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Zeros of Derivatives of Solutions to Singular \((p, n - p)\) Conjugate BVPs

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Abstract

Positive solutions of the singular \((p, n - p)\) conjugate BVP are studied. The set of all zeros of their derivatives up to order \(n - 1\) is described. By means of this, estimates from below of the solutions and the absolute values of their derivatives up to order \(n - 1\) on the considered interval are reached. Such estimates are necessary for the application of the general existence principle to the BVP under consideration.

Key words: Singular conjugate BVP, positive solutions, zeros of derivatives, estimates from below.

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1 Introduction

Let \(n, p \in \mathbb{N}, n > 2, p \leq n - 1,\) and \(T\) be a positive number. In [3] (for \(p = 1\)) and [6], the authors have considered the singular \((p, n - p)\) conjugate boundary value problem (BVP)

\[ (-1)^p x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)), \quad (1.1) \]

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where \( f \) satisfies the local Carathéodory conditions on the set \( \mathcal{D} = [0, T] \times ((0, \infty) \times \mathbb{R}_0^{n-1}) \) with \( \mathbb{R}_0 = \mathbb{R} \setminus \{0\} \) and \( f \) is singular at the value 0 of each its phase variable. They have given conditions on \( f \) guaranteeing the existence of a positive (on \((0,T)\)) solution to BVP (1.1), (1.2). The singularities of the function \( f \) in (1.1) ‘appear’ in any positive solution of BVP (1.1), (1.2) and some its derivatives at the fixed points \( t = 0, t = T \), and all its derivatives up to order \( n - 1 \) ‘pass through’ singularities of \( f \) also at inner points of the interval \((0,T)\) which are not fixed. Therefore for proving the solvability of BVP (1.1), (1.2) in the class of positive functions on \((0,T)\) it is very important to give a localization analysis of zeros of derivatives up to order \( n - 1 \) of positive solutions to BVP (1.1), (1.2). This analysis have been presented for \( p = 1 \) in [3] and for \( p = 2 \) in [6] under the assumption that \( f \geq c \) on \( \mathcal{D} \) with a positive constant \( c \).

The aim of this paper is to complete this analysis for all values of \( p \). We note that the singular differential equation

\[
(-1)^p x^{(n)}(t) = \phi(t)g(t, x(t))
\]

together with the boundary conditions (1.2) have been discussed for \( \phi(t)g(t, x) : (0,1) \times (0, \infty) \rightarrow (0, \infty) \) continuous in [1], [2], [4] and [5] (in [4] and [5] with \( \phi = 1 \)). But for BVP (1.3), (1.2) singularities of \( g \) ‘appear’ in its positive solutions only at the fixed points \( t = 0 \) and \( t = 1 \).

2 Localization analysis of zeros to solutions of BVP (1.1), (1.2)

Let \( c \) be a positive constant and let \( f \) in (1.1) satisfy \( f \geq c \) on \( \mathcal{D} \). Then the localization analysis of zeros to solutions of BVP (1.1), (1.2) and their derivatives up to order \( n - 1 \) can be studied by the localization analysis of zeros to solutions of the differential inequality

\[
(-1)^p x^{(n)}(t) \geq c
\]

satisfying the boundary conditions (1.2). By a solution of problem (2.1), (1.2) we understand a function \( x \in AC^{n-1}([0, T]) \) (functions having absolutely continuous \((n-1)\)st derivative on \([0, T]\)) satisfying (2.1) for a.e. \( t \in [0, T] \) and fulfilling (1.2).

Having a solution \( x \) of problem (2.1), (1.2) we are interested in zeros of \( x^{(k)} \), \( 0 \leq k \leq n - 1 \), belonging to \((0,T)\). Without loss of generality we can suppose

\[
p - 1 \leq n - p - 1
\]

that is \( p \leq n/2 \), because by replacing \( t \) by \( T - t \) we can transform the case \( n/2 < p \) to (2.2).

For \( p = 1, 2 \) we have already studied zeros of \( x^{(k)} \) and we have proved the following results:
Lemma 2.1 Let \( x \) be a solution of problem (2.1), (1.2) for \( p = 1 \). Then \( x > 0 \) on \( (0,T) \) and \( x^{(k)} \) has just one zero in \( (0,T) \), \( 1 \leq k \leq n-1 \).

Proof Lemma follows from [3], Lemmas 2.7 and 2.9.

Lemma 2.2 Let \( x \) be a solution of problem (2.1), (1.2) for \( p = 2 \). Then

(i) \( x > 0 \) on \( (0,T) \),

(ii) \( x^{(k)} \) has just one zero in \( (0,T) \) for \( k = 1 \) and \( k = n-1 \),

(iii) \( x^{(k)} \) has just two zeros in \( (0,T) \) for \( 2 \leq k \leq n-2 \).

Proof See [6], Lemmas 2.2.

Decomposition analysis of zeros to solutions of BVP (2.1), (1.2) with \( p \geq 3 \) is described in the next theorem.

Theorem 2.3 Let \( x \) be a solution of problem (2.1), (1.2) for \( p \geq 3 \) and let (2.2) hold. Then

(i) \( x > 0 \) on \( (0,T) \),

(ii) \( x^{(k)} \) has just \( j \) zeros in \( (0,T) \) for \( k = j \) and \( k = n-j \) where \( j = 1, 2, \ldots, p-1 \),

(iii) \( x^{(k)} \) has just \( p \) zeros in \( (0,T) \) for \( p \leq k \leq n-p \).

Proof The proof is divided into three parts.

I. Lower bounds for zeros. By (1.2) we see that \( x' \) has at least one zero \( t_1 \in (0,T) \). Hence \( x'(0) = x'(t_1^1) = x'(T) = 0 \), which implies that \( x'' \) has at least two zeros \( t_2^1, t_2^2 \in (0,T) \). So, we have \( x''(0) = x''(t_2^1) = x''(t_2^2) = x''(T) = 0 \). By induction we conclude that \( x^{(j)}(j = 3, \ldots, p-1) \), has at least \( j \) zeros \( t_1^1, \ldots, t_1^j \in (0,T) \) and, due to (1.2) and (2.2) \( x^{(j)}(0) = x^{(j)}(t_1^j) = \ldots = x^{(j)}(t_1^j) = x^{(j)}(T) = 0 \), \( j = 3, \ldots, p-1 \). Therefore \( x^{(p)} \) has at least \( p \) zeros in \( (0,T) \). Now we will distinguish two cases: \( p < n/2 \) and \( p = n/2 \).

1. Let \( p < n/2 \). Then \( p \leq n - p - 1 \) and, by (1.2),

\[
x^{(j)}(0) = 0, \quad j = p, \ldots, n - p - 1.
\]

Thus \( x^{(k)} \) has at least \( p \) zeros in \( (0,T) \) for \( k = p + 1, \ldots, n - p \).

2. Let \( p = n/2 \) (clearly \( n \) is even in this case). Then \( p = n - p \) and \( x^{(n-p)} \) has at least \( p \) zeros in \( (0,T) \).

Hence we have shown that \( x^{(n-p)} \) has at least \( p \) zeros in \( (0,T) \) in the both cases. Since for \( x^{(n-j)} \), \( 1 \leq j \leq p-1 \), we cannot already use (1.2), we deduce that \( x^{(n-j)} \) has at least \( j \) zeros in \( (0,T) \) for \( j = 1, \ldots, p-1 \). Particularly \( x^{(n-1)} \) has at least one zero in \( (0,T) \).

II. Exact number of zeros. By (2.1), \( x^{(n-1)} \) is strictly monotonous and hence it has just one zero in \( (0,T) \). Therefore, by I, we deduce that \( x^{(n-k)} \) has just \( k \) zeros in \( (0,T) \) for \( 2 \leq k \leq p-1 \) and \( x^{(k)} \) has just \( p \) zeros in \( (0,T) \) for
\[ p \leq k \leq n - p. \] Similarly, \( x^{(k)} \) has just \( k \) zeros in \( (0, T) \) for \( 1 \leq k \leq p - 1 \) and \( x \) has no zero in \( (0, T) \).

III. Positivity of \( x \). Denote by \( t^k_1 \) the first zero of \( x^{(k)} \) in \( (0, T) \), \( 1 \leq k \leq n - 1 \). Inequality (2.1) implies that \((-1)^p x^{(n-1)} < 0\) on \([0, t^{n-1}_1]\) and \((-1)^p x^{(n-2)} > 0\) on \([0, t^{n-2}_1]\). Therefore \((-1)^{p+j} x^{(n-j)} > 0\) on \((0, t^{n-j}_1)\) for \( j = 3, \ldots, p \). Particularly, we have \( x^{(n-p)} > 0 \) on \((0, t^p_1)\), wherefrom, by virtue of (1.2), we obtain \( x^{(k)} > 0 \) on \((0, t^k_1)\), \( 1 \leq k \leq n - p - 1 \), and consequently \( x > 0 \) on \((0, T)\). □

Our next theorem provides estimates from below of solutions to problem (2.1), (1.2) and of the absolute value of their derivatives up to order \( n - 1 \) on the interval \([0, T]\). These estimations are necessary to apply the general existence principle of [6] to problem (1.1), (1.2) with \( f \) in (1.1) satisfying the inequality \( f \geq c \) on \( D \).

**Theorem 2.4** Let \( x \) be a solution of problem (2.1), (1.2). Then for any \( i \in \{1, \ldots, n - 1\} \) there are \( p_i + 1 \) disjoint intervals \((a_k, a_{k+1})\), \( 0 \leq k \leq p_i \), \( p_i < (n - 1)p \), such that

\[
\bigcup_{k=0}^{p_i} (a_k, a_{k+1}) = [0, T] \tag{2.3}
\]

and for each \( k \in \{0, \ldots, p_i\} \) one of the inequalities

\[
|x^{(n-i)}(t)| \geq \frac{c}{i!} (t - a_k)^i \quad \text{for} \quad t \in [a_k, a_{k+1}] \tag{2.4}
\]

or

\[
|x^{(n-i)}(t)| \geq \frac{c}{i!} (a_{k+1} - t)^i \quad \text{for} \quad t \in [a_k, a_{k+1}] \tag{2.5}
\]

is satisfied.

**Proof** Let \( x \) be a solution of problem (2.1), (1.2) and let \( t^j_1 \in (0, T) \) be zeros of \( x^{(j)} \) described in Lemmas 2.1, 2.2 and Theorem 2.3. Integrating (2.1) we get

\[
(-1)^p x^{(n-1)}(t) \geq c(t^{n-1} - t) \quad \text{for} \quad t \in [0, t^{n-1}_1]
\]

\[
(-1)^p x^{(n-1)}(t) \geq c(t - t^{n-1}_1) \quad \text{for} \quad t \in [t^{n-1}_1, T]. \tag{2.6}
\]

Now, integrate the first inequality in (2.6) from \( t \in [0, t^{n-2}_1] \) to \( t^{n-2}_1 \), we have

\[
(-1)^p x^{n-2}(t) \geq \frac{c}{2!} \left( - (t^{n-1} - t^{n-2}_1)^2 + (t^{n-1} - t)^2 \right) \geq \frac{c}{2!} (t^{n-2}_1 - t)^2.
\]

Hence, we get in such a way

\[
(-1)^p x^{(n-2)}(t) \geq \frac{c}{2!} (t^{n-2}_1 - t)^2 \quad \text{for} \quad t \in [0, t^{n-2}_1]
\]

\[
(-1)^p x^{(n-2)}(t) \geq \frac{c}{2!} (t - t^{n-1}_1)^2 \quad \text{for} \quad t \in [t^{n-2}_1, t^{n-1}_1]
\]

\[
(-1)^p x^{(n-2)}(t) \geq \frac{c}{2!} (t^{n-1}_1 - t)^2 \quad \text{for} \quad t \in [t^{n-2}_1, t^{n-1}_1]
\]

\[
(-1)^p x^{(n-2)}(t) \geq \frac{c}{2!} (t - t^{n-2}_2)^2 \quad \text{for} \quad t \in [t^{n-2}_2, T]. \tag{2.7}
\]
Choose $i \in \{1, \ldots, n-1\}$ and take all different zeros of functions $x^{(n-1)}, \ldots, x^{(n-1)}$, which are in $(0,T)$. By Lemmas 2.1, 2.2 and Theorem 2.3, there is a finite number $p_i < (n-1)p$ of these zeros. Let us put them in order and denote by $a_1, \ldots, a_{p_i}$. Set $a_0 = 0, a_{p_i+1} = T$. In this way we get $p_i + 1$ disjoint intervals $(a_k, a_{k+1})$, $0 \leq k \leq p_i$, satisfying (2.3).

If $i = 1$, then for $a_1 = t_1^{n-1}$, $a_2 = T$, we get by (2.6) that $|x^{(n-1)}(t)| \geq c(a_1-t)$ for $t \in [a_0, a_1]$ and $|x^{(n-1)}(t)| \geq c(t-a_1)$ for $t \in [a_1, a_2]$.

If $i = 2$, we put $t_1^{n-1} = a_1$, $t_1^{n-2} = a_2$, $t_2^{n-2} = a_3$, $T = a_4$, and then (2.7) gives (2.4) or (2.5).

If $i > 2$ and we integrate the inequalities in (2.7) $(i-2)$-times, we get that on each $[a_k, a_{k+1}]$, $k \in \{0, \ldots, p_i\}$ either (2.4) or (2.5) has to be fulfilled. \qed

References


