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On Eliminating Transformations for Nuisance Parameters in Multivariate Linear Model *

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Abstract

The multivariate linear model, in which the matrix of the first order parameters is divided into two matrices: to the matrix of the useful parameters and to the matrix of the nuisance parameters, is considered. We examine eliminating transformations which eliminate the nuisance parameters without loss of information on the useful parameters and on the variance components.

Key words: Multivariate linear regression model, useful and nuisance parameters, LBLUE, eliminating transformation.

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1 Notations, auxiliary statements

The following notations will be used throughout the paper:
- \( R^n \) the space of all \( n \)-dimensional real vectors;
- \( u_p \) the real column \( p \)-dimensional vector;
- \( A_{m,n} \), \( Tr(A) \) the real \( m \times n \) matrix, the trace of the matrix \( A \);
- \( A', r(A) \) the transpose, the rank of the matrix \( A \);
- \( A^{(j)} \) \( j \)-th column of the matrix \( A \);
- \( vec(A) \) the column vector \( ((A^{(1)})', \ldots, (A^{(n)})')' \);
- \( A \otimes B \) the Kronecker (tensor) product of the matrices \( A, B \);
- \( M(A) \) the range of the matrix \( A \);
- \( A^- \) a generalized inverse of the matrix \( A \) (satisfying \( AA^- A = A \));
- \( A^+ \) the Moore-Penrose generalized inverse of the matrix \( A \) (satisfying \( AA^+ A = A \), \( A^+ A A^+ = A^+ \), \( (AA^+)' = AA^+ \), \( (A^+ A)' = A^+ A \));
- \( P_A \) the orthogonal projector onto \( M(A) \);
- \( M_A = I - P_A \) the orthogonal projector onto \( M(\bar{A}) = Ker(A') \);
- \( I_k \) the \( k \times k \) identity matrix;
- \( 0_{m,n} \) the \( m \times n \) null matrix;
- \( o \) the null element.

If \( M(A) \subset M(V) \), \( V \) p.s.d., then the symbol \( P_A^V \) denotes the projector on the subspace \( M(A) \) in the \( V \)-seminorm given by the matrix \( V, ||x||_V = \sqrt{x'Vx} \);
\( N_A = I - P_A = I - A(A'VA)^{-1}A'V \). Let \( N_{m,n} \) be p.d. (p.s.d.) matrix and \( A_{m,n} \) an arbitrary matrix, then the symbol \( A_{m,n}^- \) denotes the matrix satisfying \( AA_{m,n}^- A = A \) and \( N A_{m,n}^- A = [N A_{m,n}^- A]' \). \( AA_{m,n}^- A = [N A_{m,n}^- A]' \) is a solution of the consistent system \( A x = y \) whose \( N \)-seminorm is minimal, see [4], p. 151). \( A_{m,n}^- \) is called a minimum \( N \)-seminorm g-inverse of the matrix \( A \). It holds
\[
M(A) \subset M(N) \Rightarrow A_{m,n}^- N A^{-1} A' (A N A^{-1} A')^{-1}.
\]

Assertion 1 (see [3], Lemma 16)
\[
(M_S \Sigma M_S)^+ = \Sigma^{-1} - \Sigma^{-1} S (S' \Sigma^{-1} S)^{-1} S' \Sigma^{-1} = \Sigma^{-1} M_S^{-1}, \text{ if } \Sigma \text{ is p.d.,}
\]
\[
(M_S \Sigma M_S)^+ = \Sigma^+ - \Sigma^+ S (S' \Sigma^{-1} S)^{-1} S' \Sigma^+, \text{ if } \Sigma \text{ is p.s.d. and } M(S) \subset M(\Sigma).
\]

Assertion 2 If \( \Sigma \) is p.d. matrix, \( W \) p.s.d. and \( S \) such matrices, that
\[
M(S') = M(S' W S),
\]
then (see [6], Lemma 1)
\[
(M_S^W)' [M_S^W \Sigma (M_S^W)']^+ M_S^W = (M_S \Sigma M_S)^+.
\]
2 Multivariate linear model with nuisance parameters

Let
\[ Y_{n,m} = X_{n,k} B_{k,l} Z_{l,m} + \epsilon_{n,m} \]  
(1)
be a multivariate linear model under consideration. Here \( Y \) is an observation matrix, \( X, Z \), are known nonzero matrices, \( \epsilon \) is a random matrix and \( B \) is a matrix of unknown parameters

\[ B = (B_1, B_2), \]

where \( B_1 \) is a \( k \times r \) matrix of useful parameters which (or their functions) has to be estimated from the observation matrix \( Y \) and \( B_2 \) is a \( k \times s \) matrix of nuisance parameters. Thus we consider the model

\[ Y = X(B_1, B_2) \left( \begin{array}{c} Z_1 \\ Z_2 \end{array} \right) + \epsilon. \]  
(2)

Lemma 1 The model (2) can be equivalently written in the form

\[ \text{vec}(Y) = [Z'_1 \otimes X, Z'_2 \otimes X] \left( \begin{array}{c} \text{vec}(B_1) \\ \text{vec}(B_2) \end{array} \right) + \text{vec}(\epsilon). \]  
(3)
where a \( r \times m \) matrix \( Z_1 \) and a \( s \times m \) matrix \( Z_2 \) are known nonzero matrices.

Proof is obvious by virtue of the following statement

\[ \text{vec}(ABC) = (C' \otimes A) \text{vec}(B), \]  
(4)
valid for all matrices of corresponding types. \( \square \)

Suppose that

1. the observation vector \( \text{vec}(Y) \) has the mean value

\[ E[\text{vec}(Y)] = [Z'_1 \otimes X, Z'_2 \otimes X] \left( \begin{array}{c} \text{vec}(B_1) \\ \text{vec}(B_2) \end{array} \right), \]
and the covariance matrix

\[ \text{var}[\text{vec}(Y)] = \Sigma_{\vartheta} \otimes I_n, \]
where \( m \times m \) matrix \( \Sigma_{\vartheta} \) (the covariance matrix of any column of the matrix \( Y \)) is such a matrix that

2. \( \Sigma_{\vartheta} = \sum_{i=1}^{p} \vartheta_i V_i, \forall \vartheta = (\vartheta_1, \ldots, \vartheta_p) \in \vartheta \subset R^p, \ V_1, \ldots, V_p \) given symmetric matrices,
3. \( \vartheta \subset R^p \) contains an open sphere in \( R^p \),
4. if \( \vartheta \in \vartheta \) the matrix \( \Sigma_{\vartheta} \) is positive definite,
5. the matrix \( \Sigma_{\vartheta} \) is not a function of the matrix \( B = (B_1, B_2), \)
6. suppose that
\[ M(Z_1' \otimes X, Z_2' \otimes X) \subset M(\Sigma_\varphi \otimes I); \] (5)
this condition is warranted by
\[ M(Z_1) \subset M(\Sigma_\varphi) \land \ldots \]
and it means that
\[ \text{vec}(Y) \in M(\Sigma_\varphi \otimes I) \text{ (a.s.)}. \]

Remark 1 A parametric function \( p'\text{vec}(B_1), p \in R^k \), is said to be unbiasedly estimable under the model (2) if there exists an estimator \( L'\text{vec}(Y), L \in R^m \), such that \( E[L'\text{vec}(Y)] = p'\text{vec}(B_1), \forall \text{vec}(B_1), \forall \text{vec}(B_2). \)

The equality
\[ E[L'\text{vec}(Y)] = L'(Z_1' \otimes X)\text{vec}(B_1) + L'(Z_2' \otimes X)\text{vec}(B_2) = p'\text{vec}(B_1), \]
\( \forall \text{vec}(B_1), \forall \text{vec}(B_2), \) is fulfilled if and only if
\[ p = (Z_1 \otimes X)L \ \& \ \ (Z_2 \otimes X)L = o, \]
that is equivalent to
\[ p = (Z_1 \otimes X')M_{Z_2' \otimes X}u, \ u \in R^m. \]
Thus the class of all unbiasedly estimable linear functions \( p'\text{vec}(B_1) \) of the useful parameters in the model (2) is given by
\[ \mathcal{E}_1 = \{p'\text{vec}(B_1) : p \in \mathcal{M}[(Z_1 \otimes X')M_{Z_2' \otimes X}] = \mathcal{M}[Z_1M_{Z_2' \otimes X'}]\}. \]
Obviously the class of all unbiasedly estimable linear functions \( q'\text{vec}(B_2) \) of the nuisance parameters in the model (2) is given by
\[ \mathcal{E}_2 = \{q'\text{vec}(B_2) : q \in \mathcal{M}[(Z_2 \otimes X')M_{Z_1' \otimes X}] = \mathcal{M}[Z_2M_{Z_1' \otimes X'}]\}. \]

Notation 1 Denote \( \text{vec}(B_1) \) and \( \text{vec}(B_2) \) an \( (\Sigma_\varphi^{-1} \otimes I)\)-LS estimator of the vector parameter \( \text{vec}(B_1) \) and \( \text{vec}(B_2) \) respectively computed under the linear model (2) (see [1], p. 161). According to the assumption (6) \( p'\text{vec}(B_1), p \in \mathcal{M}[(Z_1 \otimes X')M_{Z_2' \otimes X}], \text{and} \ q'\text{vec}(B_2), q \in \mathcal{M}[(Z_2 \otimes X')M_{Z_1' \otimes X}], \) are the BLUEs of the function \( p'\text{vec}(B_1) \) and \( q'\text{vec}(B_2) \) respectively (see [1], Theorem 5.3.2., p. 162).

Theorem 1
\[ \left( \begin{array}{c} \text{vec}(B_1) \\ \text{vec}(B_2) \end{array} \right) = \]
\[ = \left( \begin{array}{c} (Z_1[M_{Z_2'} \Sigma_\varphi M_{Z_2'}]^+ Z_1' - Z_1[M_{Z_2'} \Sigma_\varphi M_{Z_2'}]^+ (X'X)^{-} X') \\ (Z_2 \Sigma_\varphi Z_2' - Z_2 \Sigma_\varphi M_{Z_1' M_{Z_2}} (M_{Z_2'} \Sigma_\varphi M_{Z_2'})^+ \otimes (X'X)^{-} X') \end{array} \right) \text{vec}(Y). \]
On eliminating transformations for nuisance parameters...  

**Proof** According to [1], Theorem 5.3.1 we have under the model (2)

\[
\begin{pmatrix}
\text{vec}(B_1) \\
\text{vec}(B_2)
\end{pmatrix} = 
\begin{pmatrix}
(Z_1 \otimes X, Z_2 \otimes X)'(\Sigma_\theta \otimes I) - (Z_1' \otimes X, Z_2' \otimes X)\end{pmatrix}^{-1}
\begin{pmatrix}
Z_1 \otimes X' \\
Z_2 \otimes X'
\end{pmatrix} (\Sigma_\theta \otimes I)^{-\text{vec}(Y}) 
\]

Using the following Rohde's formula for generalized inverse of partitioned p.s.d. matrix (see [3], Lemma 13, p.68)

\[
\begin{pmatrix}
A, & B \\
B', & C
\end{pmatrix}^{-1} = 
\begin{pmatrix}
A^{-1} + A^{-1}B(C - B'A^{-1}B)^{-1}B'A^{-1}, & -A^{-1}B(C - B'A^{-1}B)^{-1} \\
-(C - B'A^{-1}B)^{-1}B'A^{-1}, & (C - B'A^{-1}B)^{-1}
\end{pmatrix}
\]

we get the blocks of the g-inverse matrix in (8):

\[
A_{11} = (Z_1[M_{Z_2} \Sigma_\theta M_{Z_2}]^+ Z_1') \otimes (X'X)^-, \\
A_{12} = -[(Z_1[M_{Z_2} \Sigma_\theta M_{Z_2}]^+ Z_1') Z_1 \Sigma_\theta Z_2' (Z_2 \Sigma_\theta Z_2)^- \otimes (X'X)^- (X'X)(X'X)^-], \\
A_{21} = (A_{12})', \\
A_{22} = [(Z_2 \Sigma_\theta Z_2')^- \otimes (X'X)^-]
\]

After some calculations we get

\[
\begin{align*}
\text{vec}(B_1) &= [(Z_1[M_{Z_2} \Sigma_\theta M_{Z_2}]^+ Z_1') Z_1[M_{Z_2} \Sigma_\theta M_{Z_2}]^+ \otimes (X'X)^- X]'\text{vec}(Y), \\
\text{vec}(B_2) &= [(Z_2 \Sigma_\theta Z_2')^- Z_2 \Sigma_\theta M_{Z_2}^+[M_{Z_2'} \Sigma_\theta M_{Z_2'}]^+ \otimes (X'X)^- X]'\text{vec}(Y).
\end{align*}
\]

The estimates obtained by substitution \(\text{vec}(B_1)\) into unbiasedly estimable functions \(p'\text{vec}(B_1)\) are given uniquely. It can be proved if we take the following assertion (see [3], Lemma 8, p.65)

\[
AB^{-1}C \text{ is invariant to the choice of the g-inverse } B^{-1} 
\]

\[
\iff M(A') \subset M(B') \quad \& \quad M(C) \subset M(B), \quad (9)
\]

into account. \(\square\)
Theorem 2 Let us denote \( \Sigma_0 = \sum_{i=1}^{p} \vartheta_{0,i} V_i \).

a) In model (2) the function \( g'\vartheta = \sum_{i=1}^{p} g_i \vartheta_i, \vartheta \in \vartheta \), is unbiasedly, quadratically and invariantly estimable (i.e. the estimator has the form \( [\text{vec}(Y)]'[A[\text{vec}(Y)] \), where \( A_{mn,mn} \) is symmetric matrix, the estimator is invariant with respect to the change of the matrix \( B \) if and only if

\[
g \in \mathcal{M} \left( S(M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ \right),
\]

where

\[
\{ S(M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ \}_{i,j} = \]

\[
= Tr[(M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ (V_i \otimes I) (M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) \]

\[
\times M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ (V_j \otimes I)], \ i, j = 1, \ldots, p.
\]

b) If the function \( g'\vartheta \) satisfies the condition from a), then the \( \vartheta_0 \)-MINQUE of \( g'\vartheta \) is given as

\[
\widehat{g'\vartheta} = \sum_{i=1}^{p} \lambda_i (\text{vec}(Y))' [M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ (V_i \otimes I)
\]

\[
\times [M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ \text{vec}(Y),
\]

where the vector \( \lambda = (\lambda_1, \ldots, \lambda_p)' \) is a solution of the system of equations

\[
S(M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ \lambda = g.
\]

Proof see [4], Theorem IV.1.11.

Remark 2 The matrix \( S(M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ \) is called the criterion matrix for the estimability of the function \( g'\vartheta \). As \( M_{Z_1^i \otimes X, Z_2^i \otimes X} = M_{Z_1^i \otimes X} M_{Z_2^i \otimes X} \), it holds

\[
\{ S(M_{Z_1^i \otimes X, Z_2^i \otimes X} \Sigma_0 \otimes I) M_{Z_1^i \otimes X, Z_2^i \otimes X})^+ \}_{i,j} = \]

\[
= Tr[(M_{M_{Z_1^i \otimes X}(Z_1^i \otimes X)} \Sigma_0 \otimes I) M_{Z_2^i \otimes X} M_{M_{Z_1^i \otimes X}(Z_1^i \otimes X)})^+ (V_i \otimes I)
\]

\[
\times (M_{M_{Z_1^i \otimes X}(Z_1^i \otimes X)} \Sigma_0 \otimes I) M_{Z_2^i \otimes X} M_{M_{Z_1^i \otimes X}(Z_1^i \otimes X)})^+ (V_j \otimes I)]
\]

\[
= Tr[(M_{Z_1^i \otimes X} Z_2^i \otimes X \Sigma_0 \otimes I) M_{Z_2^i \otimes X} M_{Z_1^i \otimes X})^+ (V_i \otimes I)
\]

\[
\times (M_{Z_1^i \otimes X} Z_2^i \otimes X (\Sigma_0 \otimes I) M_{Z_2^i \otimes X} M_{Z_1^i \otimes X})^+ (V_j \otimes I)], i, j = 1, \ldots, p,
\]

where the equality

\[
[M_{M_{Z_1^i \otimes X}(Z_1^i \otimes X)} M_{Z_2^i \otimes X (\Sigma_0 \otimes I) M_{Z_2^i \otimes X} M_{Z_1^i \otimes X})^+]
\]

\[
= [M_{Z_1^i \otimes X} M_{Z_2^i \otimes X (\Sigma_0 \otimes I) M_{Z_2^i \otimes X} M_{Z_1^i \otimes X})^+],
\]

was used.
3 Eliminating transformations

There are situations in the practice, that the number of nuisance parameters is
much more greater than the number of useful parameters. This fact could cause
difficulties in the course of calculations.

There exist two approaches to the problem of nuisance parameters. One
of them is to eliminate the nuisance parameters by a transformation of the
observation vector provided this transformation is not allowed to cause a loss
of information of the useful parameters.

Our task is to eliminate in the model (2) the matrix $Z'_2 \otimes X$, belonging to
the vector $vec(B_2)$ of nuisance parameters, i.e. we consider the following class
of matrices

$$\mathcal{T} = \{T : T(Z'_2 \otimes X) = 0\},$$

that leads us to linear models

$$[T vec(Y), T(Z'_1 \otimes X)vec(B_1), T(\Sigma_\varnothing \otimes I)T'].$$  \hspace{1cm} (10)

The general solution of the matrix equation $T(Z'_2 \otimes X) = 0$ is of the form

$$T = A[I - (Z'_2 \otimes X)(Z'_2 \otimes X)^-],$$

where $A$ is an arbitrary matrix of the corresponding type, $(Z'_2 \otimes X)^-$ is some
version of generalized inverse of the matrix $Z'_2 \otimes X$.

If we choose $(Z'_2 \otimes X)^- = [(Z'_2 \otimes X)'W(Z'_2 \otimes X)]^{-1}(Z'_2 \otimes X)'W$, where
$W = W_1 \otimes W_2$ is an arbitrary p.s.d. matrix such that

$$\mathcal{M}(Z_2 \otimes X') = \mathcal{M}[(Z_2 \otimes X')W(Z'_2 \otimes X)],$$  \hspace{1cm} (11)

then $T = AM_{Z'_2 \otimes X}^W$, where $M_{Z'_2 \otimes X}^W$ is given uniquely.

First we consider the transformation matrix $T = M_{Z'_2 \otimes X}^W$, i.e. we consider linear
model

$$[M_{Z'_2 \otimes X}^W vec(Y), M_{Z'_2 \otimes X}^W (Z'_1 \otimes X)vec(B_1), M_{Z'_2 \otimes X}^W (\Sigma_\varnothing \otimes I)(M_{Z'_2 \otimes X}^W)', \Sigma_\varnothing \text{ p.d.}]$$  \hspace{1cm} (12)

**Remark 3** As $M_{Z'_2 \otimes X}^{W_1 \otimes W_2} vec(Y) = (I_m \otimes I_n)vec(Y) - (P_{Z'_2}^{W_1} \otimes P_X^{W_2})vec(Y)$,
we can write $Y^{trans} = Y - P_X^{W_2}Y(P_{Z'_2}^{W_1})'$.

**Lemma 2** Let $W$ is p.s.d. matrix such that (11) is valid. Then

$$\mathcal{M}(M_{Z'_2 \otimes X}^W) = \mathcal{M}([M_{Z'_2 \otimes X}^W]').$$

**Proof** see [7], Lemma 2. \hfill \Box

Thus

$$\mathcal{M}[(Z_1 \otimes X')M_{Z'_2 \otimes X}] = \mathcal{M}[(Z_1 \otimes X')(M_{Z'_2 \otimes X}^W)'],$$

i.e. the classes of the estimable functions $p'vec(B_1)$ in the model (2) and in the
model (12) are identical.
Theorem 3 The \( \vartheta \)-LBLUE of the estimable function \( p' \text{vec}(B_1) \), where \( p \in \mathcal{M}[(Z_1 \otimes X')M_{Z_1' \otimes X}] \) in the model (12) is given as

\[
p' \text{vec}(B_1) = \left[ (Z_1[M_{Z_1'} \Sigma \theta M_{Z_1'}]^+ + (X'X)^- X' \right] \text{vec}(Y),
\]

i.e. it is the same as in the model (2), (see Theorem 1).

Proof According to [2], Theorem 3.1.3 the \( \vartheta \)-LBLUE in the model (12) is given as

\[
p' \text{vec}(B_1) = \left[ \left( M_{Z_1' \otimes X}^{W}(Z_1' \otimes X) \right) \right]^{-1} m_{(M_{Z_1' \otimes X}^{W}(\Sigma \theta \otimes I)(M_{Z_1' \otimes X}^{W})')} \right]^{'} M_{Z_1' \otimes X}^{W} \text{vec}(Y)
\]

Using Assertion 2 and Assertion 1 we get

\[
p' \text{vec}(B_1) = \left[ (Z_1 \otimes X') \left( M_{Z_1' \otimes X}^{W}(\Sigma \theta \otimes I)(M_{Z_1' \otimes X}^{W}) \right) \right]^{-1} (Z_1' \otimes X' \text{vec}(Y)
\]

The validity of \( \mathcal{M}[(Z_1 \otimes X')M_{Z_1' \otimes X}] \) follows from (5) and from regularity of \( \Sigma \theta \).

Lemma 3

\[
(M_{Z_1' \otimes X}^{W})' \left[ M_{M_{Z_1' \otimes X}^{W}}^{W}(Z_1' \otimes X) \right] M_{Z_1' \otimes X}^{W}(\Sigma \theta \otimes I)(M_{Z_1' \otimes X}^{W})' M_{M_{Z_1' \otimes X}^{W}}^{W}(Z_1' \otimes X) + M_{Z_1' \otimes X}^{W} = [M_{Z_1' \otimes X}^{W} M_{Z_1' \otimes X}^{W}(\Sigma \theta \otimes I) M_{Z_1' \otimes X}^{W} M_{Z_1' \otimes X}^{W}]^{+}. 
\]
Proof Using Assertions 1, 2 we have
\[
(M^W_{Z_1^i \otimes X})'(M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)) M^W_{Z_1^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})' M^W_{Z_2^i \otimes X}(Z_1^i \otimes X) + M^W_{Z_1^i \otimes X}
\]
\[= (M^W_{Z_2^i \otimes X})'(M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})')' + M^W_{Z_2^i \otimes X}
\]
\[-(M^W_{Z_2^i \otimes X})' [M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})'] + M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)
\]
\[\times \{(Z_1 \otimes X')(M^W_{Z_1^i \otimes X})' [M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})'] + M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)\}^{-1}
\]
\[\times (Z_1 \otimes X')(M^W_{Z_1^i \otimes X})(M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)M^W_{Z_2^i \otimes X} + (Z_1^i \otimes X)\] -
\[\times (Z_1 \otimes X')(M^W_{Z_1^i \otimes X}(\Sigma_0 \otimes I)M^W_{Z_2^i \otimes X})^+
\]
\[= [M^W_{Z_1^i \otimes X} M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)M^W_{Z_2^i \otimes X} M^W_{Z_1^i \otimes X}]^+ .
\]

\[\square\]

Theorem 4 A linear function \( g' \vartheta \) of the vector parameter \( \vartheta \in \mathcal{V} \subset R^p \), unbiasedly estimable in the model (2), unbiasedly estimable in the transformed model (12).

Proof The (i,j)-th element of the criterional matrix in the model (12) is given by
\[
\left\{ S_{(M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)) M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})' M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)} \right\}_{i,j}
\]
\[= Tr\left\{ [M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)] M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})' M^W_{Z_2^i \otimes X}(Z_1^i \otimes X) \right\}^+
\]
\[\times M^W_{Z_2^i \otimes X}(V_i \otimes I)(M^W_{Z_2^i \otimes X})'
\]
\[\times [M^W_{M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)] M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})' M^W_{Z_2^i \otimes X}(Z_1^i \otimes X) \right\}^+
\]
\[\times M^W_{Z_2^i \otimes X}(V_j \otimes I)(M^W_{Z_2^i \otimes X})'
\]
\[= Tr\left\{ [M^W_{Z_2^i \otimes X}] [M^W_{M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)] M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})' M^W_{Z_2^i \otimes X}(Z_1^i \otimes X) \right\}^+
\]
\[\times M^W_{Z_2^i \otimes X}(V_i \otimes I)(M^W_{Z_2^i \otimes X})'
\]
\[\times [M^W_{M^W_{Z_2^i \otimes X}(Z_1^i \otimes X)] M^W_{Z_2^i \otimes X}(\Sigma_0 \otimes I)(M^W_{Z_2^i \otimes X})' M^W_{Z_2^i \otimes X}(Z_1^i \otimes X) \right\}^+
\]
\[\times M^W_{Z_2^i \otimes X}(V_j \otimes I) \right\} .
\]
By Lemma 3 then
\[
\left\{ S(M_{M_{22}^{W}} M_{Z_{2}^{W} \otimes X} (Z_{1}^{W} \otimes X) M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) (M_{Z_{2}^{W} \otimes X} (Z_{1}^{W} \otimes X))^{+}) \right\}_{i,j} = Tr \left\{ [M_{Z_{1}^{W} \otimes X} M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) M_{Z_{2}^{W} \otimes X} M_{Z_{1}^{W} \otimes X}]^{+} (V_{i} \otimes I) \right\} \times [M_{Z_{1}^{W} \otimes X} M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) M_{Z_{2}^{W} \otimes X} M_{Z_{1}^{W} \otimes X}]^{+} (V_{j} \otimes I) \right\}, \quad i,j = 1, \ldots, p.
\]

Due to the Remark 2 it is evident that the criterional matrices in the model (2) and in the model (12) are identical. □

**Theorem 5** Let \( g^{\hat{\vartheta}}, \vartheta \in \vartheta \) be an unbiasedly estimable function. Then the \( \vartheta_{0}-\text{MINQUE} \) in the model (2) and the \( \vartheta_{0}-\text{MINQUE} \) in the model (12) after elimination coincide.

**Proof** We have seen that each function \( g^{\hat{\vartheta}} \), that is unbiasedly estimable in the model (2) is unbiasedly estimable in the model (12).

According to Theorem 2 the \( \vartheta_{0}-\text{MINQUE} \) in the model (12) is given by

\[
\hat{g}^{\vartheta} = \sum_{i=1}^{p} \lambda_{i}(vec(Y))^{\prime} (M_{Z_{1}^{W} \otimes X}^{W})^{\prime}
\]

\[
\left[ [M_{M_{22}^{W}} M_{Z_{2}^{W} \otimes X} (Z_{1}^{W} \otimes X) M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) M_{Z_{2}^{W} \otimes X} (Z_{1}^{W} \otimes X)]^{+} \right. \times M_{Z_{2}^{W} \otimes X} (V_{i} \otimes I) (M_{Z_{2}^{W} \otimes X}^{W})^{\prime}
\]

\[
\left. \times [M_{M_{22}^{W}} M_{Z_{2}^{W} \otimes X} (Z_{1}^{W} \otimes X) M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) M_{Z_{2}^{W} \otimes X} (Z_{1}^{W} \otimes X)]^{+} \right. \times M_{Z_{1}^{W} \otimes X} vec(Y)
\]

\[
= \sum_{i=1}^{p} \lambda_{i}(vec(Y))^{\prime} [M_{Z_{1}^{W} \otimes X} M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) M_{Z_{2}^{W} \otimes X} M_{Z_{1}^{W} \otimes X}]^{+} (V_{i} \otimes I)
\]

\[
\times [M_{Z_{1}^{W} \otimes X} M_{Z_{2}^{W} \otimes X} (\Sigma_{0} \otimes I) M_{Z_{2}^{W} \otimes X} M_{Z_{1}^{W} \otimes X}]^{+} vec(Y),
\]

i.e. this estimator is identical to the estimator in the model (2)—see Remark 2. Lemma 3 has been taken into account. □

**Lemma 4**

\[
[M_{Z_{1}^{W} \otimes X} (\Sigma_{\vartheta} \otimes I) M_{Z_{1}^{W} \otimes X}]^{+} = (\Sigma_{\vartheta}^{-1} \otimes I) - (P_{Z_{1}^{W}}^{\Sigma_{\vartheta}^{-1}} \otimes P_{X}). \quad (13)
\]
Proof With respect to Assertion 1

\[
[M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ = \\
= (\Sigma_\theta^{-1} \otimes I) - (\Sigma_\theta^{-1} Z'_1 (Z_1 \Sigma_\theta^{-1} Z'_1) - Z_1 \Sigma_\theta^{-1} \otimes X[X'X] - X') \\
= (\Sigma_\theta^{-1} \otimes I) - (\Sigma_\theta^{-1} P_{Z'_1}^{\Sigma_\theta^{-1}} \otimes P_X).
\]

\[\square\]

Lemma 5

\[M_{Z'_2 \otimes X}^{[M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+} = M_{Z'_2 \otimes X}^{[M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \otimes I}. \tag{14}\]

Proof With respect to \(M^V_A = I - P^V_A = I - A(A'VA)^{-1}A'V\) and using Lemma 4 we get

\[
[M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \\
= (I \otimes I) - (Z'_2 \otimes X)[(Z_2 \otimes X')\{(\Sigma_\theta^{-1} \otimes I) - (\Sigma_\theta^{-1} P_{Z'_1}^{\Sigma_\theta^{-1}} \otimes P_X)\}(Z'_2 \otimes X)] \\
\times (Z_2 \otimes X')\{(\Sigma_\theta^{-1} \otimes I) - (\Sigma_\theta^{-1} P_{Z'_1}^{\Sigma_\theta^{-1}} \otimes P_X)\} \\
= (I \otimes I) - (P_{Z'_2}^{[M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \otimes P_X}) = M_{Z'_2 \otimes X}^{[M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \otimes I}. \tag{14}\]

\[\square\]

Lemma 6

\[P_{Z'_1 \otimes X}\{M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \otimes P_{Z'_1 \otimes X} = 0.
\]

Proof With respect to Lemma 5

\[P_{Z'_2 \otimes X}\{M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ = P_{Z'_1 \otimes X} = P_{Z'_1 \otimes X}^{M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \otimes P_X,
\]

analogously

\[P_{Z'_2 \otimes X}\{M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ = P_{Z'_1}^{M_{Z'_2 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_2 \otimes X}]^+ \otimes P_X. \tag{14}\]

Since

\[P_{Z'_2}^{M_{Z'_2 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_2 \otimes X}]^+ = P_{Z'_2}^{M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ \otimes P_{Z'_2}^{M_{Z'_1 \otimes X}(\Sigma_\theta \otimes I)M_{Z'_1 \otimes X}]^+ = 0,
\]

we get the statements. \[\square\]
Lemma 7

\[ M_{\Sigma^{-1}_\vartheta \otimes I}(Z'_1 \otimes X, Z'_2 \otimes X) = (I \otimes I) - \left(P_{Z'_1}^{[M_{Z'_2}^{\vartheta} \Sigma_{\vartheta} M_{Z'_2}^{\vartheta}]} \otimes P_X\right) - \left(P_{Z'_2}^{[M_{Z'_1}^{\vartheta} \Sigma_{\vartheta} M_{Z'_1}^{\vartheta}]} \otimes P_X\right) \]

\[ = (I \otimes I) - P_{Z'_1}^{[M_{Z'_2}^{\vartheta} \otimes (\Sigma_{\vartheta} \otimes I) M_{Z'_2}^{\vartheta}]} \otimes P_{Z'_2}^{[M_{Z'_1}^{\vartheta} \otimes (\Sigma_{\vartheta} \otimes I) M_{Z'_1}^{\vartheta}]} \]

\[ = M_{Z'_1 \otimes X}^{[M_{Z'_2}^{\vartheta} \otimes (\Sigma_{\vartheta} \otimes I) M_{Z'_2}^{\vartheta}]} \cdot M_{Z'_1 \otimes X}^{[M_{Z'_1}^{\vartheta} \otimes (\Sigma_{\vartheta} \otimes I) M_{Z'_1}^{\vartheta}]} . \]

Proof

\[ M_{\Sigma^{-1}_\vartheta \otimes I}(Z'_1 \otimes X, Z'_2 \otimes X) = (I \otimes I) - (Z'_1 \otimes X, Z'_2 \otimes X) \]

\[ \times \left[ \left( Z_1 \otimes X' \right) \left( \Sigma^{-1}_\vartheta \otimes I \right) (Z'_1 \otimes X, Z'_2 \otimes X) \right] = (I \otimes I) - (Z'_1 \otimes X, Z'_2 \otimes X) \]

\[ \times \left( Z_1 \Sigma^{-1}_\vartheta Z'_1 \otimes X' X, Z_2 \Sigma^{-1}_\vartheta Z'_2 \otimes X' X \right) - \left( Z_1 \Sigma^{-1}_\vartheta X', Z_2 \Sigma^{-1}_\vartheta X' \right) \]

\[ = (I \otimes I) - (Z'_1 \otimes X, Z'_2 \otimes X) \left( A_{11}, A_{12} \right) \left( A_{21}, A_{22} \right) \left( Z_1 \Sigma^{-1}_\vartheta X', Z_2 \Sigma^{-1}_\vartheta X' \right), \]

where (using the second Rohde’s formula)

\[ A_{11} = (Z_1 [M_{Z'_2}^{\vartheta} \Sigma_{\vartheta} M_{Z'_2}^{\vartheta}]+Z'_1)^- \otimes (X' X)^-, \]

\[ A_{12} = -[(Z_1 [M_{Z'_2}^{\vartheta} \Sigma_{\vartheta} M_{Z'_2}^{\vartheta}]+Z'_1)^- Z_1 \Sigma^{-1}_\vartheta Z'_2 (Z_2 \Sigma^{-1}_\vartheta Z'_2^-) \otimes (X' X)^-(X' X)(X' X)^-], \]

and (using the first Rohde’s formula)

\[ A_{21} = -[(Z_2 [M_{Z'_1}^{\vartheta} \Sigma_{\vartheta} M_{Z'_1}^{\vartheta}]+Z'_2)^- Z_2 \Sigma^{-1}_\vartheta Z'_1 (Z_1 \Sigma^{-1}_\vartheta Z'_1^-) \otimes (X' X)^-(X' X)(X' X)^-], \]

\[ A_{22} = (Z_2 [M_{Z'_1}^{\vartheta} \Sigma_{\vartheta} M_{Z'_1}^{\vartheta}]+Z'_2)^- \otimes (X' X)^-. \]

Substituting these expressions we get the first assertion. The rest of the proof is evident (with respect to Lemma 5 and Lemma 6). \( \square \)

If we use in the eliminating transformation \( T = M_{Z'_2 \otimes X}^{W} \) the following matrix

\[ W = [M_{Z'_1 \otimes X}^{[\Sigma_{\vartheta} \otimes I] M_{Z'_1 \otimes X}^{\vartheta}]} ^{+}, \]

we get the transformation matrix (see (14))

\[ T = M_{Z'_1 \otimes X}^{[M_{Z'_2}^{\vartheta} \otimes (\Sigma_{\vartheta} \otimes I) M_{Z'_2}^{\vartheta}]} \cdot M_{Z'_1 \otimes X}^{[M_{Z'_1}^{\vartheta} \otimes (\Sigma_{\vartheta} \otimes I) M_{Z'_1}^{\vartheta}]} \cdot M_{Z'_1 \otimes X}^{[M_{Z'_1}^{\vartheta} \otimes I]} \cdot M_{Z'_2 \otimes X}^{[M_{Z'_1}^{\vartheta} \otimes I]} . \]
that is very useful. It eliminates the nuisance parameters and does not change the design matrix belonging to the vector of useful parameters, i.e. this transformation yields the following model

\[
\begin{bmatrix}
M_{Z_2' \otimes X}^{[M_{Z_1'} \Sigma_{\theta} M_{Z_1'}]^{+} \otimes I} & vec(Y), (Z_1' \otimes X)vec(B_1), \\
M_{Z_2' \otimes X}^{[M_{Z_1'} \Sigma_{\theta} M_{Z_1'}]^{+} \otimes I} & (\Sigma_{\theta} \otimes I)(M_{Z_2' \otimes X}^{[M_{Z_1'} \Sigma_{\theta} M_{Z_1'}]^{+} \otimes I})', \quad \Sigma_{\theta} \text{ p.d.}
\end{bmatrix}
\]

(15)

**Remark 4** a) The matrix \( W = [M_{Z_1' \otimes X}(\Sigma_{\theta} \otimes I)M_{Z_1' \otimes X}]^{+} \) satisfies the assumption (11), see [2], page 189.

b) Theorem 3, Theorem 4 and Theorem 5 are true in the model (15).

Let us consider the more general model

\[
\begin{bmatrix}
AM_{Z_2' \otimes X}^{[M_{Z_1'} \Sigma_{\theta} M_{Z_1'}]^{+} \otimes I} & vec(Y), A(Z_1' \otimes X)vec(B_1), \\
AM_{Z_2' \otimes X}^{[M_{Z_1'} \Sigma_{\theta} M_{Z_1'}]^{+} \otimes I} & (\Sigma_{\theta} \otimes I)(M_{Z_2' \otimes X}^{[M_{Z_1'} \Sigma_{\theta} M_{Z_1'}]^{+} \otimes I})'A', \quad \Sigma_{\theta} \text{ p.d.,}
\end{bmatrix}
\]

where \( A \) is such that

\[
\mathcal{M}[(Z_1 \otimes X')A'] = \mathcal{M}[(Z_1 \otimes X')M_{Z_2' \otimes X}],
\]

(17)

i.e. the classes of the unbiasedly estimable functions in the model (2) and in the model (16) coincide.

It holds

\[
E\left(AP_{Z_1' \otimes X}^{[M_{Z_2' \otimes X}(\Sigma_{\theta} \otimes I)M_{Z_2' \otimes X}]^{+}} vec(Y)\right) = E\left(AP_{Z_1' \otimes X}^{[M_{Z_2'} \Sigma_{\theta} M_{Z_2'}]^{+} \otimes I} vec(Y)\right)
\]

\[
= AP_{Z_1' \otimes X}^{[M_{Z_2'} \Sigma_{\theta} M_{Z_2'}]^{+} \otimes I} \left[(Z_1' \otimes X)vec(B_1) + (Z_2' \otimes X)vec(B_2)\right]
\]

\[
= A(Z_1' \otimes X)vec(B_1),
\]

i.e. \( AP_{Z_1' \otimes X}^{[M_{Z_2'} \Sigma_{\theta} M_{Z_2'}]^{+} \otimes I} vec(Y) \) is an unbiased estimator of the vector function \( A(Z_1' \otimes X)vec(B_1) \) for each matrix \( A \).

**Lemma 8**

\[
AP_{Z_1' \otimes X}^{[M_{Z_2'} \Sigma_{\theta} M_{Z_2'}]^{+} \otimes I} vec(Y) = AP_{Z_1' \otimes X}^{[M_{Z_2' \otimes X}(\Sigma_{\theta} \otimes I)M_{Z_2' \otimes X}]^{+}} vec(Y)
\]

is the best estimator of its mean value.
Proof We use the basic lemma on the locally best estimators (see [4], p. 84).

The class of the estimators of the null parametric function in the model (2) can be expressed in the form

$$\mathcal{H}_0 = \{ u'M^{(\Sigma^{-1} \otimes I)}_{(Z'_1 \otimes X, Z'_2 \otimes X)} vec(Y), \forall u \in R^{mn} \},$$

as

$$E[L' vec(Y)] = L' (Z'_1 \otimes X, Z'_2 \otimes X) \begin{pmatrix} vec(B_1) \\ vec(B_2) \end{pmatrix} = 0,$$

$$\forall vec(B_1) \in R^{kr}, \forall vec(B_2) \in R^{ks},$$

$$\iff L' (Z'_1 \otimes X, Z'_2 \otimes X) = o'$$

$$\iff L \in M(\mathcal{M}(Z'_1 \otimes X, Z'_2 \otimes X)) = \mathcal{M}(1 (\Sigma'_{\otimes I} \otimes I)M_{Z'_2 \otimes x}^{*})^+, \quad$$

$$\text{cov}(AP_{Z'_1 \otimes x}^{[M_{Z'_2 \otimes x}^{*(\Sigma \otimes I)M_{Z'_2 \otimes x}^{*})^+]} vec(Y), u'M^{(\Sigma^{-1} \otimes I)}_{(Z'_1 \otimes X, Z'_2 \otimes X)} vec(Y))$$

$$= AP_{Z'_1 \otimes x}^{[M_{Z'_2 \otimes x}^{*(\Sigma \otimes I)M_{Z'_2 \otimes x}^{*})^+]} \begin{pmatrix} \Sigma_{\otimes I} \\ \Sigma_{\otimes I} \\ \Sigma_{\otimes I} \end{pmatrix} (\Sigma'_{\otimes I} \otimes I) u = o, \forall u \in R^{mn},$$

for each matrix A, as according to Lemma 6, Lemma 7

$$P_{Z'_1 \otimes x}^{[M_{Z'_2 \otimes x}^{*(\Sigma \otimes I)M_{Z'_2 \otimes x}^{*})^+]} M_{Z'_1 \otimes x}^{(\Sigma_{\otimes I} \otimes I)} = 0. \quad \Box$$

Theorem 6 In the model (16) the estimators

$$A P_{Z'_1 \otimes x}^{[M_{Z'_2 \otimes x}^{*(\Sigma \otimes I)M_{Z'_2 \otimes x}^{*})^+]} vec(Y),$$

where A is an arbitrary matrix such that

$$\mathcal{M}([Z_1 \otimes X']A') = \mathcal{M}([Z_1 \otimes X') M_{Z'_1 \otimes x}],$$

create the class of all optimal estimators of the vector function

$$A(Z'_1 \otimes X') vec(B_1).$$

Proof Let us denote

$$B = M_{Z'_2 \otimes x}^{[M_{Z'_1 \otimes x}^{*(\Sigma \otimes I)M_{Z'_2 \otimes x}^{*})^+]} \otimes I. \quad \text{According to [2], Theorem 3.1.3, the } \beta\text{-LBLUE of the vector function } A(Z'_1 \otimes X') vec(B_1) \text{ in the model (16) is}$$

$$A(Z'_1 \otimes X') vec(B_1) = A(Z'_1 \otimes X) \left\{ \left[ (Z'_1 \otimes X') A' \right]^{-1} \right\}^{AB vec(Y)}$$

$$= A(Z'_1 \otimes X) \left\{ (Z'_1 \otimes X') A' [AB(\Sigma_{\otimes I})B' A']^{-1} A(Z'_1 \otimes X) \right\} (Z'_1 \otimes X') A'$$

$$\times [AB(\Sigma_{\otimes I})B' A']^{-1} AB vec(Y)$$

$$= AB(Z'_1 \otimes X) \left\{ (Z'_1 \otimes X') B' A' [AB(\Sigma_{\otimes I})B' A']^{-1} AB(Z'_1 \otimes X) \right\} (Z'_1 \otimes X')$$

$$\times B' A' [AB(\Sigma_{\otimes I})B' A']^{-1} AB vec(Y) = P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_{\otimes I})B' A']^{-1} AB vec(Y)}. $$

$$\quad \Box$$
On eliminating transformations for nuisance parameters . . .

It is the best unbiased estimator. With respect to the basic lemma on the best estimators
\[
cov \left\{ P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_\phi \otimes I)B'A']^\top} ABvec(Y), u'M_{(Z'_1 \otimes X, Z'_2 \otimes X)}^{\Sigma_{\phi}^{-1} \otimes I} vec(Y) \right\} = 0, \ \forall u \in R^{mn},
\]
is valid, i.e.
\[
P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_\phi \otimes I)B'A']^\top} = A\Sigma_\phi \otimes I (M_{(Z'_1 \otimes X, Z'_2 \otimes X)}^{\Sigma_{\phi}^{-1} \otimes I} )' u'
\]
\[
= P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_\phi \otimes I)B'A']^\top} A\Sigma_\phi \otimes I (M_{(Z'_1 \otimes X, Z'_2 \otimes X)}^{\Sigma_{\phi}^{-1} \otimes I} )' u' = 0, \ \forall u \in R^{mn}.
\]

Thus
\[
P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_\phi \otimes I)B'A']^\top} ABvec(Y) = P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_\phi \otimes I)B'A']^\top} A\Sigma_\phi \otimes I (M_{(Z'_1 \otimes X, Z'_2 \otimes X)}^{\Sigma_{\phi}^{-1} \otimes I} )' vec(Y).
\]

Let us denote
\[
C = (Z_1 \otimes X')B'A'[AB(\Sigma_\phi \otimes I)B'A']^\top AB(Z'_1 \otimes X).
\]

Then
\[
A(Z'_1 \otimes X)vec(B_1) = P_{AB(Z'_1 \otimes X)}^{[AB(\Sigma_\phi \otimes I)B'A']^\top} ABvec(Y)
\]
\[
= AB(Z'_1 \otimes X)C C [(Z_1 \otimes X') [(M_{Z'_2}^{\Sigma_\phi} M_{Z'_2}^{\top} \otimes I)] (Z'_1 \otimes X)]^{-}
\]
\[
\times (Z_1 \otimes X') [(M_{Z'_2}^{\Sigma_\phi} M_{Z'_2}^{\top} \otimes I)] vec(Y)
\]
\[
= A(Z'_1 \otimes X) [(Z_1 \otimes X') [(M_{Z'_2}^{\Sigma_\phi} M_{Z'_2}^{\top} \otimes I)] (Z'_1 \otimes X)]^{-}
\]
\[
\times (Z_1 \otimes X') [(M_{Z'_2}^{\Sigma_\phi} M_{Z'_2}^{\top} \otimes I)] vec(Y)
\]
\[
= A\Sigma_\phi \otimes I (M_{Z'_1 \otimes X}^{\Sigma_{\phi}^{-1} \otimes I} )' vec(Y),
\]

(the best estimator of its mean value \( A(Z'_1 \otimes X)vec(B_1) \) according to Lemma 8).

The following equivalence has been taken into account
\[
A(M_{Z'_2}^{\Sigma_{\phi} M_{Z'_2}^{\top} \otimes I} (Z'_1 \otimes X) C C = A(M_{Z'_2}^{\Sigma_{\phi} M_{Z'_2}^{\top} \otimes I} (Z'_1 \otimes X) = AB(Z'_1 \otimes X)
\]
\[
\iff \mathcal{M} \left( A(M_{Z'_2}^{\Sigma_{\phi} M_{Z'_2}^{\top} \otimes I} (Z'_1 \otimes X) \right) \subset \mathcal{M}(C').
\]

The g-inverse matrix in the matrix \( C \) can be chosen arbitrarily. If we chose it positive definite, the condition on the right side of the equivalence is obvious.

\[\square\]
Example 1 Let us consider following situation (see [5]). When laying the foundations for a large building it is necessary to determine the moment at which the subsoil (after large landscaping has been done) stabilizes to the point that it is possible to continue construction without risk of following damage.

There are \( n \) points chosen at the building site and their heights are repeatedly measured at the moments \( t_1, \ldots, t_m \). It is necessary to create a model describing the subsidence of the subsoil at the chosen points and to estimate the unknown parameters of this model on the basis of the results of the repeated measurements.

The result of the measurement at the \( i \)-th point in the \( j \)-th epoch could be described as follows:

\[
\eta_i(t_j) = \kappa_i - \beta_1(1 - e^{-\beta_2 t_j}) + \varepsilon_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \tag{18}
\]

where \( \kappa_i \) is the height of the \( i \)-th point at time \( t_0 \), the function \( \beta_1(1 - e^{-\beta_2 t}) \) describes the movement of the earth-strata at each point. The parameters \( \beta_1 > 0, \beta_2 > 0 \) are the same at the different points, i.e. we suppose that the geological composition of the subsoil is homogenous. The aim is to estimate the unknown parameters \( \beta_1, \beta_2 \) and \( \kappa_i, \forall i = 1, \ldots, n \).

The civil engineer needs to know when it is possible to continue the construction, i.e. when the subsidence of the subsoil at the points is insignificant. It means that it is necessary to determine such \( \tau \) that

\[
\beta_1(1 - e^{-\beta_2 \tau}) \geq C\beta_1,
\]

where \( 0 < C < 1 \) is a suitable constant which is sufficiently close to 1. It is possible to continue the construction at the time \( t \geq \tau \).

The model (18) is not linear in parameters; we linearize it by using the first two members of the Taylor expansion of the function \( \beta_1(1 - e^{-\beta_2 t}) \) at the suitable point \((\beta_{1,0}, \beta_{2,0})\), \( \beta_{1,0} > 0, \beta_{2,0} > 0 \).

We get the model

\[
\eta_i(t_j) = \kappa_i - \beta_{1,0}(1 - e^{-\beta_{2,0} t_j}) + (1 - e^{-\beta_{2,0} t_j})(\beta_1 - \beta_{1,0}) + \beta_{1,0} t_j e^{-\beta_{2,0} t_j} (\beta_2 - \beta_{2,0}) + \varepsilon_{ij},
\]

\[
i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]

Denote

\[
Y_{i}^{(j)} = \eta_i(t_j) + \beta_{1,0}(1 - e^{-\beta_{2,0} t_j}), \quad \varphi_1(t) = -(1 - e^{-\beta_{2,0} t}), \quad \varphi_2(t) = -\beta_{1,0} t e^{-\beta_{2,0} t},
\]

\[
\delta \beta_1 = \beta_1 - \beta_{1,0}, \quad \delta \beta_2 = \beta_2 - \beta_{2,0}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]

Thus

\[
Y_{i}^{(j)} = \kappa_i + \varphi_1(t_j) \delta \beta_1 + \varphi_2(t_j) \delta \beta_2 + \varepsilon_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]

Let us consider the observation vector

\[
Y = (Y^{(1)}, \ldots, Y^{(m)}), \quad Y^{(j)} = (Y_{1}^{(j)}, \ldots, Y_{n}^{(j)}).
\]
The model described above could be rewritten in the form

\[ Y = X(B_1, B_2) \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \varepsilon, \]

where

\[ X = I_k, \quad B_1 = \begin{pmatrix} \delta \beta_1, \delta \beta_2 \\ \delta \beta_1, \delta \beta_2 \\ \vdots \\ \delta \beta_1, \delta \beta_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}, \]

\[ Z_1 = \begin{pmatrix} \varphi_1(t_1), \varphi_1(t_2), \ldots, \varphi_1(t_m) \\ \varphi_2(t_1), \varphi_2(t_2), \ldots, \varphi_2(t_m) \end{pmatrix}, \quad Z_2 = (1, 1, \ldots, 1). \]

The \( n \times 2 \) matrix \( B_1 \) is a matrix of useful parameters, the \( n \times 1 \) matrix \( B_2 \) is a matrix of nuisance parameters.

Let us choose \( n = 2, \quad m = 2, \quad t_1 = 1, \quad t_2 = 6, \quad \beta_{1,0} = 1, \quad \beta_{2,0} = 1, \)

\[ Z_1 = \begin{pmatrix} -0.6321 & -0.9975 \\ -0.3679 & -0.0149 \end{pmatrix}, \quad Z_2 = (1, 1). \]

\[ B_1 = \begin{pmatrix} \delta \beta_1 & \delta \beta_2 \\ \delta \beta_1 & \delta \beta_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}, \]

For the sake of simplicity let us choose \( W = I, \quad \Sigma = \sigma^2 I \), then we have for \( X = I \)

\[ M_{Z_2^T \otimes X}^W = I - [Z_2^T Z_2]^{-1} Z_2 \otimes I = \begin{pmatrix} 0.5 & 0 & -0.5 & 0 \\ 0 & 0.5 & 0 & -0.5 \\ -0.5 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0 & 0.5 \end{pmatrix}, \]

\[ M_{Z_2^T \otimes X}^{[M_{Z_1^T \Sigma M_{Z_1}^T}]^T \otimes I} = I - [Z_2^T (Z_2 M_{Z_1} Z_2^T)^{-1} Z_2 M_{Z_1} \otimes I] = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ P_{Z_1^T \otimes X}^{[M_{Z_2^T \Sigma M_{Z_2}^T}]^T \otimes I} = Z_1^T (Z_1 M_{Z_2} Z_1^T)^{-1} Z_1 M_{Z_2} \otimes I \]

\[ = \begin{pmatrix} -0.3917 & 0 & 0.3917 & 0 \\ 0 & -0.3917 & 0 & 0.3917 \\ -1.3917 & 0 & 1.3917 & 0 \\ 0 & -1.3917 & 0 & 1.3917 \end{pmatrix}. \]

All these matrices eliminate the nuisance parameters.

**Remark 5** Papers [3], [6] deal with univariate model, in [7] there is the multivariate linear model (2) with \( \text{var}[\text{vec}(Y)] = I \otimes \Sigma_\theta \) considered.
References


