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Further Results for some Third Order Differential Systems with Nonlinear Dissipation *

AWAR SIMON UKPERA

Department of Mathematics,
Obafemi Awolowo University, Ile-Ife, Nigeria
e-mail: aukpera@oauife.edu.ng

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Abstract

We formulate nonuniform nonresonance criteria for certain third order differential systems of the form \(X''' + AX'' + G(t, X') + CX = P(t)\), which further improves upon our recent results in [12], given under sharp nonresonance considerations. The work also provides extensions and generalisations to the results of Ezeilo and Omari [5], and Minhós [9] from the scalar to the vector situations.

Key words: Nonlinear dissipation, sharp and nonuniform nonresonance.

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1 Introduction

An investigation of the solvability circumstances for the nonlinear differential system

\[X''' + AX'' + G(t, X') + CX = P(t)\]  \hspace{1cm} (1.1)

subject to the \(T\)-periodic boundary conditions

\[X(0) - X(T) = X'(0) - X'(T) = X''(0) - X''(T) = 0\] \hspace{1cm} (1.2)

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on $[0, T]$ with $T > 0$, was initiated in our recent paper [12]. Our basic motivation has been to provide vector analogues to some existing results in the literature for several scalar prototypes such as those contained in [1], [2], [4] and [5]. For instance, Ezeilo and Omari [5] studied firstly the $2\pi$-periodic solutions associated with the scalar version of (1.1), with $g = g(x')$, satisfying the sharp nonresonance conditions

\[(g_1) \quad k^2 + \alpha^- (|y|) < \frac{g(y)}{y} < (k + 1)^2 - \alpha^+(|y|), \quad k \in \mathbb{N},\]

where $\alpha^\pm : (0, +\infty) \to \mathbb{R}$ are two nonincreasing functions such that

\[\lim_{|y| \to +\infty} |y| \alpha^\pm(|y|) = +\infty,\]

This result has been improved by Minhós [9] by weakening the condition on the oscillation of $g$, with the condition $(g_1)$ replaced by the two conditions

\[(g_2) \quad k^2 \leq \liminf_{|y| \to \pm \infty} \frac{g(y)}{y} \leq \limsup_{|y| \to \pm \infty} \frac{g(y)}{y} \leq (k + 1)^2\]

and

\[(G) \quad k^2 < \limsup_{y \to +\infty} \frac{2G(y)}{y^2}, \quad \liminf_{y \to +\infty} \frac{2G(y)}{y^2} < (k + 1)^2\]

where $G$ denotes the primitive of the nonlinear function $g$, that is,

\[G(y) = \int_0^y g(\tau) d\tau\]

Here, the ratio $\frac{g(y)}{y}$ may interact with the spectrum $\{k^2, \ k \in \mathbb{N}\}$, although $(G)$ imposes some ‘density’ control given by the asymptotic behaviour of the primitive of $g$.

Moreover, when $g = g(t, x')$, nonuniform assumptions

\[(g_3) \quad k^2 \leq \gamma^-(t) \leq \liminf_{|y| \to \pm \infty} \frac{g(t, y)}{y} \leq \limsup_{|y| \to \pm \infty} \frac{g(t, y)}{y} \leq \gamma^+(t) \leq (k + 1)^2\]

uniformly in $y \in \mathbb{R}$ for a.e. $t \in [0, 2\pi]$, where $\gamma^\pm \in L^1(0, 2\pi)$ such that strict inequalities hold on subsets of $[0, 2\pi]$ of positive measure; were also established in [5] for the existence of $2\pi$-periodic solutions, with accompanying uniqueness results given by appropriate modification of these conditions.

Our earlier objective, in [12], to generalise some of these results has been partially addressed with the generation of the sharp nonresonance hypotheses

\[(G_1) \quad k^2 \omega^2 + \alpha^- (\|Y\|) \leq \frac{(G(t, Y), Y)}{\|Y\|^2} \leq (k + 1)^2 \omega^2 - \alpha^+(\|Y\|),\]
uniformly in $Y \in \mathbb{R}^n$ with $\|Y\| \geq r > 0$, and a.e. $t \in [0, T]$, where $k \in \mathbb{N}$, 
$\omega = \frac{2\pi}{T}$, and $\alpha^\pm : \mathbb{R}^n_+ \to \mathbb{R}$ are two functions which are such that
\[
\lim_{\|Y\| \to +\infty} \frac{\|Y\| \alpha^\pm(\|Y\|)}{\|Y\|^2} = +\infty
\]
for the existence of $T$-periodic solutions to (1.1)-(1.2). These relations clearly generalise the sharp nonresonance conditions prescribed in [5].

There are however, certain equations of type (1.1) with $G$ not satisfying $(G_1)$–$(G_2)$, for which, nevertheless, $T$-periodic solvability results appear to be provable, subject to some other generalisations on $G$. An example is the system
\[
X'''' + AX'' + \frac{1}{2}((k + 1)^2\omega^2 + k^2\omega^2 + (2k + 1)\omega^2 \cos t)X' + CX = P(t) \quad (1.3)
\]
with the ratio
\[
\frac{\langle G(t, Y), Y \rangle}{\|Y\|^2} = \frac{1}{2}((k + 1)^2\omega^2 + k^2\omega^2 + (2k + 1)\omega^2 \cos t)
\]
lying in the open interval $(k^2\omega^2, (k + 1)^2\omega^2)$ for a.e. $t \in [0, T]$, but for which there do not exist functions $\alpha^\pm$ satisfying $G_2$ for which $G_1$ holds (since the ratio touches both (possible) eigenvalues as $(k + 1)^2 - k^2 = 2k + 1$). This justifies a further treatment of (1.1) incorporating $g_2$ and $g_3$ along the lines of [3], [7], [8] and [10], which clearly specifies the growth pattern and asymptotic conditions on $G$, unlike the rather arbitrary assumptions employed in [11]. This article proposes some generalisations in this direction.

Note also that condition $(G_2)$ cannot be dropped as shown by the nonlinear system
\[
X'''' + AX'' + k^2\omega^2X' + \tan^{-1}(X') + CX = P(t) \quad (1.4)
\]
Here, the ratio
\[
\frac{\langle G(t, Y), Y \rangle}{\|Y\|^2} = k^2\omega^2 + \|Y\|^{-1}\tan^{-1}(Y),
\]
with
\[
\alpha^{-}(\|Y\|) = \|Y\|^{-1}\tan^{-1}(Y) \quad \text{and} \quad \alpha^{+}(\|Y\|) = 2k\omega^2
\]
but
\[
\lim_{\|Y\| \to \infty} \|Y\|\alpha^{-}(\|Y\|) = \frac{\pi}{2} \neq +\infty,
\]
so that $(G_2)$ is not fulfilled by $\alpha^{-}$ and therefore, the system has no $T$-periodic solution.

Accordingly, $X \in \mathbb{R}^n$, $A$ and $C$ are constant real $n \times n$ nonsingular matrices, and $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $P : [0, T] \to \mathbb{R}^n$ are $n$-vectors, which are $T$-periodic in $t$. We shall assume further that $G$ satisfies the Carathéodory conditions, that is, $G(\cdot, X')$ is measurable for every $X' \in \mathbb{R}^n$; $G(t, \cdot)$ is continuous for a.e. $t \in [0, T]$, and for each $r > 0$, there exists an integrable function $\gamma_r \in L^1([0, T], \mathbb{R})$ such that $\|G(t, X')\| \leq \gamma_r(t)$, for $\|X\| \leq r$ and a.e. $t \in [0, T]$. 

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Let $X$ be a point of the Euclidean space $\mathbb{R}^n$ equipped with the usual norm $\|X\|$. For any pair $X, Y \in \mathbb{R}^n$, we shall write $\langle X, Y \rangle$ for the usual scalar product of $X$ and $Y$ so that in particular, $\langle X, X \rangle = \|X\|^2$.

It is standard result that if $D$ is a real $n \times n$ symmetric matrix, then for any $X \in \mathbb{R}^n$,

$$\delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2, \quad (1.6)$$

where $\delta_d$ and $\Delta_d$ are respectively the least and greatest eigenvalues of $D$. In general, $\lambda_i(D)$ shall denote the eigenvalues of any matrix $D$, and $\|D\|_2$ its spectral norm.

The following Banach spaces will also be frequently referred to:

(i) the classical spaces of $k$ times continuously differentiable functions $C^k([0, T], \mathbb{R}^n)$, $k \geq 0$ an integer, where $C^0 = C$ and $C^\infty = \cap_{k \geq 0} C^k$ with norms $\|X\|_{C^k}$ and $\|X\|_\infty$ respectively;

(ii) the space of $T$-periodic functions $C^k_T([0, T], \mathbb{R}^n)$ defined by

$$C^k_T = \{X : [0, T] \to \mathbb{R}^n : X \in C^k \text{ and } X \text{ is } T\text{-periodic}\}$$

with the norm on $C^k_T$;

(iii) $L^p([0, T], \mathbb{R}^n)$, $1 \leq p < +\infty$, the usual Lebesgue spaces with the norms $\|X\|_{L^p}$ and $\|X\|_\infty$ for $p = +\infty$;

(iv) the Sobolev space $W^{k,p}_T([0, T], \mathbb{R}^n)$, of $T$-periodic functions of order $k$, defined by

$$W^{k,p}_T = \{X : [0, T] \to \mathbb{R}^n : X, X', \ldots, X^{(k-1)} \text{ are absolutely continuous on } [0, T], X^{(k)} \in L^p(0, T) \text{ and } X^{(i)}(0) - X^{(i)}(T) = 0, i = 0, 1, 2, \ldots, k - 1, k \in \mathbb{N}\}$$

with corresponding norm $\|X\|_{W^{k,p}_T}$;

(v) The Hilbert space $H^1([0, T], \mathbb{R}^n)$ defined by

$$H^1(0, T) = \{X : [0, T] \to \mathbb{R}^n : X, X' \text{ is absolutely continuous on } [0, T], X' \in L^2(0, T) \text{ and } X^{(i)}(0) - X^{(i)}(T) = 0, i = 0, 1\}$$

with norm

$$\|X\|_{H^1} = \left\{ \sum_{i=1}^n \left[ \frac{1}{T} \int_0^T x_i(t) \, dt \right]^2 + \frac{1}{T} \int_0^T (x_i(t))^2 \, dt + \frac{1}{T} \int_0^T (x'_i(t))^2 \, dt \right\}^{\frac{1}{2}}.$$

Let

$$\widetilde{H}^1(0, T) = \left\{ X \in H^1(0, T) \left| \frac{1}{T} \int_0^T X(t) \, dt = 0 \right. \right\}$$
2 Previous investigations and some preliminary results

Consider the eigenvalue problem

\[ X''' + AX'' + CX = -\lambda X' \]  \hspace{1cm} (2.1)

together with (1.2), with \( A, C \) nonsingular, and \( \lambda \) a real parameter. It has been shown in [5] that

(i) any \( \lambda \neq k^2\omega^2 \), for each \( k = 1, 2, \ldots \), is not an eigenvalue; and

(ii) \( \lambda = k^2\omega^2 \), for some \( k = 1, 2, \ldots \), is an eigenvalue if and only if \( C = k^2\omega^2 A \).

Let \( \mathcal{E}_k \) be the eigenspace corresponding to the unique eigenvalue \( k^2\omega^2 \), when it exists. Then we deduce from [9] the following result:

For every \( X \in W^{3,2}_T(0, 2\pi) \), we have

\[ \int_0^T \langle X''' + AX'' + k^2\omega^2 X' + CX, X''' + AX'' + (k+1)^2\omega^2 X' + CX \rangle \, dt \geq 0, \]  \hspace{1cm} (2.2)

and the equality holds if and only if \( X = 0 \) or either \( k^2\omega^2 \) or \( (k+1)^2\omega^2 \) is an eigenvalue of (2.1) and \( X \in \mathcal{E}_k \) or \( X \in \mathcal{E}_{k+1} \), respectively.

Each of the statements (i) or (ii) has an important bearing on the solvability of the PBVP for the non-autonomous system

\[ X''' + AX'' + \lambda X' + CX = P(t) \]  \hspace{1cm} (2.3)

with \( P \in L^1 \).

It is clear for instance, from (i) and the Fredholm alternative, that a solution for (1.1)–(1.2) can be expected if the ratio \( \langle G(t, X'), X' \rangle / \|X'\|^2 \) is such that

\[ k^2\omega^2 < \frac{\langle G(t, X'), X' \rangle}{\|X'\|^2} < (k+1)^2\omega^2, \]

for \( \|X'\| \) sufficiently large, and a.e. \( t \in [0, T] \), provided that some control is put on the closeness of the ratio to \( k^2\omega^2 \) and \( (k+1)^2\omega^2 \). This expectation has resulted in the evolution of conditions \((G_1) - (G_2)\).

The main role of statement (ii) is to provide an adequate background against which the sharpness of our conditions on \( G \) can be tested. Observe that \( \alpha \pm \) considered in \((G_1)\) can be infinitesimal as \( \|Y\| \to +\infty \), but by \((G_2)\) their order must be less than one. This implies that the ratio can approach the (possible) eigenvalues \( k^2\omega^2 \) and \( (k+1)^2\omega^2 \), provided that the approach is not too fast. For instance, conditions \((G_1) - (G_2)\) admit functions \( G \) such as

\[ G(Y) = k^2Y - \|Y\|^\alpha \text{sgn}(Y), \quad m \in \mathbb{N}, \quad 0 < \alpha < 1, \]

satisfying

\[ \lim_{\|Y\| \to +\infty} \frac{\langle G(Y), Y \rangle}{\|Y\|^2} = k^2, \]
and yet by the statement (ii), (2.3)–(1.2) with \( \lambda = k^2 \), does not have a solution in general, that is, for unrestricted \( A \) and \( C \) nonsingular. Thus for (1.1), we seek conditions on \( G(t, Y) \) allowing \( \lim_{\| Y \| \to +\infty} \frac{(G(t, Y), Y)}{\| Y \|^2} \) (if it exists) to touch \( k^2, k \in \mathbb{N} \), for many values of \( t \).

In the sequel, we shall require some preliminary lemmas.

**Lemma 2.1** Consider the linear homogeneous system

\[
X'''(t) + AX''(t) + B(t)X'(t) + CX(t) = 0 \quad (2.4)
\]

where \( A \) is an arbitrary matrix, \( C \) is a nonsingular matrix and \( B(t) \equiv (b_{ij}(t)) \) is such that \( b_{ij} \in L^1(0, T) \) and

\[
(B_1) \quad k^2 \omega^2 \leq \lambda_i(B(t)) \leq (k + 1)^2 \omega^2
\]

for a.e. \( t \in [0, T] \), \( i = 1, \ldots, n \), \( k \in \mathbb{N} \), with the strict inequality holding on subsets of \([0, T]\) of positive measure.

Then, (2.4)–(1.2) has no non-trivial solution.

**Proof** Let the solution \( X(t) = \overline{X}(t) + \tilde{X}(t) \) have the Fourier expansion

\[
X(t) \sim \sum_{i=1}^{n} \left( c_{0,i} + \sum_{k=1}^{\infty} (c_{k,i} \cos k\omega t + d_{k,i} \sin k\omega t) \right),
\]

such that

\[
\overline{X} = \sum_{i=1}^{n} \left( c_{0,i} + \sum_{k=1}^{n} (c_{k,i} \cos k\omega t + d_{k,i} \sin k\omega t) \right)
\]

and

\[
\tilde{X} = \sum_{i=1}^{n} \sum_{k=n+1}^{\infty} (c_{k,i} \cos k\omega t + d_{k,i} \sin k\omega t),
\]

for some integer \( N > 0 \) with \( N^2 \omega^2 < \lambda < (N + 1)^2 \omega^2 \), where \( \omega = \frac{2\pi}{T} \).

Then, multiplying (2.4) by \( \overline{X}'(t) - \tilde{X}'(t) \) and integrating over \([0, T]\) gives,

\[
\int_{0}^{T} \left( (\tilde{X}'(t))^2 - \langle B(t)\overline{X}'(t), \overline{X}'(t) \rangle \right) dt
\]

\[
- \int_{0}^{T} \left( (\overline{X}'(t))^2 - \langle B(t)\overline{X}'(t), \overline{X}'(t) \rangle \right) dt = 0. \quad (2.5)
\]

Let \( \delta \) be a constant defined by

\[
\delta = \frac{1}{2} \left( \min_{1 \leq i \leq n} \lambda_i(B(t)) + \max_{1 \leq i \leq n} \lambda_i(B(t)) \right) \quad (2.6)
\]

for a.e. \( t \in [0, T] \). Then in fact,

\[
k^2 \omega^2 \leq \delta \leq (k + 1)^2 \omega^2, \quad \text{for a.e. } t \in [0, T],
\]

\[
k^2 \omega^2 < \delta < (k + 1)^2 \omega^2, \quad \text{on subsets of } [0, T] \text{ of positive measure}. \quad (2.7)
\]
Thus, combining (B₁), (2.6) and (2.7), (2.5) becomes
\[
0 \geq \int_0^T \left[ \left( \tilde{X}''(t) \right)^2 - \delta \left( \tilde{X}'(t) \right)^2 \right] dt - \int_0^T \left[ \left( \overline{X}''(t) \right)^2 - \delta \left( \overline{X}'(t) \right)^2 \right] dt = 0. \tag{2.8}
\]

By Parseval’s identity given by
\[
\int_0^T \|X\|^2 dt = \sum_{i=1}^n \left( c_{0,i}^2 + \frac{T}{2} \sum_{k=1}^n (c_{k,i}^2 + d_{k,i}^2) \right),
\]
(2.8) becomes
\[
\frac{T}{2} \sum_{i=1}^n \left[ \sum_{k=N+1}^{\infty} k^2 \omega^2 (k^2 \omega^2 - \delta) (c_{k,i}^2 + d_{k,i}^2) + \sum_{k=1}^N k^2 \omega^2 (\delta - k^2 \omega^2) (c_{k,i}^2 + d_{k,i}^2) \right] = 0. \tag{2.9}
\]
It follows from (2.7) that \( c_{k,i} = 0 \) (\( k = 0, 1, 2, \ldots \)) and \( d_{k,i} = 0 \) (\( k = 1, 2, \ldots \)),
for all \( i = 1, \ldots, n \). Thus, \( X \equiv 0 \), and the lemma follows. □

**Lemma 2.2** Let \( C \) be nonsingular, and assume that \( M, N \in L^1([0, T], \mathbb{R}^n) \) are nonsingular matrices which satisfy the following conditions
\[
k^2 \omega^2 \|Y\|^2 \leq \langle M(t)Y, Y \rangle \leq \langle N(t)Y, Y \rangle \leq (k + 1)^2 \omega^2 \|Y\|^2 \tag{2.10}
\]
uniformly in \( Y \in \mathbb{R}^n \), for a.e. \( t \in [0, T] \), \( k \in \mathbb{N} \), \( \omega = \frac{2\pi}{T} \), and
\[
k^2 \omega^2 \|Y\|^2 < \langle M(t)Y, Y \rangle, \quad \langle N(t)Y, Y \rangle < (k + 1)^2 \omega^2 \|Y\|^2 \tag{2.11}
\]
on subsets of \([0, T]\) of positive measure.

Then, there exists constants \( \epsilon = \epsilon(M, N, C) > 0 \) and \( \delta_0 = \delta_0(M, N, C) > 0 \)
uniformly a.e. on \([0, T]\), such that for all \( B(t) \equiv (b_{ij}(t)) \) with \( b_{ij} \in L^1([0, T], \mathbb{R}) \)
satisfying
\[
\langle M(t)Y, Y \rangle - \epsilon \|Y\|^2 \leq \langle B(t)Y, Y \rangle \leq \langle N(t)Y, Y \rangle + \epsilon \|Y\|^2 \tag{B₂}
\]
uniformly in \( Y \in \mathbb{R}^n \), a.e. on \([0, T]\), and all \( X \in W^{3,1}_T([0, T], \mathbb{R}^n) \), one has
\[
\|X''' + AX'' + B(\cdot)X' + CX\|_{L^1} \geq \delta_0 \|X\|_{W^{3,1}_T} \tag{2.12}
\]

**Proof** Let us assume that the conclusion of the Lemma does not hold, that is, \( \epsilon \)
and \( \delta_0 \) do not exist. Then, there exists a sequence \((X_n) \in W^{3,1}([0, T], \mathbb{R}^n)\)
with \( \|X_n\|_{W^{3,1}_T} = 1 \), and a sequence \((B_n) \in L^1([0, T], \mathbb{R}^n)\) of nonsingular matrices
with
\[
\langle M(t)Y, Y \rangle - \frac{1}{n} \|Y\|^2 \leq \langle B_n(t)Y, Y \rangle \leq \langle N(t)Y, Y \rangle + \frac{1}{n} \|Y\|^2, \quad n \in \mathbb{N}, \tag{2.13}
\]
uniformly in \( Y \in \mathbb{R}^n \), for a.e. \( t \in [0, T] \), such that for all \( X \in W^{3,1} \), one has
\[
\int_0^T \|X_n'''(t) + AX_n''(t) + B_n(t)X_n'(t) + CX_n\| dt < \frac{1}{n}. \tag{2.14}
\]
Let $\|B_n\|$ denote the norm of $B_n$. Then, by (2.13), there exists some $\beta \in L^1([0,T], \mathbb{R})$ such that
\[
\|B_n(t)\| \leq \beta(t), \quad n = 1, 2, \ldots
\] (2.15)
for a.e. $t \in [0,T]$, $n \in \mathbb{N}$. For example, one can take
\[
\beta(t) \equiv \frac{1}{\|Y\|^2} \left[ \| \langle M(t)Y, Y \rangle - \langle Y, Y \rangle \| + \| \langle N(t)Y, Y \rangle + \langle Y, Y \rangle \| \right].
\]

Now, by the compact embedding of $W^{3,1}([0,T], \mathbb{R}^n)$ into $W^{2,1}([0,T], \mathbb{R}^n)$ and the continuous embedding of $W^{2,1}([0,T], \mathbb{R}^n)$ into $C^1([0,T], \mathbb{R}^n)$ imply that by going to subsequences if necessary, we can assume that
\[
X_n \to X \text{ in } C^1([0,T], \mathbb{R}^n), \quad X''_n \to X'' \text{ in } L^\infty([0,T], \mathbb{R}^n) \subset L^1([0,T], \mathbb{R}^n).
\] (2.16)

Moreover, by (2.15), we deduce that
\[
B_n \rightharpoonup B \text{ in } L^1([0,T], \mathbb{R}^{n^2})
\] (2.17)
so that by (2.13),
\[
\langle M(t)Y, Y \rangle \leq \langle B(t)Y, Y \rangle \leq \langle N(t)Y, Y \rangle
\] (2.18)
for a.e. $t \in [0,T]$.

On the other hand, for every $\Phi \in L^\infty([0,T], \mathbb{R}^n)$, we have by Schwarz inequality
\[
\left\| \int_0^T \left\langle B_n(t)X_n(t) - B(t)X(t), \Phi(t) \right\rangle dt \right\|
\leq \left\| \int_0^T \langle B_n(t)(X_n(t) - X(t)), \Phi(t) \rangle dt \right\| + \left\| \int_0^T \langle (B_n(t) - B(t))X(t), \Phi(t) \rangle dt \right\|
\leq \|\Phi\|_{L^\infty} \|\beta\|_{L^1} \|X_n' - X'\|_{L^\infty} + \left\| \int_0^T \langle (B_n(t) - B(t))X(t), \Phi(t) \rangle dt \right\|.
\] (2.19)

The right hand side of (2.19) tends to zero by (2.16) and (2.17), and we deduce that
\[
B_nX_n' \rightharpoonup BX' \text{ in } L^1([0,T], \mathbb{R}^n).
\] (2.20)

By (2.14), (2.16) and (2.20), it follows that
\[
X_n''' = -AX_n' - B_n(\cdot)X_n' - CX_n \rightharpoonup -AX''' - B(\cdot)X' - CX \text{ in } L^1([0,T], \mathbb{R}^n).
\] (2.21)

Since the operator
\[
\frac{d^3}{dt^3} : W^{3,1}([0,T], \mathbb{R}^n) \subset L^1([0,T], \mathbb{R}^n) \to L^1([0,T], \mathbb{R}^n)
\]
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is weakly closed, this implies (by (2.16) and (2.21)) that $X \in W^{3,1}([0, T], \mathbb{R}^n)$, and $X''' = -AX'' - B(\cdot)X' - CX$, that is,

$$X'''(t) + AX''(t) + B(t)X'(t) + CX(t) = 0,$$

(2.22)

for a.e. $t \in [0, T]$ and $X \in W^{3,1}([0, T], \mathbb{R}^n)$.

It follows from (2.9), (2.10), (2.18), (2.22) and Lemma 2.1 that $X \equiv 0$, that is, $X_n \to 0$ in $W^{3,1}([0, T], \mathbb{R}^n)$ as $n \to \infty$. But this clearly contradicts the initial assumption that $\|X_n\|_{W^{3,1}} = 1$ for all $n$, and the proof is complete. \qed

**Lemma 2.3** Let $D \in L^1([0, T], \mathbb{R}^{n^2})$ be a nonsingular matrix such that $0 \leq \lambda_i(D(t)) \leq \omega^2$ a.e. on $[0, T]$, with the strict inequality holding on a subset of $[0, T]$ of positive measure. Then, there exists a constant $\eta = \eta(D) > 0$ such that for all $X \in H^1([0, T], \mathbb{R}^n)$, we have

$$\frac{1}{T} \int_0^T \left[ \left( \tilde{X}'(t) \right)^2 - \langle D(t)\tilde{X}(t), \tilde{X}(t) \rangle \right] dt \geq \eta \|\tilde{X}\|_{H^1}^2$$

(2.23)

**Proof** This is clearly the same as in the proof of Lemma 1 of [8] by setting $\lambda_i(D(t)) = \Gamma_i(t)$, $i = 1, 2, \ldots, n$, where $\Gamma_i \in L^1([0, T], \mathbb{R})$ satisfies $\Gamma_i(t) \leq \omega^2$ a.e. on $[0, T]$, with the strict inequality holding on a subset of $[0, T]$ of positive measure, and replacing the period $2\pi$ by $T$. \qed

3 The main results

We now present our main results:

**Theorem 3.1** Let $C$ be a nonsingular matrix. Suppose that $G$ is $L^1$-Carathéodory and satisfies

$$(G_3) \quad k^2 \omega^2 \leq \frac{\langle M(t)Y, Y \rangle}{\|Y\|^2} \leq \liminf_{\|Y\| \to \infty} \frac{\langle G(t, Y), Y \rangle}{\|Y\|^2} \leq \limsup_{\|Y\| \to \infty} \frac{\langle G(t, Y), Y \rangle}{\|Y\|^2} \leq \frac{\langle N(t)Y, Y \rangle}{\|Y\|^2} \leq (k + 1)^2 \omega^2$$

uniformly in $Y \in \mathbb{R}^n$ for a.e. $t \in [0, T]$, $k \in \mathbb{N}$ and $M, N \in L^1([0, T], \mathbb{R}^{n^2})$ are such that $k^2 \omega^2 \|Y\|^2 < \langle M(t)Y, Y \rangle$, $\langle N(t)Y, Y \rangle < (k + 1)^2 \omega^2 \|Y\|^2$ on subsets of $[0, T]$ of positive measure. Then, for any arbitrary matrix $A$, the system (1.1)–(1.2) has at least one solution for every $P \in L^1([0, T], \mathbb{R}^n)$.

**Proof** Let $\epsilon > 0$ be as in Lemma 2.2. Then, by $(G_3)$, we can fix a constant vector $\rho = \rho(\epsilon)$ with each $\rho_i > 0$ such that

$$\langle M(t)Y, Y \rangle - \epsilon \|Y\|^2 \leq \langle G(t, Y), Y \rangle \leq \langle N(t)Y, Y \rangle + \epsilon \|Y\|^2$$

(3.1)

for a.e. $t \in [0, T]$ and all $Y \in \mathbb{R}^n$ with $|y_i| \geq \rho_i$. 164
Now define $\nu(t, Y) \equiv (\nu_i(t, Y))_{1 \leq i \leq n} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ by
\[
\nu_i(t, Y) = \begin{cases} 
  y_i^{-1} g_i(t, Y), & \text{if } |y_i| \geq \rho_i; \\
  y_i \rho_i^{-2} g_i(t, y_1, \ldots, y_{i-1}, \rho_i, y_{i+1}, \ldots, y_n) + (1 - \frac{\nu_i}{\rho_i}) \beta(t), & \text{if } 0 \leq y_i < \rho_i; \\
  y_i \rho_i^{-2} g_i(t, y_1, \ldots, y_{i-1}, -\rho_i, y_{i+1}, \ldots, y_n) + (1 + \frac{\nu_i}{\rho_i}) \beta(t), & \text{if } -\rho_i \leq y_i < 0.
\end{cases}
\]
for a.e. $t \in [0, T]$, where $\beta$ is given by
\[
\beta(t) \equiv \frac{1}{\|Y\|^2} \left[ \|\langle M(t)Y, Y \rangle - \langle Y, Y \rangle \| + \|\langle N(t)Y, Y \rangle + \langle Y, Y \rangle \| \right],
\]
so that by construction and (3.1), we deduce that
\[
\langle M(t)Y, Y \rangle - \epsilon \|Y\|^2 \leq \langle \nu(t, Y), Y \rangle \leq \langle N(t)Y, Y \rangle + \epsilon \|Y\|^2
\]
for a.e. $t \in [0, T]$ and $Y \in \mathbb{R}^n$.

The function $\tilde{G} \equiv \tilde{g}_i(t, Y))_{1 \leq i \leq n} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\tilde{g}_i(t, Y) = \nu_i(t, Y) y_i$ satisfies the Carathéodory conditions, by construction. Hence, setting $\Psi(t, Y) = G(t, Y) - \tilde{G}(t, Y)$, then $\Psi(t, Y)$ is also $L^1$-Carathéodory with
\[
\|\Psi(t, Y)\| \leq \sup_{|y_i| \leq \rho_i} \|G(t, Y) - \tilde{G}(t, Y)\| \leq \varphi(t)
\]
for a.e. $t \in [0, T]$ and $Y \in \mathbb{R}^n$, for some $\varphi \in L^1([0, T], \mathbb{R})$ depending only on $M, N$ and $\gamma_r$ mentioned at the beginning in association with $G$. Then, the problem (1.1) is equivalent to
\[
X'''(t) + AX''(t) + \tilde{G}(t, X'(t)) + \Psi(t, X'(t)) + CX(t) = P(t)
\]
By the Leray–Schauder technique (see Mawhin [6]), the proof of the Theorem now follows by showing that there is a constant $K > 0$, independent of $\lambda \in (0, 1)$, such that $\|X\|_{C^2} < K$, for all possible solutions $X$ of the homotopy
\[
X''' + AX'' + (1 - \lambda) N(t)X' + \lambda \tilde{G}(t, X') + \lambda \Psi(t, X') + CX = \lambda P(t)
\]
We observe from (3.3) that
\[
\langle M(t)Y, Y \rangle - \epsilon \|Y\|^2 \leq \langle (1 - \lambda) N(t)Y + \lambda \tilde{G}(t, Y), Y \rangle \leq \langle N(t)Y, Y \rangle + \epsilon \|Y\|^2
\]
for a.e. $t \in [0, T]$, $Y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Thus, we may set $(1 - \lambda) N(t)X' + \lambda \tilde{G}(t, X') \equiv B(t)X'$, for a.e. $t \in [0, T]$, $X' \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, where, by (3.7), $B(t)$ is such that
\[
\langle M(t)X', X' \rangle - \epsilon \|X'\|^2 \leq \langle B(t)X', X' \rangle \leq \langle N(t)X', X' \rangle + \epsilon \|X'\|^2
\]
for a.e. $t \in [0, T]$, $X' \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. 
Further results for some third order differential systems...

Thus (3.6) becomes
\[ 0 \geq \|X''' + AX'' + B(\cdot)X' + CX\|_{L^1} - \|\Psi(\cdot, X')\|_{L^1} - \|P(\cdot)\|_{L^1} \] (3.9)

Using Lemma 2.2 and (3.4) finally gives
\[ 0 \geq \delta_0 \|X\|_{W^{3,1}} - \|\delta\|_{L^1} - \|P\|_{L^1} \] (3.10)

which yields a constant \( K_0 > 0 \) such that \( \|X\|_{W^{3,1}} \leq K_0 \). Hence, we obtain the required constant \( K > 0 \) such that \( \|X\|_{C^2} < K \), following a standard procedure just as in [2], and the conclusion follows. \( \square \)

**Remark 3.1** The result of Theorem 3.1 can be extended to nonlinear systems of the form
\[ X''' + \frac{d}{dt} \text{grad} f(X') + G(t, X') + H(X) = P(t), \] (3.11)

under suitable assumptions on \( G \) satisfying some requirements in respect of the first (possible) eigenvalue \( \lambda = \omega^2 \) of (2.1)–(1.2).

Here, \( f : \mathbb{R}^n \to \mathbb{R} \) is a \( C^2 \)-function, \( H : \mathbb{R}^n \to \mathbb{R}^n \) is continuous and satisfies a sign condition, while \( G \) and \( P \) are as specified earlier.

**Theorem 3.2** Assume that \( G \) satisfies
\[ (G_4) \quad \lim_{\|Y\| \to +\infty} \frac{\langle G(t,Y),Y \rangle}{\|Y\|^2} \leq \frac{\langle N(t)Y,Y \rangle}{\|Y\|^2} \leq \omega^2 \]

uniformly in \( Y \in \mathbb{R}^n \) for a.e. \( t \in [0,T] \), where \( N \in L^1([0,T],\mathbb{R}^{n^2}) \) is such that \( \langle N(t)Y,Y \rangle < \omega^2 \|Y\|^2 \) on subsets of \([0,T]\) of positive measure.

Moreover, suppose that \( H \) satisfies
\[ (H) \quad \lim_{\|X\| \to +\infty} \text{sgn}(X) H(X) = +\infty. \]

Then, (3.11)–(1.2) has at least one solution for every \( P \in L^1([0,T],\mathbb{R}^n) \).

**Proof** As in the preceding proof, for each \( \epsilon > 0 \), there exists \( \rho = \rho(\epsilon) > 0 \) such that
\[ \langle G(t,Y),Y \rangle \leq \langle N(t)Y,Y \rangle + \epsilon \|Y\|^2 \]
for a.e. \( t \in [0,T] \) and all \( Y \in \mathbb{R}^n \) with \( |y_i| \geq \rho_i \).

Then, define \( \tilde{G}(t,Y) \) and \( \Psi(t,Y) \) as before, so that the relations
\[ \langle (1 - \lambda)N(t)Y + \lambda \tilde{G}(t,Y),Y \rangle \leq \langle N(t)Y,Y \rangle + \epsilon \|Y\|^2, \quad \lambda \in [0,1] \]
and \[ \|\Psi(t,Y)\| \leq \phi(t) \]
hold, for a.e. \( t \in [0,T] \) and every \( Y \in \mathbb{R}^n \).
It suffices to establish the necessary (or appropriate) a-priori bounds for the \( \lambda \)-dependent family of systems
\[
X'' + \lambda \frac{d}{dt} \text{grad } f(X') + \lambda N(t)X' + \lambda \tilde{G}(t, X') + \lambda \Psi(t, X') + (1 - \lambda)CX + \lambda H(X) = \lambda P(t),
\]
for \( \lambda \in [0, 1] \), where \( C \) is a fixed nonsingular and positive definite matrix.

Let \( X \) be a solution of (3.12)–(1.2). Taking the scalar product of (3.12) with \( X' \) and integrating over \([0, T]\) using (1.2) gives
\[
\int_0^T \|X''\|^2 dt = \int_0^T \langle (1 - \lambda)N(t)X' + \lambda \tilde{G}(t, X'), X' \rangle dt + \langle \Psi(\cdot, X') - P(\cdot), X' \rangle_{L^2}
\]
That is, from above
\[
\|X''\|^2_{L^2} \leq \int_0^T \langle N(t)X'(t), X'(t) \rangle dt + \epsilon \|X'\|^2_{L^2} + (\|\varphi\|_{L^1} + \|P\|_{L^1})\|X'\|_{L^1}
\]
Noting that by Lemma 2.3,
\[
\|X''\|^2_{L^2} - \int_0^T \langle N(t)X'(t), X'(t) \rangle dt =
\int_0^T ((X''(t))^2 - \langle N(t)X'(t), X'(t) \rangle) dt \geq \eta \|X'\|^2_{H^1} = \frac{\eta}{T} \|X''\|^2_{L^2},
\]
for some constant \( \eta = \eta(\Gamma) > 0 \), we obtain from (3.14)
\[
\eta \|X''\|^2_{L^2} \leq \frac{\epsilon T}{\omega^2} \|X''\|^2_{L^2} + (\|\varphi\|_{L^1} + \|P\|_{L^1})T^{\frac{3}{2}} \|X''\|_{L^2}
\]
by the Wirtinger and other standard inequalities. Hence, taking \( 0 < \epsilon T < \omega^2 \eta \), we deduce that
\[
\|X''\|_{L^2} \leq c_1,
\]
for some \( c_1 > 0 \). Thus, we have
\[
\|X'\|_{L^1} \leq \sqrt{T} \|X''\|_{L^2} \leq \sqrt{T} c_1
\]
This implies that
\[
\|X - X(t_0)\| \leq T \|X'\|_{L^1} \leq T^{\frac{3}{2}} c_1
\]
where \( t_0 \in [0, T] \) is arbitrarily fixed.

Now observe that
\[
\int_0^T (1 - \lambda)N(t)X' + \lambda \tilde{G}(t, X') dt \leq \int_0^T (N(t)X' + \epsilon X') dt = 0
\]
Then, taking the average of (3.12) on $[0, T]$, we obtain by the Mean Value Theorem,
\[
\|(1 - \lambda)X(t^*) + \lambda C^{-1}H(X(t^*))\| = \\
\|\left(1 - \lambda\right)\left(\frac{1}{T}\int_0^T X(t)\,dt\right) + \lambda\left(\frac{1}{T}\int_0^T C^{-1}H(X(t))\,dt\right)\| \\
\leq \|C^{-1}\| \left(\frac{1}{T}\|\delta\|_{L^1} + \frac{1}{T}\|P\|_{L^1}\right) := c_2
\] (3.20)
for some $t^* \in [0, T]$.

Now by hypothesis ($\mathcal{H}$), it follows that for any $k > 0$, there exists a $q = q(k) > 0$ such that
\[
\|C^{-1}H(X)\| = \|\tilde{H}(X)\| = \text{sgn}(X)\tilde{H}(X) > k, \tag{3.21}
\]
for every $\|X\| > \max\{k, q\}$, and all positive definite $C$. Hence, for any $\lambda \in (0, 1]$, we have
\[
\|(1 - \lambda)X + \lambda C^{-1}H(X)\| = \text{sgn}(X)((1 - \lambda)X + \lambda C^{-1}H(X)) \geq (1 - \lambda)k + \lambda k = k \tag{3.22}
\]
for every $\|X\| > \max\{k, q\}$. Thus, choosing $k > c_2$, it follows that
\[
\|X(t^*)\| \leq \max\{k, q\} := c_3 \tag{3.23}
\]
Combining (3.18) and (3.23) with $t_0 = t^*$, we obtain
\[
\|X\|_{\infty} \leq T^\frac{1}{2}c_1 + c_3 := c_4 \tag{3.24}
\]

Lastly, integrating (3.12) and using the continuity of $H$ and (3.24), we deduce the existence of a constant $c_5 > 0$, such that
\[
\|X''\|_{L^1} \leq c_5, \tag{3.25}
\]
so that
\[
\|X''\|_{\infty} \leq T\|X''\|_{L^1} = Tc_5 \tag{3.26}
\]
Therefore, by (3.17), (3.24) and (3.26),
\[
\|X\|_{C^2} = \|X\|_{\infty} + \|X'\|_{\infty} + \|X''\|_{\infty} \leq c_6, \tag{3.27}
\]
for some $c_6 > 0$, and we are done. \qed

As pointed out earlier, Theorem 3.2 admits solutions for periodic systems associated with
\[
X''' + \frac{d}{dt} \text{grad } f(X') + \frac{\omega^2}{2} (1 + \sin t)X' + H(X) = P(t). \tag{3.28}
\]

Finally, we conclude this study with a uniqueness criterion for the system (1.1)–(1.2). The following result holds:
Theorem 3.3 Let $C$ be nonsingular and suppose that $G$ satisfies, for some $k \in \mathbb{N}$,

$$(G_5) \quad k^2\omega^2 \leq \frac{\langle M(t)(Y_1 - Y_2), Y_1 - Y_2 \rangle}{\|Y_1 - Y_2\|^2} \leq \frac{\langle G(t, Y_1) - G(t, Y_2), Y_1 - Y_2 \rangle}{\|Y_1 - Y_2\|^2} \leq \frac{\langle N(t)(Y_1 - Y_2), Y_1 - Y_2 \rangle}{\|Y_1 - Y_2\|^2} \leq (k+1)^2\omega^2,$$

or

$$(G_6) \quad \frac{\langle G(t, Y_1) - G(t, Y_2), Y_1 - Y_2 \rangle}{\|Y_1 - Y_2\|^2} < \omega^2,$$

uniformly for a.e. $t \in [0, T]$ and $Y_1, Y_2 \in \mathbb{R}^n$ with $Y_1 \neq Y_2$.

Then, (1.1)–(1.2) has at most one solution.

Proof Case (i) $G$ subject to $(G_5)$: The PBVP satisfied by $V = Y_1 - Y_2$, for any two solutions $Y_1, Y_2$ of (1.1)–(1.2) is of the form

$$V'''(t) + AV''(t) + B^*(t, V')V'(t) + CV(t) = 0, \quad (3.28)$$

with

$$V(0) - V(T) = V'(0) - V'(T) = V''(0) - V''(T) \quad (3.29)$$

where the matrix $B^* \in L^1(0, T)$ is defined by

$$B^*(t, V(t))V(t) = \begin{cases} G(t, V + Y_2) - G(t, Y_2), & \text{if } V \neq 0 \\ M(t), & \text{if } V = 0 \end{cases}$$

and by $(G_5)$ satisfies

$$\lambda_i(M(t)) \leq \lambda_i(B^*(t, V(t))) \leq \lambda_i(N(t))$$

uniformly in $V \in \mathbb{R}^n$ for a.e. $t \in [0, T]$.

Hence, using the arguments of Lemma 2.1, we see that $V \equiv 0$, and the uniqueness, subject to $(G_5)$, is thus proved.

Case (ii) $G$ subject to $(G_6)$: We consider the PBVP (3.28)-(3.29) as before except that this time $B^*$ is defined by

$$B^*(t, V(t))V(t) = \begin{cases} G(t, V + Y_2) - G(t, Y_2), & \text{if } V \neq 0 \\ 0, & \text{if } V = 0 \end{cases}$$

so that by $(G_6)$, $\lambda_i(B^*(t, V(t))) < \omega^2$ uniformly in $V \in \mathbb{R}^n$ for $t \in [0, T]$.

Multiply now (3.28) scalarly by $V'(t)$ and integrate over $[0, T]$ using (3.29) and we get

$$\int_0^T \|V''(t)\|^2 dt = \int_0^T \langle B^*(t, V(t))V'(t), V'(t) \rangle dt \leq \int_0^T \langle \tilde{B}(t)V'(t), V'(t) \rangle dt,$$

$$\quad (3.30)$$
where we set \( \lambda_i(\tilde{B}(t)) = \max\{0, \lambda_i(B^*(t, V(t)))\} \) uniformly in \( V \) for a.e. \( t \in [0, T] \).

Clearly then, \( \tilde{B}(t) \in L^1(0, T) \) is such that \( 0 \leq \lambda_i(\tilde{B}(t)) < \omega^2 \) for a.e. \( t \in [0, T] \). Thus using Lemma 2.3 setting \( \tilde{X} = V' \), (3.30) becomes

\[
0 \geq \int_0^T \|V''(t)\|^2 \, dt - \int_0^T \langle \tilde{B}(t)V'(t), V'(t) \rangle \, dt \geq \eta \|V'\|^2_{H^1} \tag{3.31}
\]

from which we deduce that \( V' \equiv 0 \), leading to \( V \equiv 0 \), and the proof is complete. \( \square \)

References


