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A Characterization of Almost Continuity and Weak Continuity

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Abstract

It is well known that a function f from a space X into a space Y is continuous if and only if, for every set K in X the image of the closure of K under f is a subset of the closure of the image of it.

In this paper we characterize almost continuity and weak continuity by proving similar relations for the subsets K of X.

Key words: Almost continuous function, weakly continuous function.

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1 Introduction and notations

The term almost continuous function is defined in different ways by several authors [3, 4, 5, 7]. In this paper we adopt the following definition due to Singal and Singal [7].

Definition 1 *A function $f : X \to Y$ is said to be almost continuous if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x, such that $f(U) \subset \overline{V}^0$.*

The following definition of weak continuity is due to N. Levine [2].

Definition 2 *A mapping $f : X \to Y$ is said to be weakly continuous if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subset \overline{V}$.*

It is well known the following:

Proposition 1 *A function $f : X \to Y$ is continuous iff for every set K in X we have*

$$f(\overline{K}) \subset \overline{f(K)}. \tag{1}$$

Comparing definitions 1, 2 and this proposition, it is natural to look for similar to (1) relations for the other two kinds of continuity.

The aim of this paper is to prove two theorems which give such relations.

We will use the following definitions:

A set A is said to be regularly open if $A = \overline{A}^0$. Since $\overline{\overline{A}^0}^0 = \overline{A}^0$, we have that for every set A, the set $\overline{A}^0$ is regularly open.

For a set $B \subset X$ we denote by $\overline{B}^{reg}$, the regular closure of B, that is the set of points of X for which, every regularly open set which contains x, intersects B.

2 Almost continuous functions

From definitions 1, 2 we have that continuity implies almost continuity implies weak continuity, see also [1, 6].

If Y is a regular space then all these kinds of continuity coincide.

The analogous relation to (1) for the almost continuity is given by the following:

Theorem 1 *A function $f : X \to Y$ is almost continuous iff for every $K \subset X$ we have*

$$f(\overline{K}) \subset \overline{f(K)}^{reg}.$$

Proof ($\Leftarrow$) Suppose, in contrary, that f is not almost continuous. Then there exists an open set V containing $f(x)$ such that for every open set U containing x we have

$$f(U) \not\subset \overline{V}^0.$$

Therefore in every such U, there exists a point y_U such that $f(y_U) \notin \overline{V}^0$. These points y_U define a net, which converges to x, and such that $f(y_U) \notin \overline{V}^0$, for every U.

Let $K = \{y_U : U \text{ is a neighborhood of } x, f(y_U) \notin \overline{V}^0\}$. Since $x \in \overline{K}$, it follows that $f(x) \in f(\overline{K})$. Now there exists a regularly open set containing $f(x)$, namely the set $\overline{V}^0$, which does not intersect $f(K)$, i.e.

$$f(x) \notin \overline{f(K)}^{reg}.$$

So it follows that,

$$f(\overline{K}) \not\subset \overline{f(K)}^{reg},$$

a contradiction.

($\Rightarrow$) Suppose, in contrary, that there exists a subset K of X with $f(\overline{K}) \not\subset \overline{f(K)}^{reg}$.

Then we can find a point $x \in \overline{K}$ such that $f(x) \notin \overline{f(K)}^{reg}$. It follows that there is a regularly open set $V = \overline{V}^0$ containing $f(x)$ with

$$V \cap f(K) = \emptyset. \tag{2}$$

Let U be an open neighborhood of x. Since $x \in \overline{K}$ it follows that $U \cap K \neq \emptyset$, which gives that

$$f(U) \cap f(K) \neq \emptyset. \tag{3}$$

By (2) $f(K) \subset V^c$, so (3) imply that

$$f(U) \cap V^c \neq \emptyset$$

i.e.

$$f(U) \not\subset V = \overline{V}^0.$$

Therefore f is not almost continuous, a contradiction. □

3 Weak continuity

The corresponding to (1) characterization of weakly continuous functions is given by the following:

Theorem 2 *A function $f : X \to Y$ is weakly continuous if and only if for every subset K of X we will have*

$$f(\overline{K}) \subset \cap\{\overline{U(f(K))} : U(f(K)) \textit{ is an open subset of } Y \textit{ containing } f(K)\}. \tag{4}$$

Proof ($\Leftarrow$) Suppose that f is not weakly continuous. Then there exists an x and an open subset V of Y containing $f(x)$, such that for every open subset U of X containing x we have

$$f(U) \not\subset \overline{V}.$$

Choose a y_U in every member U of an open base at x, such that

$$f(y_U) \notin \overline{V}. \tag{5}$$

In this way we take a net y_U of X converging to x. If K is the set all of these y_U, then $x \in \overline{K}$ and so $f(x) \in f(\overline{K})$.

We will show that there is an open set W containing $f(K)$ such that $f(x) \notin \overline{W}$, which contradicts (4).

Actually (5) implies that

$$f(K) \subset \overline{V}^c \subset V^c. \tag{6}$$

Put $W = \overline{V}^c$. Then $\overline{W} = \overline{\overline{V}^c} \subset V^c$ because of V^c is a closed set. Now since $f(x) \in V$ it follows that

$$f(x) \notin \overline{W}.$$

($\Rightarrow$) Suppose that (4) does not hold. Then there exists a set $K \subset X$ and an open set U containing $f(K)$, with

$$f(\overline{K}) \not\subset \overline{U}.$$

So for some point x in $\overline{K}$ we have $f(x) \notin \overline{U}$, i.e.

$$f(x) \in \overline{U}^c.$$

Put

$$V = \overline{U}^c.$$

We assert that for the open set V, which contains $f(x)$, there does not exist a W containing x with

$$f(W) \subset \overline{V} \tag{7}$$

which contradicts the weak continuity of f.

Suppose, in contrary, that such a W exists.

Since $x \in \overline{K}$ we have $W \cap K \neq \emptyset$, so $f(W) \cap f(K) \neq \emptyset$.

By (7) this implies that

$$f(K) \cap \overline{V} \neq \emptyset. \tag{8}$$

But $V = \overline{U}^c \subset U^c$ and the last set is closed, so $\overline{V} \subset U^c$. Since U contains $f(K)$, it follows that $\overline{V} \cap f(K) = \emptyset$ which contradicts (8). Thus such a W does not exist, and this proves our assertion, which completes the proof. □

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