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Further Ultimate Boundedness of Solutions of some System of Third Order Nonlinear Ordinary Differential Equations

A. U. AFUWAPE\textsuperscript{1}, M. O. OMEIKE\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
e-mail: aafuwape@oauife.edu.ng

\textsuperscript{2}Department of Mathematical Sciences, University of Agriculture Abeokuta, Nigeria
e-mail: moomeike@yahoo.com

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Abstract

In this paper, we shall give sufficient conditions for the ultimate boundedness of solutions for some system of third order non-linear ordinary differential equations of the form

$$\ddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

where $X, F(\ddot{X}), G(\dot{X}), H(X), P(t, X, \dot{X}, \ddot{X})$ are real $n$-vectors with $F, G, H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous in their respective arguments. We do not necessarily require that $F(\ddot{X}), G(\dot{X})$ and $H(X)$ are differentiable. Using the basic tools of a complete Lyapunov Function, earlier results are generalized.

Key words: Ultimate boundedness, complete Lyapunov functions, nonlinear third order system.

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1 Introduction

In a sequence of results, Afuwape [1, 2, 3], Ezeilo [5], Ezeilo and Tejumola [8, 9], Meng [10] and Tiryaki [12] studied particular cases of the third-order nonlinear system of differential equations of the form

\[ \ddot{X} + F(\dot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}) \] (1.1)

where \( X, F(\dot{X}), G(\dot{X}), H(X), P(t, X, \dot{X}, \ddot{X}) \) are real \( n \)-vectors with \( F, G, H : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) continuous in the respective arguments.

Boundedness and Periodicity results were discussed by imposing differentiability conditions in [5, 8, 9, 12] on the nonlinear functions in the particular cases of (1.1), while not necessarily differentiable conditions were imposed in [1, 3, 10] for the study of ultimate boundedness of particular cases of (1.1). Furthermore, the Lyapunov second method was used with the aid of a suitable differentiable Lyapunov function.

For \( n = 1 \) and \( f(\ddot{x}) = a\ddot{x}, g(\dot{x}) = b\dot{x} \) this reduces to

\[ \ddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}) \] (1.2)

which was studied by Ezeilo [6, 7]. In [7], Ezeilo studied the ultimate boundedness and convergence of solutions of (1.2) by assuming

\[ \frac{h(\xi + \eta) - h(\eta)}{\xi} \in I_0 \] (1.3)

for some designated \( \xi, \eta \neq 0 \) with \( I_0 = [\delta, kab] \) where \( \delta > 0 \) is an arbitrary constant and \( 0 < k < 1 \). \( I_0 \) is a subset of the generalized Routh–Hurwitz interval \((0, ab)\).

When \( \eta = 0, \xi \neq 0 \) in (1.3) we have

\[ H_0 = H_0(\xi) = \frac{\{h(\xi) - h(0)\}}{\xi} \] (1.4)

and

\[ H_0 = \frac{h(\xi)}{\xi} \quad \text{if} \quad h(0) = 0. \] (1.5)

On the other hand if \( F(\dot{X}) = A\dot{X}, G(\dot{X}) = B\dot{X} \) in (1.1) we have

\[ \dddot{X} + A\dddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}) \] (1.6)

where \( A, B \) are real symmetric \( n \times n \) matrices.

Afuwape [1] and Meng [10] studied (1.6) for the ultimate boundedness and periodicity of solutions for which \( H \) is of class \( C(\mathbb{R}^n) \) by satisfying

\[ H(X) = H(Y) + A(X, Y)(X - Y) \] (1.7)
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where $A(X, Y)$ is a real $n \times n$ operator for any $X, Y$ in $\mathbb{R}^n$, and having real eigenvalues $\lambda_i(A(X, Y))$ ($i = 1, 2, \ldots, n$).

It was assumed that these eigenvalues satisfy

$$0 < \delta_h \leq \lambda_i(A(X, X)) \leq \Delta_h$$

(1.8)

with $\delta_h, \Delta_h$ as fixed constants.

Moreover, the matrices $A, B$ have real positive eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$ respectively with $\delta_a = \min \lambda_i(A), \delta_b = \min \lambda_i(B), \Delta_a = \max \lambda_i(A), \Delta_b = \max \lambda_i(B), i = 1, 2, \ldots, n$ and that for some constant $k(< 1)$ the “generalized” Routh–Hurwitz condition,

$$\Delta_h \leq k\delta_a\delta_g$$

(1.9)

was satisfied. Furthermore, when $F(\ddot{X}) = A\ddot{X}$ in (1.1) we have

$$\ddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

(1.10)

where $A$ is a real symmetric $n \times n$ matrix.

In [3], Afuwape studied (1.10) for the ultimate boundedness of solutions for which $G, H$ are of class $C(\mathbb{R}^n)$ by satisfying

$$G(Y_1) = G(Y_2) + B_g(Y_1, Y_2)(Y_1 - Y_2)$$

(1.11a)

$$H(X_1) = H(X_2) + C_h(X_1, X_2)(X_1 - X_2)$$

(1.11b)

where $B_g(Y_1, Y_2), C_h(X_1, X_2)$ are $n \times n$ real continuous operators, having real eigenvalues $\lambda_i(B_g(Y_1, Y_2)), \lambda_i(C_h(X_1, X_2)), (i = 1, 2, \ldots, n)$ respectively and which satisfy

$$0 < \delta_g \leq \lambda_i(B_g(Y_1, Y_2)) \leq \Delta_g$$

(1.12a)

$$0 < \delta_h \leq \lambda_i(C_h(X_1, X_2)) \leq \Delta_h$$

(1.12b)

with $\delta_g, \delta_h, \Delta_g, \Delta_h$ as fixed constants.

Also, the matrix $A$ has real positive eigenvalues $\lambda_i(A)$ with $\delta_a = \min \lambda_i(A), \Delta_a = \max \lambda_i(A), i = 1, 2, \ldots, n$ and that for some constant $k(< 1)$ the “generalized” Routh Hurwitz condition (1.9) was satisfied.

In this paper, we shall extend earlier results of [1, 3, 5, 8, 9, 10, 12] to systems of the form (1.1) and for which generalized Routh–Hurwitz condition (1.9) is satisfied. A new differentiable Lyapunov function which is a modification of the one used in [10] is used to prove ultimate boundedness of solutions of (1.1). In addition to (1.11a) and (1.11b) we assume that $F$ is of class $C(\mathbb{R}^n)$ and satisfies

$$F(Z_1) = F(Z_2) + A_f(Z_1, Z_2)(Z_1 - Z_2)$$

(1.11c)

where $A_f(Z_1, Z_2)$ is $n \times n$ real continuous operator having real eigenvalues $\lambda_i(A_f(Z_1, Z_2)) (i = 1, 2, \ldots, n)$. These real eigenvalues satisfy

$$0 < \delta_f \leq \lambda_i(A_f(Z_1, Z_2)) \leq \Delta_f$$

(1.12c)

with $\delta_f, \Delta_f$ as fixed constants.
Furthermore, these eigenvalues satisfy, for some constant $k (k < 1$, defined later) the “generalized” Routh–Hurtwitz condition (1.9).

Finally, we shall assume that $P(t, X, Y, Z)$ satisfies

$$
\|P(t, X, Y, Z)\| \leq p_1(t) + p_2(t) \left\{ \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2 \right\}^{p/2} 
+ p_3(t) \left\{ \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2 \right\}^{1/2} 
$$

(1.13)

for any $X, Y, Z$ in $\mathbb{R}^n$, where $p_1(t), p_2(t), p_3(t)$ are continuous functions in $t$ and $0 \leq \rho \leq 1$.

**Remark 1** The estimate (1.13) reduces to [8, 1.3 (3)] if $p_3(t) = \delta_0$. When specialized to the case $n = 1$, the estimate (1.13) reduces to estimate (4.96) of [11, p. 339] if $p_3(t) = q$.

2 Notations

We shall use the notations as given in [1]. Throughout this paper, $\delta$’s and $\Delta$’s with or without suffices will denote positive constants whose magnitudes depend on vector functions $F, G, H$ and $P$. The $\delta$’s and $\Delta$’s with numerical or alphabetical suffices shall retain fixed magnitudes, while those without suffices are not necessarily the same at each occurrence.

Finally, we shall denote the scalar product $\langle X, Y \rangle$ of any vectors $X, Y$ in $\mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. In particular, $\langle X, X \rangle = \|X\|^2$.

3 Statement of the results

Our first main result in this paper is the following:

**Theorem 1** Suppose $F(0) = G(0) = H(0) = 0$, and that

(i) there exist $n \times n$ real continuous operators

$$
A_f(Z_1, Z_2), \quad B_g(Y_1, Y_2), \quad C_h(X_1, X_2)
$$

for any vectors $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ in $\mathbb{R}^n$, such that the functions $F, G, H$ are of class $C(\mathbb{R}^n)$, satisfy (1.11a,b,c), with the eigenvalues, $\lambda_i(A_f(Z_1, Z_2)), \lambda_i(B_g(Y_1, Y_2)), \lambda_i(C_h(X_1, X_2))$ $(i = 1, 2, \ldots, n)$ satisfying (1.12a,b,c);

(ii) the operators $A_f, B_g$ and $C_h$ are associative and commute pairwise, and

(iii) the vector function $P$ satisfies inequality (1.13) for all $X, Y, Z$ in $\mathbb{R}^n$, where $p_1(t), p_2(t)$ and $p_3(t)$ are continuous functions of $t$, with $0 \leq \rho < 1$.

Then, there exist constants $\rho_3, \Delta_1, \Delta_2, \Delta_3$ such that if $|p_3(t)| \leq \rho_3$, for all $t$ in $\mathbb{R}$, with $\rho_3$ chosen small enough, then every solution $X(t)$ of (1.1) with $X(t_0) = \ldots$
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\begin{align*}
X_0, \dot{X}(t_0) = Y_0, \ddot{X}(t_0) = Z_0, \text{ and for any constant } r, \text{ whatever in the range } \\
\frac{1}{2} \leq r \leq 1, \text{ satisfies } \\
\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\}^r \leq \Delta_1 \exp\{-\Delta_2(t - t_0)\} \\
+ \Delta_3 \int_{t_0}^{t} \left\{p_1^{2r}(\tau) + p_2^{2r/(1-\rho)}(\tau)\right\} \exp\{-\Delta_2(t - \tau)\} \, d\tau; \\
(3.1)
\end{align*}

for all \( t \geq t_0 \geq 0 \), where \( \Delta_1 \equiv \Delta_1(X_0, Y_0, Z_0) \).

**Remark 2**

(1) When specialized to the case \( n = 1 \) with \( P \) dependent only on \( t \) the above estimate (3.1) reduces to the estimate (4.86) of [11, Theorem (4.24) p. 335].

(2) In fact this result generalizes Theorem 1 of [3] if \( \rho_3 = \delta_0 \): A number of quite important results can be deduced from the above. For example, we have

**Corollary 1**

If \( P \equiv 0 \) and all the conditions of Theorem 1 hold, then every solution \( X(t) \) of (1.1) satisfies

\begin{align*}
\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\} \rightarrow 0 \\
(3.2)
\end{align*}

as \( t \rightarrow \infty \), provided that \( \rho_3 \) is small enough.

Indeed by setting \( \rho_1(t) = 0 = \rho_2(t) \) in (1.13), we have that

\begin{align*}
\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\}^r \leq \Delta_1 \exp\{-\Delta_2(t - t_0)\}, \quad t \geq t_0
\end{align*}

from which (3.2) follows on letting \( t \rightarrow \infty \).

**Remark 3**

When specialized to the case \( n = 1 \) with \( p_1(t) = p_2(t) = 0 \) i.e. satisfying condition \((C'')\) of [11, Theorem 4.25] then the above estimate (3.2) reduces to the estimate (4.97) of [11, Theorem 4.25].

Further, if \( P \neq 0 \), but such that

\begin{align*}
\int_{t}^{t+\mu} \left\{p_1^{\nu}(\tau) + p_2^{\nu/(1-\rho)}(\tau)\right\} \, d\tau \rightarrow 0 \\
(3.3)
\end{align*}

as \( t \rightarrow \infty \), then we have

**Corollary 2**

Suppose that there are some fixed constants \( \nu \) \((1 \leq \nu \leq 2)\), and \( \mu > 0 \), such that (3.3) is true, and all the conditions of Theorem 1 hold. Then, every solution \( X(t) \) of (1.1) satisfies (3.2) as \( t \rightarrow \infty \).

**Remark 4**

This result is a direct generalization of [6, Theorem 2] when specialized to the case \( n = 1 \). Its proof can be obtained from (3.1) by using an obvious modification of the arguments in [6, §3.2].

The next result is on the ultimate boundedness of solutions of (1.1).
Theorem 2 Suppose that $F(0) = G(0) = H(0) = 0$ and all the conditions of Theorem 1 hold. Suppose further that $|p_3(t)| \leq \rho_3$ for all $t \in \mathbb{R}$ with $\rho_3$ sufficiently small and that the functions $p_1(t), p_2(t)$ satisfy
\[
|p_1(t)| \leq \delta_0 \quad \text{and} \quad |p_2(t)| \leq \delta_1
\]
for all $t \in \mathbb{R}$.

Then, there exists a constant $\Delta_4$ such that every solution $X(t)$ of (1.1) ultimately satisfies
\[
\{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\} \leq \Delta_4 \tag{3.4}
\]

Remark 5 (1) If $|p_1(t)| \leq \delta_0$, $|p_2(t)| \leq \delta_1$ and $|p_3(t)| \leq \rho_3$, with $\rho_3$ sufficiently small, then Theorem 2 reduces to Corollary 3 of [8] for which equation (1.6) was considered.

(2) If $\rho = 0$ in (1.13) we have the estimates (3.6) of [1, Theorem 1] which improves on estimates (3.4) of [1, Theorem 1] and (1.8) of [10, Theorem 1]. Thus, Theorem 2 reduces to Theorem 1 of [1,10] for which (1.6) was considered. Moreover, the estimate (1.13) is a generalization of all the bounds on $P(t, X, Y, Z)$ mentioned earlier.

4 Some preliminary results

We shall state, for completeness, some standard results needed in the proofs of our results.

Lemma 1 (1,§4) Let $Q, D$ be real symmetric commuting $n \times n$ matrices. Then,

(i) for any $X$ in $\mathbb{R}^n$,
\[
\delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2 \tag{4.1}
\]
where $\delta_d, \Delta_d$ are respectively, the least and greatest eigenvalues, of matrix $D$;

(ii) the eigenvalues $\lambda_i(QD)$, $(i = 1, 2, \ldots, n)$ of the product matrix $QD$ are all real and satisfy
\[
\min_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D) \leq \lambda_i(QD) \leq \max_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D) \tag{4.2}
\]

(iii) the eigenvalues $\lambda_i(Q + D)$, $(i = 1, 2, \ldots, n)$ of the sum of $Q$ and $D$ are all real and satisfy
\[
\left\{ \min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D) \right\} \leq \lambda_i(Q + D) \leq \left\{ \max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D) \right\} \tag{4.3}
\]
where $\lambda_j(Q)$ and $\lambda_k(D)$ are respectively the eigenvalues of $Q$ and $D$. 

5 The function $V$

Our main tool in the proof of the results is the continuous function $V = V(X, Y, Z)$ defined for any $X, Y, Z$ in $\mathbb{R}^n$ by

$$2V = \beta(1 - \beta)\delta_2^2\|X\|^2 + \beta\delta_g\|Y\|^2 + \alpha\delta_g\delta_f^{-1}\|Y\|^2 + \alpha\delta_f^{-1}\|Z\|^2$$
$$+ \|Z + \delta_fY + (1 - \beta)\delta_gX\|^2.$$  \hspace{1cm} (5.1)

where $0 < \beta < 1$ and $\alpha > 0$

The following result is immediate from (5.1):

**Lemma 2** Assume that all the hypothesis on vectors $F(Z), G(Y)$ and $H(X)$ in Theorem 1 are satisfied. Then, there exist positive constants $\delta_2$ and $\delta_3$ such that

$$\delta_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq 2V \leq \delta_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2)$$  \hspace{1cm} (5.2)

**Proof** The proof follows if we use Lemma 1 repeatedly and then choose

$$\delta_2 = \min \left\{ \beta(1 - \beta)\delta_2^2; \delta_g(\beta + \alpha\delta_f^{-1}); \alpha\delta_f^{-1} \right\}$$

and

$$\delta_3 = \max \left\{ \delta_g(1 - \beta)(1 + \delta_g + \delta_f); \delta_g(\beta + \alpha\delta_f^{-1}) + \delta_f[1 + \delta_g(1 - \beta) + \delta_f]; 1 + \alpha\delta_f^{-1} + \delta_f + \delta_g(1 - \beta) \right\}.$$  \hspace{1cm} \Box

6 Proof of Theorem 1

Let us replace system of differential equations of form (1.1) in the equivalent system form

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = -F(Z) - G(Y) - H(X) + P(t, X, Y, Z)$$  \hspace{1cm} (6.1)

for which a typical solution will be $(X(t), Y(t), Z(t))$.

To prove Theorem 1, it suffices to show that the function $V$ (defined in (5.1)) satisfies for any solution $(X(t), Y(t), Z(t))$ of (6.1) and for any $r$ in the range $\frac{1}{2} \leq r \leq 1$.

$$\dot{V} \leq -\delta_4\psi^2 + \delta_5 \left\{ p_1^{2r}(t) + p_2^{\frac{2r}{1-r}}(t) \right\} \psi^{2(1-r)}$$  \hspace{1cm} (6.2)

for some constants $\delta_4, \delta_5$ where $\psi^2 = \{\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2\}$. We note that from Lemma 2, (6.2) becomes

$$\dot{V} \leq -\delta_6V + \delta_7 \left\{ p_1^{2r}(t) + p_2^{\frac{2r}{1-r}}(t) \right\} V^{(1-r)}$$  \hspace{1cm} (6.3)

with $\delta_6 = \delta_2\delta_4$ and $\delta_7 = \delta_3\delta_5$. If we choose $U = V^r$, this reduces to

$$\dot{U} \leq -r\delta_6U + r\delta_7 \left\{ p_1^{2r}(t) + p_2^{\frac{2r}{1-r}}(t) \right\} .$$  \hspace{1cm} (6.3)
which can be solved for $U$ to obtain

$$U(t) \leq U(t_0) \exp \{-r\delta_6(t - t_0)\}$$

$$+ \Delta_5 \int_{t_0}^{t} \left\{ p_1^{2r}(\tau) + p_1^{2r-\rho}(\tau) \right\} \exp \{-r\delta_6(t - \tau)\} d\tau$$

(6.4)

for all $t \geq t_0$.

Rewriting this with $V^r = U$ and applying Lemma 2, we shall obtain (3.1) with

$$\Delta_1 = \delta\{\|X(t_0)\|^2 + \|Y(t_0)\|^2 + \|Z(t_0)\|^2\}^r;$$

$$\Delta_2 = r\delta_6 \text{ and } \Delta_3 = \delta\Delta_5$$

Thus the proof of Theorem 1 is complete as soon as inequality (6.2) is proved.

7 The derivative of $V$ and the proof of (6.2)

Let $(X(t), Y(t), Z(t))$ be any solution of (6.1). The total derivative of $V$, with respect to $t$ along the solution path after simplification is

$$\dot{V} = -W_1 - W_2 - W_3 - W_4 - W_5 - W_6 - W_7 + W_8$$

(7.1)

where

$$W_1 = \left\{ \gamma_1\delta_9(1 - \beta)\langle X, H(X) \rangle + \eta_1\delta_f \langle Y, G(Y) - \delta_9(1 - \beta)Y \rangle \right.$$

$$+ \xi_1\alpha\delta_f^{-1}\langle Z, F(Z) \rangle + \langle Z, F(Z) - \delta_f Z \rangle \right\}$$

$$W_2 = \left\{ \gamma_2\delta_9(1 - \beta)\langle X, H(X) \rangle + \eta_2\alpha\delta_f^{-1}\langle Z, F(Z) \rangle + (1 + \alpha\delta_f^{-1})\langle Z, H(X) \rangle \right\}$$

$$W_3 = \left\{ \gamma_3\delta_9(1 - \beta)\langle X, H(X) \rangle + \eta_3\delta_f \langle Y, G(Y) - \delta_9(1 - \beta)Y \rangle + \delta_f \langle Y, H(X) \rangle \right\}$$

$$W_4 = \left\{ \gamma_4\delta_f \langle 1 - \beta \rangle\langle X, H(X) \rangle + \xi_3\alpha\delta_f^{-1}\langle Z, F(Z) \rangle \right.$$ $$+ \delta_g(1 - \beta)\langle X, F(Z) - \delta_f Z \rangle \right\}$$

$$W_5 = \left\{ \gamma_5\delta_9(1 - \beta)\langle X, H(X) \rangle + \eta_5\delta_f \langle Y, G(Y) - \delta_9(1 - \beta)Y \rangle \right.$$ $$+ \delta_g(1 - \beta)\langle X, G(Y) - \delta_g Y \rangle \right\}$$

$$W_6 = \left\{ \xi_4\alpha\delta_f^{-1}\langle Z, F(Z) \rangle + \eta_4\delta_f \langle Y, G(Y) - \delta_9(1 - \beta)Y \rangle \right.$$ $$+ (1 + \alpha\delta_f^{-1})\langle Z, G(Y) - \delta_g Y \rangle \right\}$$

$$W_7 = \left\{ \xi_5\alpha\delta_f^{-1}\langle Z, F(Z) \rangle + \eta_5\delta_f \langle Y, G(Y) - \delta_9(1 - \beta)Y \rangle + \delta_f \langle Y, F(Z) - \delta_f Z \rangle \right\}$$

$$W_8 = \left\{ \langle (1 - \beta)\delta_g X + \delta_f Y + (1 + \alpha\delta_f^{-1})Z, P(t, X, Y, Z) \rangle \right\}$$

with $\xi_i, \eta_i, \gamma_i; \ (i = 1, 2, 3, 4, 5)$ are strictly positive constants such that

$$\sum_{i=1}^{5} \xi_i = 1; \quad \sum_{i=1}^{5} \eta_i = 1 \quad \text{and} \quad \sum_{i=1}^{5} \gamma_i = 1.$$

To arrive at (6.2), we first prove the following:
Lemma 3 Subject to a conveniently chosen value of $k$ in (1.9), we have for all $X, Y, Z$ in $\mathbb{R}^n$

$$W_j \geq 0, \quad (j = 2, 3, 4, 5, 6, 7).$$

Proof For strictly positive constants $k_1, k_2$, conveniently chosen later, we have

$$\langle (1 + \alpha \delta_f^{-1})Z, H(X) \rangle =
\|k_1(1 + \alpha \delta_f^{-1})^{1/2}Z + 2^{-1}k_1^{-1}(1 + \alpha \delta_f^{-1})^{1/2}H(X)\|^2
- \langle k_1^2(1 + \alpha \delta_f^{-1})Z, Z \rangle - \langle 4^{-1}k_1^{-2}(1 + \alpha \delta_f^{-1})H(X), H(X) \rangle$$

and

$$\langle \delta_f Y, H(X) \rangle = \|k_2 \delta_f^{1/2}Y + 2^{-1}k_2^{-1} \delta_f^{1/2}H(X)\|^2
- \langle k_2^2 \delta_f Y, Y \rangle - \langle 4^{-1}k_2^{-2} \delta_f H(X), H(X) \rangle.$$  (7.2a)

Now, using (1.11) and the assumptions that $F(0) = G(0) = H(0) = 0$, we have

$$W_2 = \|k_1(1 + \alpha \delta_f^{-1})^{1/2}Z + 2^{-1}k_1^{-1}(1 + \alpha \delta_f^{-1})^{1/2}H(X)\|^2
+ \langle Z, \xi_2 \alpha \delta_f^{-1}F(Z) - k_1^2(1 + \alpha \delta_f^{-1})Z \rangle
+ \langle H(X), \gamma_2 \delta_g(1 - \beta)X - 4^{-1}k_1^{-2}(1 + \alpha \delta_f^{-1})H(X) \rangle$$

and

$$W_3 = \|k_2 \delta_f^{1/2}Y + 2^{-1}k_2^{-1} \delta_f^{1/2}H(X)\|^2
+ \langle Y, \eta_2 \delta_f [G(Y) - \delta_g(1 - \beta)Y] - k_2^2 \delta_f Y \rangle
+ \langle H(X), \gamma_3 \delta_g(1 - \beta)X - 4^{-1}k_2^{-2} \delta_f H(X) \rangle.$$  (7.3a)

Furthermore, by using Lemma 1 repeatedly, we obtain for all $X, Z$ in $\mathbb{R}^n$, (7.4a)

$$W_2 \geq 0$$

if $k_1^2 \leq \frac{\xi_2 \alpha \delta_f}{\alpha + \delta_f}$ with

$$\Delta_h \leq \frac{4\gamma_2 \xi_2 \alpha (1 - \beta) \delta_f^2 \delta_g}{(\alpha + \delta_f)^2}$$  (7.5a)

and for all $X, Y$ in $\mathbb{R}^n$, (7.4b)

$$W_3 \geq 0.$$  (7.4b)

If $k_2^2 \leq \eta_2 \beta \delta_g$ with

$$\Delta_h \leq 4\gamma_3 \eta_2 \beta (1 - \beta) \delta_g^2 / \delta_f.$$  (7.5b)

Combining all the inequalities in (7.3) and (7.4), we have for all $X, Y, Z$ in $\mathbb{R}^n$, $W_2 \geq 0$ and $W_3 \geq 0$, if $\Delta_h \leq k \delta_f \delta_g$ with

$$k = \min \left\{ \frac{4\gamma_2 \xi_2 \alpha (1 - \beta) \delta_f}{(\alpha + \delta_f)^2}, \frac{4\eta_2 \gamma_3 \beta (1 - \beta) \delta_g}{\delta_f^2} \right\} < 1.$$  (7.6)
To complete the proof of Lemma 3, we need to show that for all \(X, Y, Z \in \mathbb{R}^n\)
\[W_i \geq 0 \quad (i = 4, 5, 6, 7).\]

By hypothesis (1.11) the assumptions that \(F(0) = G(0) = H(0) = 0\), and for strictly positive constants \(k_3, k_4, k_5, k_6\) conveniently chosen later, we have
\[
\langle \delta_g(1 - \beta)X, F(Z) - \delta_f Z \rangle = \langle \delta_g(1 - \beta)X, [A_f(Z, O) - \delta_f I]Z \rangle
\]
\[= \|2^{-1}k_3^{-1}\delta_g^{1/2}(1 - \beta)^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}X \]
\[+ k_3\delta_g^{1/2}(1 - \beta)^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}Z\|^2
\]
\[- \langle 4^{-1}k_3^{-2}\delta_g(1 - \beta)[A_f(Z, O) - \delta_f I]X, X \rangle
\]
\[- \langle k_3^2\delta_g(1 - \beta)[A_f(Z, O) - \delta_f I]Z, Z \rangle \tag{7.8a}\]

\[
\delta_g(1 - \beta)(X, G(Y) - \delta_g Y) = \langle \delta_g(1 - \beta)X, [B_g(Y, O) - \delta_g I]Y \rangle
\]
\[= \|2^{-1}k_4^{-1}\delta_g^{1/2}(1 - \beta)^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}X \]
\[+ k_4\delta_g^{1/2}(1 - \beta)^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}Y\|^2
\]
\[- \langle 4^{-1}k_4^{-2}\delta_g(1 - \beta)[B_g(Y, O) - \delta_g I]X, X \rangle
\]
\[- \langle k_4^2\delta_g(1 - \beta)[B_g(Y, O) - \delta_g I]Y, Y \rangle \tag{7.8b}\]

\[
(1 + \alpha\delta_f^{-1})(Z, G(Y) - \delta_g Y) = \langle (1 + \alpha\delta_f^{-1})Z, [B_g(Y, O) - \delta_g I]Y \rangle
\]
\[= \|2^{-1}k_5^{-1}(1 + \alpha\delta_f^{-1})^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}Z \]
\[+ k_5(1 + \alpha\delta_f^{-1})^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}Y\|^2
\]
\[- \langle 4^{-1}k_5^{-2}(1 + \alpha\delta_f^{-1})[B_g(Y, O) - \delta_g I]Z, Z \rangle
\]
\[- \langle k_5^2(1 + \alpha\delta_f^{-1})[B_g(Y, O) - \delta_g I]Y, Y \rangle \tag{7.8c}\]

\[
\delta_f(Y, F(Z) - \delta_f Z) = \langle \delta_f Y, [A_f(Z, O) - \delta_f I]Z \rangle
\]
\[= \|2^{-1}k_6^{-1}\delta_f^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}Y + k_6\delta_f^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}Z\|^2
\]
\[- \langle 4^{-1}k_6^{-2}\delta_f[A_f(Z, O) - \delta_f I]Y, Y \rangle
\]
\[- \langle k_6^2\delta_f[A_f(Z, O) - \delta_f I]Z, Z \rangle. \tag{7.8d}\]

Thus,
\[
W_4 = \|2^{-1}k_3^{-1}\delta_g^{1/2}(1 - \beta)^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}X \]
\[+ k_3\delta_g^{1/2}(1 - \beta)^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}Z\|^2
\]
\[+ \langle X, \gamma_4\delta_g(1 - \beta)C_h(X, O) - 4^{-1}k_3^{-2}\delta_g(1 - \beta)[A_f(Z, O) - \delta_f I] \rangle X \]
\[+ \langle Z, \xi_3\alpha\delta_g^{-1}A_f(Z, O) - k_3^2\delta_g(1 - \beta)[A_f(Z, O) - \delta_f I] \rangle Z \tag{7.8a}\]
Further ultimate boundedness of solutions . . .

\[
W_5 = \|2^{-1}k_4^{-1}\delta_g^{1/2}(1 - \beta)^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}X \\
+ k_4\delta_g^{1/2}(1 - \beta)^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}Y\|^2 \\
+ \langle X, \{\gamma_2\delta_g(1 - \beta)C_h(X, O) - 4^{-1}k_4^{-2}\delta_g(1 - \beta)[B_g(Y, O) - \delta_g I]\}X \rangle \\
+ \langle Y, \{\eta_3\delta_f[B_g(Y, O) - \delta_g(1 - \beta)I] - k_2^2\delta_g(1 - \beta)[B_g(Y, O) - \delta_g I]\}Y \rangle
\] (7.8b)

\[
W_6 = \|2^{-1}k_5^{-1}(1 + \alpha\delta_f^{-1})^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}Z \\
+ k_5(1 + \alpha\delta_f^{-1})^{1/2}[B_g(Y, O) - \delta_g I]^{1/2}Y\|^2 \\
+ \langle Z, \{\xi_4\alpha\delta_g^{-1}A_f(Z, O) - 4^{-1}k_5^{-2}(1 + \alpha\delta_f^{-1})[B_g(Y, O) - \delta_g I]\}Z \rangle \\
+ \langle Y, \{\eta_4\delta_f[B_g(Y, O) - \delta_f(1 - \beta)I] \\
- k_5^2(1 + \alpha\delta_f^{-1})[B_g(Y, O) - \delta_g I]\}Y \rangle
\] (7.8c)

and

\[
W_7 = \|2^{-1}k_6^{-1}\delta_f^{1/2}[A_f(Z, O) - \delta_f I]^{1/2}Y + k_6\delta_f^{1/2}[A_f(Z, 0) - \delta_f I]^{1/2}Z\|^2 \\
+ \langle Y, \{\eta_5\delta_f[B_g(Y, O) - \delta_g(1 - \beta)I] - 4^{-1}k_6^{-2}\delta_f[A_f(Z, O) - \delta_f I]\}Y \rangle \\
+ \langle Z, \{\xi_5\alpha\delta_f^{-1}A_f(Z, O) - k_5^2\delta_f[A_f(Z, O) - \delta_f I]\}Z \rangle.
\] (7.8d)

Thus, for all \(X, Z\) in \(\mathbb{R}^n\)

\[
W_4 \geq 0
\] (7.9a)

if

\[
\frac{\Delta_f - \delta_f}{4\gamma_4\delta_h} \leq k_3^2 \leq \frac{\xi_3\alpha}{(1 - \beta)(\delta_g - \delta_f)}.
\] (7.10a)

For all \(X, Y\) in \(\mathbb{R}^n\)

\[
W_5 \geq 0
\] (7.9b)

if

\[
\frac{\Delta_g - \delta_g}{4\gamma_5\delta_h} \leq k_4^2 \leq \frac{\eta_3\beta\delta_f}{(1 - \beta)(\Delta_g - \delta_g)}.
\] (7.10b)

For all \(Y, Z\) in \(\mathbb{R}^n\)

\[
W_6 \geq 0
\] (7.9c)

if

\[
\frac{\delta_g(\alpha + \delta_f)(\Delta_g - \delta_g)}{4\xi_4\alpha\delta_f^2} \leq k_5^2 \leq \frac{\beta\eta_4\delta_g\delta_f^2}{(\alpha + \delta_f)(\Delta_g - \delta_g)}.
\] (7.10c)

Also, for all \(Y, Z\) in \(\mathbb{R}^n\)

\[
W_7 \geq 0
\] (7.9d)

if

\[
\frac{\Delta_f - \delta_f}{4\eta_5\beta\delta_g} \leq k_6^2 \leq \frac{\alpha\xi_5}{\delta_f(\Delta_f - \delta_f)}.
\] (7.10d)

This completes the proof of Lemma 3. \(\square\)
We are now left with the estimates for $W_1$ and $W_8$.

From (7.1), we clearly have

$$W_1 \geq \gamma_1 \delta g \delta h (1 - \beta) \|X\|^2 + \eta_1 \delta f \delta g \beta \|Y\|^2 + \xi_1 \alpha \|Z\|^2$$

$$\geq \delta \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)$$

(7.11)

where $\delta = \min \{\gamma_1 \delta g \delta h; \eta_1 \delta f \delta g \beta; \xi_1 \alpha\}$. For the remaining part of the proof of (6.2); let us for convenience denote $(\|X\|^2 + \|Y\|^2 + \|Z\|^2)$ by $\psi^2$.

Since $P(t, X, Y, Z)$ satisfies (1.5), Schwarz's inequality gives for $W_8$.

$$|W_8| \leq \left\{(1 - \beta)\delta \|X\| + \delta f \|Y\| + (1 + \alpha \delta_1^{-1}) \|Z\|\right\} \|P(t, X, Y, Z)\|$$

$$\leq 3^{1/2} \delta \left\{p_3(t) \psi^2 + p_2(t) \psi^{(1+\rho)} + p_1(t) \psi\right\};$$

(7.12)

where $\delta = \max \{(1 - \beta)\delta; \delta f; (1 + \alpha \delta_1^{-1})\}$.

Combining inequalities (7.3), (7.11) and (7.13) with the assumption that $|p_3(t)| \leq \rho_3$ for all $t$ in $\mathbb{R}$, we obtain from (7.1) that

$$\dot{V} \leq -\delta_8 - 3^{1/2} \delta_9 \psi^2 + 3^{1/2} \delta_9 \left\{p_2(t) \psi^{(1+\rho)} + p_1(t) \psi\right\}.$$  

(7.14)

This we can rewrite as

$$\dot{V} \leq -\delta_{10} \psi^2 + \psi_1 + \psi_2$$  

(7.15)

where

$$3\delta_{10} = \delta_8 - 3^{1/2} \delta_9 \rho_3, \quad \psi_1 = \left\{\delta_{11} p_1(t) - \delta_{10} \psi\right\};$$

and

$$\psi_2 = \left\{\delta_{11} p_2(t) \psi^{(1+\rho)} - \delta_{10} \psi^2\right\}.$$  

If we choose $\rho_3$ small enough such that $\delta_{10} > 0$ (following [6, p. 306]), with the necessary modification we obtain

$$\psi_1 \leq \delta_{12} \psi^{2(1-r)} p_1^{2r}(t)$$

(7.16a)

and

$$\psi_2 \leq \delta_{13} \psi^{2(1-r)} p_2^{2r/(1-\rho)}(t)$$

(7.16b)

for any constant $r$ in the range $\frac{1}{2} \leq r \leq 1$.

Thus, (7.15) reduces to

$$\dot{V} \leq -\delta_{10} \psi^2 + \delta_{14} \left\{p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t)\right\} \psi^{2(1-r)}$$

(7.17)

with

$$\delta_{14} = \max \{\delta_{12}; \delta_{13}\}$$

This is (6.2) with $\delta_4 = \delta_{10}$ and $\delta_5 = \delta_{14}$. 
8 Proof of Theorem 2

As pointed out in [1], to prove Theorem 2, it suffices to prove that the function \( V \) satisfies

(i) \( V(X, Y, Z) \to \infty \) as \( (\|X\|^2 + \|Y\|^2 + \|Z\|^2) \to \infty \); and

(ii) \( \dot{V} \leq -1 \)

along paths of any solution \((X(t), Y(t), Z(t))\) of (6.1) for which \( (\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2) \) is large enough. We only need to concern ourselves with property (ii), since by Lemma 2, inequality (5.3), property (i) has been taken care of.

If all the conditions of Theorem 1 are satisfied, then, for any solution \((X(t), Y(t), Z(t))\) of (6.1), \( \dot{V} \) satisfies inequality (7.17). That is

\[
\dot{V} \leq -\delta_{10} \psi^2 + \delta_{14} \left\{ p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t) \right\} \psi^{2(1-r)}
\]

for any \( r \) in the range \( \frac{1}{2} \leq r \leq 1 \).

Now, if \( p_1(t) \) and \( p_2(t) \) are bounded for all \( t \) in \( \mathbb{R} \), then there exists some constant \( \delta_{15} > 0 \) such that

\[
\dot{V} \leq -\delta_{10} \psi^2 + \delta_{15} \psi^{2(1-r)} \leq -1
\]

if

\[
\psi \geq \delta_{16} > (\delta_{10} \delta_{15})^{1/2r}.
\]

Thus property (ii) is proved for \( V \), and this completes the proof of Theorem 2.

References


[7] Ezeilo, J. O. C.: New properties of the equation \( x''' + ax'' + bx' + h(x) = p(t, x, \dot{x}, x''') \) for certain special values of the incremental ratio \( y^{-1}\{h(x+y) - h(x)\} \). In: Equations Differentielles et Functionelles Non-lineares (P. Janssens, J. Mawhin and N. Rouche, eds.), Hermann, Paris, 1973, 447–462.


