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Remarks on Ideals in Lower-Bounded Dually Residuated Lattice-Ordered Monoids

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Abstract

Lattice-ordered groups, as well as GMV-algebras (pseudo MV-algebras), are both particular cases of dually residuated lattice-ordered monoids ($DR\ell$-monoids for short). In the paper we study ideals of lower-bounded $DR\ell$-monoids including GMV-algebras. Especially, we deal with the connections between ideals of a $DR\ell$-monoid A and ideals of the lattice reduct of A.

Key words: $DR\ell$-monoid, ideal, prime ideal.

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In 1965, K. L. N. Swamy [11] introduced the notion of a (commutative) dually residuated lattice-ordered semigroup in order to capture the common features of Abelian lattice-ordered groups and Brouwerian algebras. It turns out that well-known MV-algebras [1], an algebraic version of the Łukasiewicz infinite valued propositional logic, can be considered as certain bounded commutative $DR\ell$-monoids [7, 8]. The present concept of a (non-commutative) $DR\ell$-monoid is due to T. Kovář [3]:

Definition 1 An algebra $(A; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of type $\langle 2, 0, 2, 2, 2, 2\rangle$ is said to be a *dually residuated lattice-ordered monoid* (simply, a *$DR\ell$-monoid*) if

(i) $(A; +, 0, \vee, \wedge)$ is an ℓ-monoid, i.e., $(A; +, 0)$ is a monoid, $(A; \vee, \wedge)$ is a lattice and the monoid operation distributes over the lattice operations;

(ii) for any $a, b \in A$, $a \rightharpoonup b$ is the least $x \in A$ such that $x + b \geqslant a$, and $a \leftharpoondown b$ is the least $y \in A$ such that $b + y \geqslant a$;

(iii) A fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leqslant x \vee y, \quad y + ((x \leftharpoondown y) \vee 0) \leqslant x \vee y,$$
$$x \rightharpoonup x \geqslant 0, \quad x \leftharpoondown x \geqslant 0.$$

Recently, J. Rachůnek [10] established the notion of a GMV-algebra as a non-commutative generalization of MV-algebras. Non-commutative structures named pseudo MV-algebras extending MV-algebras were independently introduced also by G. Georgescu and A. Iorgulescu [2]. The relationship between GMV-algebras and $DR\ell$-monoids is similar to the commutative case [10, 6]: every GMV-algebra can be regarded as a bounded $DR\ell$-monoid satisfying certain additional conditions, and conversely, any bounded $DR\ell$-monoid that fulfils those conditions is in fact a GMV-algebra. Other examples come from lattice-ordered groups: every ℓ-group, as well as the positive cone of any ℓ-group, is a $DR\ell$-monoid. Therefore, dually residuated lattice-ordered monoids constitute a wide generalization of ℓ-groups and GMV-algebras. We should remark that there exist also other algebraic structures related to logic (for instance, pseudo BL-algebras) that are equivalent to particular $DR\ell$-monoids.

In this paper we deal with ideals of lower-bounded $DR\ell$-monoids (by [3], a $DR\ell$-monoid A is lower-bounded iff $0 \leqslant x$ for all $x \in A$). We will focus especially the connections between ideals in A and those in $\ell(A)$, the lattice reduct of A. The motivation is the following:

(1) When regarded to be a $DR\ell$-monoid, every GMV-algebra is a lower-bounded $DR\ell$-monoid;

(2) T. Kovář [3] proved that every $DR\ell$-monoid is isomorphic to the direct product of an ℓ-group and a $DR\ell$-monoid with 0 at the bottom.

Let us recall basic properties of dually residuated ℓ-monoids [3] and necessary facts about ideals [4].

Lemma 2 [3] *In any $DR\ell$-monoid we have:*

(i) $x \rightharpoonup x = 0 = x \leftharpoondown x$;

(ii) $((x \rightharpoonup y) \vee 0) + y = x \vee y = y + ((x \leftharpoondown y) \vee 0)$;

(iii) $x \rightharpoonup (y + z) = (x \rightharpoonup z) \rightharpoonup y$, $x \leftharpoondown (y + z) = (x \leftharpoondown y) \leftharpoondown z$;

(iv) *if* $x \leqslant y$ *then* $x \rightharpoonup z \leqslant y \rightharpoonup z$ *and* $x \leftharpoondown z \leqslant y \leftharpoondown z$;

(v) *if* $x \leqslant y$ *then* $z \rightharpoonup x \geqslant z \rightharpoonup y$ *and* $z \leftharpoondown x \geqslant z \leftharpoondown y$;

(vi) $x \leqslant y$ *iff* $x \rightharpoonup y \leqslant 0$ *iff* $x \leftharpoondown y \leqslant 0$;

(vii) $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \vee (x \rightharpoonup z)$, $x \leftharpoondown (y \wedge z) = (x \leftharpoondown y) \vee (x \leftharpoondown z)$;

(viii) $(x \vee y) \rightharpoonup z = (x \rightharpoonup z) \vee (y \rightharpoonup z)$, $(x \vee y) \leftharpoondown z = (x \leftharpoondown z) \vee (y \leftharpoondown z)$.

Remark 3 In Definition 1, the condition (ii) can be equivalently replaced by the following identities [3, 10]:

$$\begin{aligned} (x \rightharpoonup y) + y &\geqslant x, & y + (x \leftharpoondown y) &\geqslant x, \\ x \rightharpoonup y &\leqslant (x \vee z) \rightharpoonup y, & x \leftharpoondown y &\leqslant (x \vee z) \leftharpoondown y, \\ (x + y) \rightharpoonup y &\leqslant x, & (y + x) \leftharpoondown y &\leqslant x. \end{aligned}$$

Letting $|x| = x \vee (0 \rightharpoonup x)$ we define the *absolute value* of $x \in A$. It is easily seen that $0 \leqslant x$ iff $x = |x|$, and hence in the special case that we are dealing with lower-bounded $DR\ell$-monoids, this concept is redundant.

Let $I \subseteq A$. Then I is said to be an *ideal* in A if (i) $0 \in I$, (ii) $x + y \in I$ for all $x, y \in I$, and (iii) $|y| \leqslant |x|$ implies $y \in I$ for all $x \in I$ and $y \in A$.

We use $\mathrm{Id}(A)$ to denote the set of all ideals in A; it is partially ordered by set-inclusion. Obviously, $\mathrm{Id}(A)$ is a complete lattice and for any $X \subseteq A$ there exists the smallest ideal, $I(X)$, including X. It can be easily shown that

$$I(X) = \{a \in A : |a| \leqslant |x_1| + \dots + |x_n| \text{ for some } x_1, \dots, x_n \in X, n \in \mathbb{N}\}.$$

In addition, the ideal lattice $\mathrm{Id}(A)$ is algebraic and distributive.

We define an ideal I to be *prime* if for all $J, K \in \mathrm{Id}(A)$, if $J \cap K \subseteq I$ then $J \subseteq I$ or $K \subseteq I$. Every ideal equals the intersection of all primes exceeding it, and $I \in \mathrm{Id}(A)$ is prime if and only if $|x| \wedge |y| \in I$ entails $x \in I$ or $y \in I$, for all $x, y \in A$.

An ideal I in A is called *normal* if $(x \rightharpoonup y) \vee 0 \in I$ iff $(x \leftharpoondown y) \vee 0 \in I$ for all $x, y \in A$. Equivalently, an ideal I is normal if and only if $x + I^+ = I^+ + x$ for every $x \in A$, where $I^+ = \{a \in I : 0 \leqslant a\}$. The normal ideals of any $DR\ell$-monoid correspond one-to-one to its congruence relations.

We shall write $\ell(A)$ for $(A; \vee, \wedge)$, the lattice reduct of A. As usual, for any $X \subseteq A$, $(X]$ denotes the lattice ideal generated by X. It is worth adding that by [3, Theorem 1.1.23], $\ell(A)$ is a distributive lattice.

From this moment on, A stands for a lower-bounded $DR\ell$-monoid!

Theorem 4 *For any $I \subseteq A$ such that $0 \in I$, the following conditions are equivalent:*

(i) *I is an ideal in A;*

(ii) *if $x \in I$ and $y \rightharpoonup x \in I$ then $y \in I$;*

(iii) *if $x \in I$ and $y \leftharpoondown x \in I$ then $y \in I$.*

Proof We are going to show (i) $\Leftrightarrow$ (ii); the proof of (i) $\Leftrightarrow$ (iii) is parallel.

(i) $\Rightarrow$ (ii): If $x \in I$ and $y \rightharpoonup x \in I$ then $y \leqslant x \vee y = (y \rightharpoonup x) + x \in I$, whence $y \in I$.

(ii) $\Rightarrow$ (i): For $x, y \in I$ we have

$$((x + y) \rightharpoonup y) \rightharpoonup x = (x + y) \rightharpoonup (x + y) = 0 \in I$$

which yields $(x + y) \rightharpoonup y \in I$ and therefore $x + y \in I$. If $y \leqslant x \in I$ then $y \rightharpoonup x = 0 \in I$, and so $y \in I$. □

Theorem 5 *Every ideal in A is an ideal in $\ell(A)$. Moreover, if I is a prime ideal in A then I is a prime ideal in $\ell(A)$.*

Proof Let $I \in \mathrm{Id}(A)$. Then clearly I is non-empty, $y \leqslant x$ entails $y \in I$ whenever $x \in I$, and we have also $x \vee y \in I$ for all $x, y \in I$ since $x \vee y \leqslant x + y$. The latter claim is evident. □

The converse statement fails to be true in general. However, we shall prove that if I is a lattice ideal generated by a set of additively idempotent elements or I is a minimal prime ideal in $\ell(A)$, then it is an ideal in A.

Let $Idem(A) = \{a \in A : a = a + a\}$.

Lemma 6 *For all $a \in Idem(A)$ and $x \in A$ we have:*

(i) $a + x = a \vee x = x + a$,

(ii) $x \rightharpoonup a = x \leftharpoondown a$.

Proof (i) To see that $a + x = a \vee x$, compute

$$\begin{aligned} a + x &= a \vee (a + x) = (a + a) \vee (a + x) \\ &= a + (a \vee x) = a + a + (x \leftharpoondown a) \\ &= a + (x \leftharpoondown a) = a \vee x. \end{aligned}$$

(ii) For every $y \in A$, $y \geqslant x \rightharpoonup a$ iff $a + y = y + a \geqslant x$ iff $y \geqslant x \leftharpoondown a$, so $x \rightharpoonup a = x \leftharpoondown a$. □

Theorem 7 *Let $X \subseteq Idem(A)$. Then $(X]$ is a normal ideal in A.*

Proof We have $a \in (X]$ iff $a \leqslant x_1 \vee ... \vee x_n$ for some $x_1, ..., x_n \in X$ and $a \in I(X)$ iff $a \leqslant x_1 + \cdots + x_m = x_1 \vee \ldots \vee x_m$ for some $x_1, \ldots, x_m \in X$, and therefore $I(X) = (X]$.

If $a \rightharpoonup b \in I(X)$ then $a \rightharpoonup b \leqslant x_1 + \cdots + x_n$, where $x_1, \ldots, x_n \in X$, which implies $a \leqslant x_1 + \cdots + x_n + b = b + x_1 + \cdots + x_n$, and so $a \leftharpoondown b \leqslant x_1 + \cdots + x_n$ proving $a \leftharpoondown b \in I(X)$. Similarly $a \leftharpoondown b \in I(X)$ entails $a \rightharpoonup b \in I(X)$, and consequently, $(X]$ is a normal ideal in A. □

We turn now to minimal prime ideals.

Theorem 8 *(i) Let I be a proper ideal in $\ell(A)$. For $x \in A \setminus I$, let us put*

$$\Phi(I, x) = \{a \in A : x \rightharpoonup a \notin I\}$$

and

$$\Phi(I) = \bigcap \{\Phi(I, x) : x \in A \setminus I\}.$$

Then $\Phi(I)$ is an ideal in A such that $\Phi(I) \subseteq I$. In addition, if I is prime then so is $\Phi(I)$.

(ii) Let I be a proper ideal in $\ell(A)$. For $x \in A \setminus I$, let us put

$$\Psi(I, x) = \{a \in A : x \leftharpoondown a \notin I\}$$

and

$$\Psi(I) = \bigcap\{\Psi(I, x) : x \in A \setminus I\}.$$

Then $\Psi(I)$ is an ideal in A such that $\Psi(I) \subseteq I$. In addition, if I is prime then so is $\Psi(I)$.

Proof (i) Let $a \in \Phi(I)$. If $a \notin I$ then $a \in \Phi(I, a)$, so $0 = a \rightharpoonup a \notin I$, a contradiction. Thus $a \in I$ and we have $\Phi(I) \subseteq I$.

We shall now prove that $\Phi(I) \in \mathrm{Id}(A)$. It is obvious that $0 \in \Phi(I)$ as $x \rightharpoonup 0 = x \notin I$ for all $x \in A \setminus I$. Further, let $a, b \in \Phi(I)$ and take any $x \in A \setminus I$. Then $x \rightharpoonup b \notin I$ and hence $x \rightharpoonup (a+b) = (x \rightharpoonup b) \rightharpoonup a \notin I$ since $a \in \Phi(I, x \rightharpoonup b)$; thus $a + b \in \Phi(I, x)$ for all $x \in A \setminus I$ and consequently, $a + b \in \Phi(I)$. If now $a \in \Phi(I)$ and $b \leqslant a$ then $x \rightharpoonup a \leqslant x \rightharpoonup b$ for every $x \in A \setminus I$, and therefore $x \rightharpoonup b \notin I$ since $x \rightharpoonup b \in I$ would imply $x \rightharpoonup a \in I$. Thus $b \in \Phi(I, x)$ for any $x \in A \setminus I$, i.e. $b \in \Phi(I)$.

For the latter statement we shall need two claims.

Claim A: If $x \leqslant y$ then $\Phi(I, x) \subseteq \Phi(I, y)$.

For every $a \in \Phi(I, x)$, $x \rightharpoonup a \leqslant y \rightharpoonup a$ entails $y \rightharpoonup a \notin I$, so $a \in \Phi(I, y)$.

Claim B: If $a \wedge b \in \Phi(I, x)$ then $a \in \Phi(I, x)$ or $b \in \Phi(I, x)$.

We have $a \wedge b \in \Phi(I, x)$ iff $(x \rightharpoonup a) \vee (x \rightharpoonup b) = x \rightharpoonup (a \wedge b) \notin I$ which yields $x \rightharpoonup a \notin I$ or $x \rightharpoonup b \notin I$.

Let now I be a prime ideal in $\ell(A)$ and assume that $a \wedge b \in \Phi(I)$ for $a, b \in A$. If neither a nor b belongs to $\Phi(I)$ then certainly $a \notin \Phi(I, x)$ and $b \notin \Phi(I, y)$ for some $x, y \in A \setminus I$. Since I a prime ideal in $\ell(A)$, it is obvious that $x \wedge y \notin I$. By Claim A we have $\Phi(I, x \wedge y) \subseteq \Phi(I, x) \cap \Phi(I, y)$, and so $a \wedge b \in \Phi(I)$ yields $a \wedge b \in \Phi(I, x \wedge y) \subseteq \Phi(I, x) \cap \Phi(I, y)$. Hence by Claim B, $a \in \Phi(I, x \wedge y) \subseteq \Phi(I, x) \cap \Phi(I, y)$ or $b \in \Phi(I, x \wedge y) \subseteq \Phi(I, x) \cap \Phi(I, y)$, a contradiction with $a \notin \Phi(I, x)$ and $b \notin \Phi(I, y)$. Thus $a \wedge b \in \Phi(I)$ implies $a \in \Phi(I)$ or $b \in \Phi(I)$.

By replacing "$\rightharpoonup$" by "$\leftharpoondown$" we obtain (ii). □

Remark 9 If $I \in \mathrm{Id}(A)$ then $I = \Phi(I) = \Psi(I)$. Indeed, by Theorem 4 (ii), $a \in I$ and $x \notin I$ yield $x \rightharpoonup a \notin I$. Thus $I \subseteq \Phi(I)$.

Corollary 10 *For every $I \subseteq A$, I is a minimal prime ideal in A if and only if it is a minimal prime ideal in $\ell(A)$.*

Proof If I is a minimal prime ideal in A, then it is a prime ideal in $\ell(A)$ by Theorem 5, and by Theorem 8, I is minimal prime.

Conversely, if I is a minimal prime ideal in $\ell(A)$ then, again by Theorem 8, $\Phi(I)$ is a minimal prime ideal in A and obviously $I = \Phi(I)$. □

Remark 11 Let I be an ideal in $\ell(A)$. If I is a normal subset of A, that is, $x \rightharpoonup y \in I$ iff $x \leftharpoondown y \in I$ for all $x, y \in A$, then one can easily show that $\Phi(I) = \Psi(I)$. Conversely, an ideal I in $\ell(A)$ satisfying $\Phi(I) = \Psi(I)$ need not be normal.

Lemma 12 *If $z \leqslant x + y$ then $z = x_1 + y_1$ for some $x_1 \leqslant x$ and $y_1 \leqslant y$.*

Proof Let $x_1 = x \wedge z \leqslant x$ and $y_1 = z \leftharpoondown x_1$. Then

$$x_1 + y_1 = x_1 + (z \leftharpoondown x_1) = z \vee x_1 = z,$$

where $y_1 = z \leftharpoondown (x \wedge z) = (z \leftharpoondown x) \vee (z \leftharpoondown z) = z \leftharpoondown x \leqslant y$ as desired. □

Corollary 13 *If I, J are normal ideals in A then*

$$I \vee J = \{a \in A : a = x + y \text{ for some } x \in I, y \in J\}.$$

Proof Since I, J are normal ideals, $a \in I \vee J$ iff $a \leqslant x + y$ for $x \in I$ and $y \in J$, and so by Lemma 12, $a = x_1 + y_1$ for some $x_1 \leqslant x, y_1 \leqslant y$, i.e. $x_1 \in I$ and $y_1 \in J$. □

Let A be a bounded $DR\ell$-monoid with the greatest element 1. Let us denote by $B(A)$ the set of all $a \in A$ having the complement a' in $\ell(A)$.

Lemma 14 If $x \wedge y = 0$ then $x + y = x \vee y$.

Proof Let $x \wedge y = 0$. Then

$$x = x \rightharpoonup (x \wedge y) = (x \rightharpoonup x) \vee (x \rightharpoonup y) = x \rightharpoonup y$$

which yields $x + y = (x \rightharpoonup y) + y = x \vee y$. □

Lemma 15 *$B(A) \subseteq Idem(A)$.*

Proof Let $a \in B(A)$, i.e. $a \wedge a' = 0$ and $a \vee a' = 1$ for some $a' \in A$. Note that $a + a' = 1$ since $a \vee a' \leqslant a + a'$. Then

$$a = a + (a \wedge a') = (a + a) \wedge (a + a') = (a + a) \wedge 1 = a + a,$$

so $a \in Idem(A)$. □

Remark 16 Observe that if $a \in B(A)$ then $(a]$ and $(a']$ are normal ideals in A such that $(a] \cap (a'] = \{0\}$ and $(a] \vee (a'] = A$, and therefore we can easily see that A is isomorphic with the direct product of $(a]$ and $(a']$.

Theorem 17 *$B(A)$ is a $DR\ell$-submonoid of A in which $a + b = a \vee b$ and $a \rightharpoonup b = a \leftharpoondown b = a \wedge b'$.*

Proof One readily sees that $B(A)$ is a sublattice of $\ell(A)$ since $\ell(A)$ is a distributive lattice.

By Lemma 6, $a \rightharpoonup b = a \leftharpoondown b$ and $x \geqslant a \rightharpoonup b$ iff $x \vee b = x + b \geqslant a$, whence $a \wedge b' \leqslant (x \vee b) \wedge b' = x \wedge b' \leqslant x$. Conversely, if $x \geqslant a \wedge b'$ then $x + b = x \vee b \geqslant (a \wedge b') \vee b = a \vee b \geqslant a$, thus $x \geqslant a \rightharpoonup b$. Altogether, $x \geqslant a \rightharpoonup b$ iff $x \geqslant a \wedge b'$ for any $x \in A$. Therefore $(a \rightharpoonup b)' = a' \vee b$ and so $a \rightharpoonup b \in B(A)$. □

Corollary 18 $(B(A); \vee, \wedge, {}', 0, 1)$ *is a Boolean algebra, where* $a' = 1 \rightharpoonup a$.

By [6, Theorem 2.3], A is a GMV-algebra if and only if the identities

$$x \wedge y = x \rightharpoonup (x \leftharpoondown y) = x \leftharpoondown (x \rightharpoonup y)$$

hold in A. Therefore, let

$$GMV(A) = \{a \in A : a \wedge x = x \rightharpoonup (x \leftharpoondown a) = x \leftharpoondown (x \rightharpoonup a) \text{ for all } x \in A\}.$$

Lemma 19 *The following identities hold in any* $DR\ell$*-monoid:*

(i) $y \geqslant x \rightharpoonup (x \leftharpoondown y)$, $y \geqslant x \leftharpoondown (x \rightharpoonup y)$,

(ii) $x \leftharpoondown (x \rightharpoonup (x \leftharpoondown y)) = x \leftharpoondown y$, $x \rightharpoonup (x \leftharpoondown (x \rightharpoonup y)) = x \rightharpoonup y$.

Proof (i) Obviously, $y \geqslant x \rightharpoonup (x \leftharpoondown y)$ iff $x \vee y = y + (x \leftharpoondown y) \geqslant x$.

(ii) From $y \geqslant x \rightharpoonup (x \leftharpoondown y)$ we obtain

$$x \leftharpoondown y \leqslant x \leftharpoondown (x \rightharpoonup (x \leftharpoondown y))$$

and by replacing y by $x \leftharpoondown y$ in (i) we immediately have

$$x \leftharpoondown y \geqslant x \leftharpoondown (x \rightharpoonup (x \leftharpoondown y)).$$ □

Theorem 20 $B(A) = Idem(A) \cap GMV(A)$.

Proof If $a \in Idem(A) \cap GMV(A)$ then

$$(1 \rightharpoonup a) \vee a = (1 \rightharpoonup a) + a = 1 \vee a = 1$$

and

$$\begin{aligned}(1 \rightharpoonup a) \wedge a &= (1 \rightharpoonup a) \leftharpoondown ((1 \rightharpoonup a) \rightharpoonup a) = (1 \rightharpoonup a) \leftharpoondown (1 \rightharpoonup (a + a)) \\ &= (1 \rightharpoonup a) \leftharpoondown (1 \rightharpoonup a) = 0,\end{aligned}$$

so $a \in B(A)$.

Conversely, let $a \in B(A) \subseteq Idem(A)$, that is, $a \wedge a' = 0$. In view of Lemma 19 (i) we have $x \rightharpoonup (x \leftharpoondown a) \leqslant x \wedge a$. However,

$$\begin{aligned}x \rightharpoonup (x \leftharpoondown a) &= (x \rightharpoonup (x \leftharpoondown a)) + (a \wedge a') \\ &= ((x \rightharpoonup (x \leftharpoondown a)) + a) \wedge ((x \rightharpoonup (x \leftharpoondown a)) + a') \geqslant a \wedge x\end{aligned}$$

since $(x \rightharpoonup (x \leftharpoondown a)) + a \geqslant a$ and $a' = 1 \leftharpoondown a \geqslant x \leftharpoondown a = x \leftharpoondown (x \rightharpoonup (x \leftharpoondown a))$ by Lemma 2 (iv) and Lemma 19 (ii), which implies $(x \rightharpoonup (x \leftharpoondown a)) + a' \geqslant x$. Therefore, $a \in Idem(A) \cap GMV(A)$. □

Lemma 21 $B(A) = \{a \in A : a \wedge (1 \rightharpoonup a) = 0\} = \{a \in A : a \wedge (1 \leftharpoondown a) = 0\}$.

Proof If $a \wedge (1 \rightharpoonup a) = 0$ then

$$(1 \rightharpoonup a) \vee a = (1 \rightharpoonup a) + a = 1 \vee a = 1$$

by Lemma 14. Thus $a' = 1 \rightharpoonup a$ is the complement of a in $\ell(A)$. □

Corollary 22 *Let I be a normal ideal in A. Then A/I is a Boolean algebra if and only if $a \wedge (1 \rightharpoonup a) \in I$ for all $a \in A$.*

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