which are contained in K_0. If we place the closed Euclidean balls with the radius r into the centre of every "small" cube $\prod_{s=1}^{n} (\alpha_s \cdot 2^{-(k+i)}, (\alpha_s + 1) \cdot 2^{-(k+i)})$, then we obtain 2^{kn} pairwise disjoint closed Euclidean balls with radius r. This follows from the inequality $r < 2^{-(i+k+1)}$, i.e. $2r < 2^{-(i+k)}$. We have an inequality $[K_0: B_r] \geq (2^k)^n$. From the inequality $2^{-(i+k+2)} \leq r$ it follows that $2^k \geq r^{-1} 2^{-(i+2)}$.

Therefore $[K_0: B_r] \geq 2^{-n(i+2)} r^{-n}$, which means

$$[U: B_r] \geq m \cdot 2^{-n(i+2)} r^{-n}.$$

The second inequality holds also for $r \in (0, \sqrt{n} \cdot 2^{-i})$ not only for $r \in (0, 2^{-(i+1)})$ and may be proved in a similar way. The third inequality is a consequence of the preceding ones.

Proposition 15. *Let* X *be* R^n *with the standard topology and* $\mathcal{G}$ *be the group of all translations. Then* $\mathcal{G}$ *satisfies the condition* (C).

This is a consequence of Propositions 13 *and* 14.

Theorem 16. *Let (X, d) be a locally compact metric space and $\mathscr{G}$ be a transitive group of isometric homeomorphisms of X.*

Let (X, d) satisfy the condition:

there exist nonempty open $W \subset R^n$, $W' \subset X$, a homeomorphism $\Phi: W \to W'$ and $C_1, C_2 > 0$ such that

$$C_1 \|x - y\| \geq d(\Phi(x), \Phi(y)) \geq C_2 \|x - y\| \text{ for all } x, y \in W.$$

Then $\mathscr{G}$ satisfies the condition (C).

Proof. Let $x_0 \in X$ be arbitrary but fixed. Consider the system $\{A_r\}_{r>0}$, where $A_r = \{y \in X: d(y, x_0) \leq r\}$. Since X is locally compact, for some $r_0 > 0$ the set A_{r_0} is compact. So, we have a filter base $\{A_r\}_{0<r\leq r_0}$ which consists of the bounded Borel sets with nonempty interiors. By Proposition 13 the system $\mathscr{S}_W$ is separating on W. Then the system $\mathscr{S}' = \{\Phi(U): U \in \mathscr{S}_W\}$ is separating on W'. We need to prove that

$$\exists \alpha \in (0, 1)\ \forall U' \in \mathscr{S}'\ \exists r \in (0, r_0\rangle: \forall s: 0 < s \leq r \Rightarrow [U' : A_s] \geq \alpha(U' : A_s).$$

Let $U' \in \mathscr{S}'$. Then $U' = \Phi(U)$, where $\bar{U} \subset W$ and $U \in \mathscr{S}_i$ for some natural number i.

Let $s \in (0, r_0\rangle$ be such that

$$C_2^{-1}s < 2^{-(i+1)},\quad s < \inf\{d(x, y): x \in U',\ y \in X \setminus W'\}.$$

The infinum in the last inequality is positive because the set $\bar{U}$ is compact in R^n, $\bar{U} \subset W$ and $U' = \Phi(U) \subset \Phi(\bar{U})$. By Proposition 14 there exist N pairwise disjoint closed Euclidean balls $B_1, \ldots, B_N$ with the radius $C_2^{-1}s$ which are contained in U and $N \geq m \cdot 2^{-n(i+2)}(C_2^{-1} \cdot s)^{-n}$. ($m$ denotes the number of the cubes from $\mathscr{R}_i$ from which the set U is constructed.)

Let the centre of the ball B_j be a point $b_j \in U$ for $j = 1, \ldots, N$. Put $a_j = \Phi(b_j)$. Since $\mathscr{G}$ is transitive on X there exists a sequence $F_1, \ldots, F_N \in \mathscr{G}$ such that $a_j = F_j(x_0)$ for $j = 1, \ldots, N$. Since F_j are isometric homeomorphisms,

$$F_j(A_s) = \{y \in X: d(a_j, y) \leq s\}.$$

Moreover $F_j(A_s) = \{y \in X: d(a_j, y) \leq s\} \subset W'$ because $a_j = \Phi(b_j) \in \Phi(U) = U'$ and s was chosen so that $s < d(U', X \setminus W)$. We shall show that $F_j(A_s) \subset \Phi(B_j)$ for $j = 1, \ldots, N$. Let $y_0 \in F_j(A_s)$, i.e. $y_0 \in \{y: d(a_j, y) \leq s\} \subset W'$. Then $y_0 = \Phi(z_0)$ some $z_0 \in W$. Therefore

$$\|z_0 - b_j\| = \|\Phi^{-1}(y_0) - \Phi^{-1}(a_j)\| \leq C_2^{-1} d(y_0, a_j) \leq C_2^{-1} \cdot s,$$

which means that $z_0 \in B_j$, i.e. $y_0 \in \Phi(B_j)$.

Since $F_j(A_s) \subset \Phi(B_j)$ and the balls B_j are pairwise disjoint the sets $F_j(A_s)$ are pairwise disjoint. Since B_j are contained in U, $F_j(A_s)$ are contained in $\Phi(U) = U'$.

Therefore $[U:A_s] \geq N \geq m \cdot 2^{-n(i+2)} \cdot (C_2^{-1}s)^{-n}$ for every s such that $s \in \langle 0, r_0 \rangle \cap \cap (0, d(U', X \setminus W')) \cap (0, C_2^{-1} \cdot 2^{-(i+1)})$. If $s < C_1 \cdot 2^{-(i+1)}$ and $s < C_1 \cdot \inf\{\|x - - y\| : x \in U, y \in R^n \setminus W\}$, then in a similar way it may be proved that $(U:A_s) \leq \leq m(\sqrt{n})^n \cdot 2^{-i} \cdot (C_1^{-1}s)^{-n}$. Whenever

$$0 < s < \min(C_2 \cdot 2^{-(i+1)}, d(U', X \setminus W'), C_1 \cdot 2^{-(i+1)}, C_1 \cdot d(U, R^n \setminus W))$$

we have both inequalities:

$$[U':A_s] \geq m \cdot 2^{-n(i+2)} \cdot (C_2^{-1}s)^{-n}$$
$$(U':A_s) \leq m \cdot (\sqrt{n})^n \cdot 2^{-in} \cdot (C_1^{-1}s)^{-n},$$

which imply:

$$[U':A_s] \geq (4^{-1} \cdot \sqrt{n^{-1}} \cdot C_2 \cdot C_1^{-1})^n \cdot (U':A_s).$$

There is a natural question: Which locally compact spaces satisfy the assumption of Theorem 16? The Riemann spaces satisfy this condition which is explicitly proved in [2, p. 281], but this fact is well known.

Theorem 17. *Let X be a locally compact space and $\mathcal{G}$ be a group of homeomorphisms of X which contains some minimal group $\mathcal{G}'$. If $\mathcal{G}'$ satisfies the condition (C), then a nonzero $\mathcal{G}$-invariant Borel measure on X exists if and only if $\mathcal{G}$ satisfies the condition (N).*

Proof. The necessity of the condition (N) follows from Theorem 7. Theorem 12 and Proposition 11 say that (N) is also sufficient.

Theorem 18. *Let (X, d) be a Riemann space and $\mathcal{G}$ be a group of homeomorphisms which contains some transitive group of isometric homeomorphisms. Then a nonzero $\mathcal{G}$-invariant Borel measure exists if and only if $\mathcal{G}$ satisfies the condition (N).*

This Theorem follows from Theorems 16 and 17.

Corollary. *Let $\mathcal{G}$ be a group of homeomorphisms of R^n which contains all translations. A $\mathcal{G}$-invariant Borel measure exists if and only if $\mathcal{G}$ satisfies the condition (N). In this case the $\mathcal{G}$-invariant measure may be only the Lebesgue measure or its constant multiple.*

Now we are going to show that Theorem 12 gives a possibility to characterized the class of unimodular Lie groups in terms of the general topology.

A Lie group is a group which is simultaneously a smooth manifold such that group operations (multiplication and inversion) are smooth mappings. Every Lie group G is locally compact and there is a left and a right Har measure on G. This two measures need not be the same. See [4, p. 259]. If the left Haar measure is also a right Haar measure, then this measure is T-invariant for every mapping $T: G \to G$ of the form $T(x) = g \cdot x \cdot h$, where $g, h \in G$. Such Lie groups are called unimodular.

Theorem 19. *Let G be a Lie group and $\mathcal{G}$ be the group of all homeomorphisms*

T of G of the form $T(x) = g \cdot x \cdot h$, where $g, h \in G$. The following properties are equivalent:

(i) *G is unimodular*

(ii) *$\mathscr{G}$ satisfies the condition (N)*

(iii) *For any compact subset $K \subset G$ with the nonempty interior and any $a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}, c_1, \ldots, c_n, d_1, \ldots, d_n \in G$:*

$$a_i \cdot K \cdot b_i \cap a_j \cdot K \cdot b_j = \emptyset \text{ for all } i \neq j \Rightarrow \bigcup_{i=1}^{n+1} a_i \cdot K \cdot b_i \not\subset \bigcup_{i-1}^{n} c_i \cdot K \cdot d_i.$$

Proof. The implications (i) ⇒ (ii), (ii) ⇒ (iii) are obvious. We shall proved the implication (iii) ⇒ (i). If G satisfies (iii), then $[U: K] \leq (U: K)$ for every open set $U \subset G$ and every compact set $K \subset G$ with the nonempty interior. $(U: K)$ and $[U: K]$ are constructed by $\mathscr{G}$, (see Definition 6).

Theorem 12 says that for the existence of a $\mathscr{G}$-invariant measure it suffices to find α, W, $\mathscr{S}$, $\mathscr{B}$ such that:
$\mathscr{S}$ is a separating system on a nonempty open subset $W \subset G$,
$\mathscr{B}$ is a filter base which consists of the compact sets with nonempty interiors,

$$\alpha \in (0, 1) \quad \text{and}$$

$$\forall U \in \mathscr{S} \ \exists A \in \mathscr{B}: \ \forall A' \in \mathscr{B}: \ A' \subset A \Rightarrow [U: A'] \geq \alpha \cdot (U: A').$$

It suffices to find such α, W, $\mathscr{S}$, $\mathscr{B}$ for some transitive subgroup $\mathscr{G}'$ of $\mathscr{G}$ (see statement (ix) of Theorem 7). Let $\mathscr{G}'$ be a group of all left translations T on G, i.e.

$$T(x) = g \cdot x.$$

Obviously $\mathscr{G}' \subset \mathscr{G}$ and $\mathscr{G}'$ is transitive. There is a left invariant Riemann metric on G [6, p. 186]. By Theorem 16 there exist α, $\mathscr{S}$, W, $\mathscr{B}$ with the required properties, which completes the proof.

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ИНВАРИАНТНЫЕ МЕРЫ НА ЛОКАЛЬНО КОМПАКТНЫХ ПРОСТРАНСТВАХ И ТОПОЛОГИЧЕСКАЯ ХАРАКТЕРИЗАЦИЯ УНИМОДУЛЯРНЫХ ЛИЕВЫХ ГРУПП

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Резюме

В работе рассмотрены меры на локально компактном пространстве инвариантные относительно заданной группы гомеоморфизмом. Найдены два условия достаточные для существования инвариантной меры. Первое из них является также необходимым. Приведены следствия доказанных теорем. Одним из них является топологическая характеризация унимодулярных лиевых групп.