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Further Results for some Third Order Differential Systems with Nonlinear Dissipation *

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Abstract

We formulate nonuniform nonresonance criteria for certain third order differential systems of the form $X^{'''} + AX^{''} + G(t, X^{'}) + CX = P(t)$, which further improves upon our recent results in [12], given under sharp nonresonance considerations. The work also provides extensions and generalisations to the results of Ezeilo and Omari [5], and Minhós [9] from the scalar to the vector situations.

Key words: Nonlinear dissipation, sharp and nonuniform nonresonance.

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1 Introduction

An investigation of the solvability circumstances for the nonlinear differential system

$$X^{'''} + AX^{''} + G(t, X^{'}) + CX = P(t) \tag{1.1}$$

subject to the T-periodic boundary conditions

$$X(0) - X(T) = X^{'}(0) - X^{'}(T) = X^{''}(0) - X^{''}(T) = 0 \tag{1.2}$$

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on $[0,T]$ with $T > 0$, was initiated in our recent paper [12]. Our basic motivation has been to provide vector analogues to some existing results in the literature for several scalar prototypes such as those contained in [1], [2], [4] and [5]. For instance, Ezeilo and Omari [5] studied firstly the 2π-periodic solutions associated with the scalar version of (1.1), with $g = g(x^{'})$, satisfying the sharp nonresonance conditions

$$(g_1) \qquad k^2 + \alpha^-(|y|) < \frac{g(y)}{y} < (k+1)^2 - \alpha^+(|y|), \quad k \in \mathbb{N},$$

where $\alpha^{\pm} : (0,+\infty) \to \mathbb{R}$ are two nonincreasing functions such that

$$\lim_{|y|\to+\infty} |y|\, \alpha^{\pm}(|y|) = +\infty,$$

This result has been improved by Minhós [9] by weakening the condition on the oscillation of g, with the condition (g_1) replaced by the two conditions

$$(g_2) \qquad k^2 \leq \liminf_{|y|\to\pm\infty} \frac{g(y)}{y} \leq \limsup_{|y|\to\pm\infty} \frac{g(y)}{y} \leq (k+1)^2$$

and

$$(\mathcal{G}) \qquad k^2 < \limsup_{y\to+\infty} \frac{2\mathcal{G}(y)}{y^2}, \qquad \liminf_{y\to+\infty} \frac{2\mathcal{G}(y)}{y^2} < (k+1)^2)$$

where $\mathcal{G}$ denotes the primitive of the nonlinear function g, that is,

$$\mathcal{G}(y) = \int_0^y g(\tau)\, d\tau$$

Here, the ratio $\frac{g(y)}{y}$ may interact with the spectrum $\{k^2,\ k \in \mathbb{N}\}$, although $(\mathcal{G})$ imposes some 'density' control given by the asymptotic behaviour of the primitive of g.

Moreover, when $g = g(t, x^{'})$, nonuniform assumptions

$$(g_3) \qquad k^2 \leq \gamma^-(t) \leq \liminf_{|y|\to\infty} \frac{g(t,y)}{y} \leq \limsup_{|y|\to\infty} \frac{g(t,y)}{y} \leq \gamma^+(t) \leq (k+1)^2$$

uniformly in $y \in \mathbb{R}$ for a.e. $t \in [0, 2\pi]$, where $\gamma^{\pm} \in L^1(0, 2\pi)$ such that strict inequalities hold on subsets of $[0, 2\pi]$ of positive measure; were also established in [5] for the existence of 2π-periodic solutions, with accompanying uniqueness results given by appropriate modification of these conditions.

Our earlier objective, in [12], to generalise some of these results has been partially addressed with the generation of the sharp nonresonance hypotheses

$$(\mathcal{G}_1) \qquad k^2\omega^2 + \alpha^-(\|Y\|) \leq \frac{\langle G(t,Y), Y\rangle}{\|Y\|^2} \leq (k+1)^2\omega^2 - \alpha^+(\|Y\|)\ ,$$

uniformly in $Y \in \mathbb{R}^n$ with $\|Y\| \geq r > 0$, and a.e. $t \in [0,T]$, where $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, and $\alpha^{\pm} : \mathbb{R}^n_+ \to \mathbb{R}$ are two functions which are such that

$$(\mathcal{G}_2) \qquad \lim_{\|Y\| \to +\infty} \|Y\| \, \alpha^{\pm}(\|Y\|) = +\infty$$

for the existence of T-periodic solutions to (1.1)–(1.2). These relations clearly generalise the sharp nonresonance conditions prescribed in [5].

There are however, certain equations of type (1.1) with G not satisfying $(\mathcal{G}_1)$–$(\mathcal{G}_2)$, for which, nevertheless, T-periodic solvability results appear to be provable, subject to some other generalisations on G. An example is the system

$$X^{'''} + AX^{''} + \frac{1}{2}\big((k+1)^2\omega^2 + k^2\omega^2 + (2k+1)\omega^2 \cos t\big)X^{'} + CX = P(t) \quad (1.3)$$

with the ratio

$$\frac{\langle G(t,Y), Y\rangle}{\|Y\|^2} = \frac{1}{2}\big((k+1)^2\omega^2 + k^2\omega^2 + (2k+1)\omega^2 \cos t\big)$$

lying in the open interval $\big(k^2\omega^2, (k+1)^2\omega^2\big)$ for a.e. $t \in [0,T]$, but for which there do not exist functions $\alpha^{\pm}$ satisfying $(\mathcal{G}_2)$ for which $(\mathcal{G}_1)$ holds (since the ratio touches both (possible) eigenvalues as $(k+1)^2 - k^2 = 2k+1$). This justifies a further treatment of (1.1) incorporating g_2 and g_3 along the lines of [3], [7], [8] and [10], which clearly specifies the growth pattern and asymptotic conditions on G, unlike the rather arbitrary assumptions employed in [11]. This article proposes some generalisations in this direction.

Note also that condition $(\mathcal{G}_2)$ cannot be dropped as shown by the nonlinear system

$$X^{'''} + AX^{''} + k^2\omega^2 X^{'} + \tan^{-1}(X^{'}) + CX = P(t) \quad (1.4)$$

Here, the ratio

$$\frac{\langle G(t,Y), Y\rangle}{\|Y\|^2} = k^2\omega^2 + \|Y\|^{-1}\tan^{-1}(Y)\ ,$$

with

$$\alpha^{-}(\|Y\|) = \|Y\|^{-1}\tan^{-1}(Y) \quad \text{and} \quad \alpha^{+}(\|Y\|) = 2k\omega^2$$

but

$$\lim_{\|Y\| \to \infty} \|Y\|\alpha^{-}(\|Y\|) = \frac{\pi}{2} \neq +\infty,$$

so that $(\mathcal{G}_2)$ is not fulfilled by α^{-} and therefore, the system has no T-periodic solution.

Accordingly, $X \in \mathbb{R}^n$, A and C are constant real $n \times n$ nonsingular matrices, and $G : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $P : [0,T] \to \mathbb{R}^n$ are n-vectors, which are T-periodic in t. We shall assume further that G satisfies the Carathéodory conditions, that is, $G(\cdot, X^{'})$ is measurable for every $X^{'} \in \mathbb{R}^n$; $G(t,\cdot)$ is continuous for a.e. $t \in [0,T]$, and for each $r > 0$, there exists an integrable function $\gamma_r \in L^1([0,T],\mathbb{R})$ such that $||G(t,X^{'})|| \leq \gamma_r(t)$, for $||X^{'}|| \leq r$ and a.e. $t \in [0,T]$.

Let X be a point of the Euclidean space $\mathbb{R}^n$ equipped with the usual norm $\|X\|$. For any pair $X, Y \in \mathbb{R}^n$, we shall write $\langle X, Y\rangle$ for the usual scalar product of X and Y so that in particular, $\langle X, X\rangle = \|X\|^2$.

It is standard result that if D is a real $n \times n$ symmetric matrix, then for any $X \in \mathbb{R}^n$,

$$\delta_d||X||^2 \leq \langle DX, X\rangle \leq \Delta_d||X||^2, \tag{1.6}$$

where δ_d and Δ_d are respectively the least and greatest eigenvalues of D. In general, $\lambda_i(D)$ shall denote the eigenvalues of any matrix D, and $\|D\|_2$ its spectral norm.

The following Banach spaces will also be frequently refered to:

(i) the classical spaces of k times continuously differentiable functions $C^k([0,T], \mathbb{R}^n), k \geq 0$ an integer, where $C^0 = C$ and $C^\infty = \cap_{k\geq 0} C^k$ with norms $\|X\|_{C^k}$ and $\|X\|_\infty$ respectively;

(ii) the space of T-periodic functions $C_T^k([0,T], \mathbb{R}^n)$ defined by

$$C_T^k = \{X : [0,T] \to \mathbb{R}^n : X \in C^k \text{ and } X \text{ is } T\text{-periodic}\}$$

with the norm on C^k ;

(iii) $L^p([0,T], \mathbb{R}^n)$, $1 \leq p < +\infty$, the usual Lebesgue spaces with the norms $\|X\|_{L^p}$ and $\|X\|_\infty$ for $p = +\infty$;

(iv) the Sobolev space $W_T^{k,p}([0,T], \mathbb{R}^n)$, of T-periodic functions of order k, defined by

$$\begin{aligned} W_T^{k,p} = \{X : [0,T] \to \mathbb{R}^n : X, X', \ldots, X^{(k-1)} &\text{ are absolutely continuous} \\ \text{on } [0,T],\ X^{(k)} \in L^p(0,T) &\text{ and } X^{(i)}(0) - X^{(i)}(T) = 0, \\ &i = 0,1,2,\ldots,k-1,\ k \in \mathbb{N}\} \end{aligned}$$

with corresponding norm $\|X\|_{W_T^{k,p}}$;

(v) The Hilbert space $H^1([0,T], \mathbb{R}^n)$ defined by

$$\begin{aligned} H^1(0,T) = \{X : [0,T] \to \mathbb{R}^n : X, &\text{ is absolutely continuous on } [0,T], \\ X' \in L^2(0,T) &\text{ and } X^{(i)}(0) - X^{(i)}(T) = 0,\ i = 0,1\} \end{aligned}$$

with norm

$$\|X\|_{H^1} = \left\{\sum_{i=1}^n \left[\left(\frac{1}{T}\int_0^T x_i(t)\,dt\right)^2 + \frac{1}{T}\int_0^T \big(x_i(t)\big)^2 dt + \frac{1}{T}\int_0^T \big(x_i'(t)\big)^2 dt\right]\right\}^{\frac{1}{2}}.$$

Let

$$\widetilde{H}^1(0,T) = \left\{X \in H^1(0,T) \,\Big|\, \frac{1}{T}\int_0^T X(t)\,dt = 0\right\}$$

2 Previous investigations and some preliminary results

Consider the eigenvalue problem

$$X^{'''} + AX^{''} + CX = -\lambda X^{'} \tag{2.1}$$

together with (1.2), with A, C nonsingular, and λ a real parameter. It has been shown in [5] that

(i) any $\lambda \neq k^2\omega^2$, for each $k = 1, 2, \ldots$, is not an eigenvalue; and

(ii) $\lambda = k^2\omega^2$, for some $k = 1, 2, \ldots$, is an eigenvalue if and only if $C = k^2\omega^2 A$.

Let $\mathcal{E}_k$ be the eigenspace corresponding to the unique eigenvalue $k^2\omega^2$, when it exists. Then we deduce from [9] the following result:

For every $X \in W_T^{3,2}(0, 2\pi)$, we have

$$\int_0^T \langle X^{'''} + AX^{''} + k^2\omega^2 X^{'} + CX,\ X^{'''} + AX^{''} + (k+1)^2\omega^2 X^{'} + CX\rangle\, dt \geq 0, \tag{2.2}$$

and the equality holds if and only if $X = 0$ or either $k^2\omega^2$ or $(k+1)^2\omega^2$ is an eigenvalue of (2.1) and $X \in \mathcal{E}_k$ or $X \in \mathcal{E}_{k+1}$, respectively.

Each of the statements (i) or (ii) has an important bearing on the solvability of the PBVP for the non-autonomous system

$$X^{'''} + AX^{''} + \lambda X^{'} + CX = P(t) \tag{2.3}$$

with $P \in L^1$.

It is clear for instance, from (i) and the Fredholm alternative, that a solution for (1.1)–(1.2) can be expected if the ratio $\langle G(t, X^{'}), X^{'}\rangle / \|X^{'}\|^2$ is such that

$$k^2\omega^2 < \frac{\langle G(t, X^{'}), X^{'}\rangle}{\|X^{'}\|^2} < (k+1)^2\omega^2,$$

for $\|X^{'}\|$ sufficiently large, and a.e. $t \in [0, T]$, provided that some control is put on the closeness of the ratio to $k^2\omega^2$ and $(k+1)^2\omega^2$. This expectation has resulted in the evolution of conditions $(\mathcal{G}_1) - (\mathcal{G}_2)$.

The main role of statement (ii) is to provide an adequate background against which the sharpness of our conditions on G can be tested. Observe that $\alpha^{\pm}$ considered in $(\mathcal{G}_1)$ can be infinitesimal as $\|Y\| \to +\infty$, but by $(\mathcal{G}_2)$ their order must be less than one. This implies that the ratio can approach the (possible) eigenvalues $k^2\omega^2$ and $(k+1)^2\omega^2$, provided that the approach is not too fast. For instance, conditions $(\mathcal{G}_1) - (\mathcal{G}_2)$ admit functions G such as

$$G(Y) = k^2 Y - \|Y\|^{\alpha} \operatorname{sgn}(Y), \quad m \in \mathbb{N},\ 0 < \alpha < 1,$$

satisfying

$$\lim_{\|Y\| \to +\infty} \frac{\langle\, G(Y), Y\,\rangle}{\|Y\|^2} = k^2,$$

and yet by the statement (ii), (2.3)–(1.2) with $\lambda = k^2$, does not have a solution in general, that is, for unrestricted A and C nonsingular. Thus for (1.1), we seek conditions on $G(t, Y)$ allowing $\lim_{\|Y\|\to+\infty} \frac{\langle G(t,Y),Y\rangle}{\|Y\|^2}$ (if it exists) to touch k^2, $k \in \mathbb{N}$, for many values of t.

In the sequel, we shall require some preliminary lemmas.

Lemma 2.1 *Consider the linear homogeneous system*

$$X'''(t) + AX''(t) + B(t)X'(t) + CX(t) = 0 \tag{2.4}$$

where A is an arbitrary matrix, C is a nonsingular matrix and $B(t) \equiv (b_{ij}(t))$ is such that $b_{ij} \in L^1(0, T)$ and

$$(\mathcal{B}_1) \qquad k^2\omega^2 \le \lambda_i(B(t)) \le (k+1)^2\omega^2$$

for a.e. $t \in [0, T]$, $i = 1, \ldots, n$, $k \in \mathbb{N}$, with the strict inequality holding on subsets of $[0, T]$ of positive measure.

Then, (2.4)–(1.2) has no non-trivial solution.

Proof Let the solution $X(t) = \overline{X}(t) + \widetilde{X}(t)$ have the Fourier expansion

$$X(t) \sim \sum_{i=1}^{n}\Big(c_{0,i} + \sum_{k=1}^{\infty}(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t)\Big),$$

such that

$$\overline{X} = \sum_{i=1}^{n}\Big(c_{0,i} + \sum_{k=1}^{N}(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t)\Big)$$

and

$$\widetilde{X} = \sum_{i=1}^{n}\sum_{k=N+1}^{\infty}(c_{k,i}\cos k\omega t + d_{k,i}\sin k\omega t\,),$$

for some integer $N > 0$ with $N^2\omega^2 < \lambda < (N+1)^2\omega^2$, where $\omega = \frac{2\pi}{T}$.

Then, multiplying (2.4) by $\overline{X}'(t) - \widetilde{X}'(t)$ and integrating over $[0.T]$ gives,

$$\int_0^T \Big(\big(\widetilde{X}''(t)\big)^2 - \big\langle B(t)\widetilde{X}'(t), \widetilde{X}'(t)\big\rangle\Big)\,dt - \int_0^T \Big(\big(\overline{X}''(t)\big)^2 - \big\langle B(t)\overline{X}'(t), \overline{X}'(t)\big\rangle\Big)\,dt = 0. \tag{2.5}$$

Let δ be a constant defined by

$$\delta = \frac{1}{2}\left(\min \lambda_i\left(B(t)\right) + \max \lambda_i\left(B(t)\right)\right) \tag{2.6}$$

for a.e. $t \in [0, T]$. Then in fact,

$$\begin{array}{ll} k^2\omega^2 \le \delta \le (k+1)^2\omega^2, & \text{for a.e. } t \in [0,T], \text{ and} \\ k^2\omega^2 < \delta < (k+1)^2\omega^2, & \text{on subsets of } [0,T] \text{ of positive measure.} \end{array} \tag{2.7}$$

Thus, combining ($\mathcal{B}_1$), (2.6) and (2.7), (2.5) becomes

$$0 \geq \int_0^T \left[\left(\widetilde{X}''(t) \right)^2 - \delta \left(\widetilde{X}'(t) \right)^2 \right] dt - \int_0^T \left[\left(\overline{X}''(t) \right)^2 - \delta \left(\overline{X}'(t) \right)^2 \right] dt = 0. \tag{2.8}$$

By Parseval's identity given by

$$\int_0^T \|X\|^2 \, dt = \sum_{i=1}^n \Big(c_{0,i}^2 T + \frac{T}{2} \sum_{k=1}^{\infty} (c_{k,i}^2 + d_{k,i}^2) \Big),$$

(2.8) becomes

$$\frac{T}{2} \sum_{i=1}^n \Big[\sum_{k=N+1}^{\infty} k^2\omega^2(k^2\omega^2 - \delta)(c_{k,i}^2 + d_{k,i}^2) + \sum_{k=1}^{N} k^2\omega^2(\delta - k^2\omega^2)(c_{k,i}^2 + d_{k,i}^2) \Big] = 0. \tag{2.9}$$

It follows from (2.7) that $c_{k,i} = 0$ $(k = 0, 1, 2, \ldots)$ and $d_{k,i} = 0$ $(k = 1, 2, \ldots)$, for all $i = 1, \ldots, n$. Thus, $X \equiv 0$, and the lemma follows. □

Lemma 2.2 *Let C be nonsingular, and assume that $M, N \in L^1([0,T], \mathbb{R}^{n^2})$ are nonsingular matrices which satisfy the following conditions*

$$k^2\omega^2\|Y\|^2 \leq \langle M(t)Y, Y\rangle \leq \langle N(t)Y, Y\rangle \leq (k+1)^2\omega^2\|Y\|^2 \tag{2.10}$$

uniformly in $Y \in \mathbb{R}^n$, for a.e. $t \in [0,T]$, $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, and

$$k^2\omega^2\|Y\|^2 < \langle M(t)Y, Y\rangle, \qquad \langle N(t)Y, Y\rangle < (k+1)^2\omega^2\|Y\|^2 \tag{2.11}$$

on subsets of $[0,T]$ of positive measure.

Then, there exists constants $\epsilon = \epsilon(M, N, C) > 0$ and $\delta_0 = \delta_0(M, N, C) > 0$ uniformly a.e. on $[0,T]$, such that for all $B(t) \equiv (b_{ij}(t))$ with $b_{ij} \in L^1([0,T], \mathbb{R})$ satisfying

$$(\mathcal{B}_2) \qquad \langle M(t)Y, Y\rangle - \epsilon\|Y\|^2 \leq \langle B(t)Y, Y\rangle \leq \langle N(t)Y, Y\rangle + \epsilon\|Y\|^2$$

uniformly in $Y \in \mathbb{R}^n$, a.e. on $[0,T]$, and all $X \in W_T^{3,1}([0,T], \mathbb{R}^n)$, one has

$$\|X''' + AX'' + B(\cdot)X' + CX\|_{L^1} \geq \delta_0 \|X\|_{W_T^{3,1}} \tag{2.12}$$

Proof Let us assume that the conclusion of the Lemma does not hold, that is, ϵ and δ_0 do not exist. Then, there exists a sequence $(X_n) \in W^{3,1}([0,T], \mathbb{R}^n)$ with $\|X_n\|_{W^{3,1}} = 1$, and a sequence $(B_n) \in L^1([0,T], \mathbb{R}^{n^2})$ of nonsingular matrices with

$$\langle M(t)Y, Y\rangle - \frac{1}{n}\|Y\|^2 \leq \langle B_n(t)Y, Y\rangle \leq \langle N(t)Y, Y\rangle + \frac{1}{n}\|Y\|^2, \quad n \in \mathbb{N}, \tag{2.13}$$

uniformly in $Y \in \mathbb{R}^n$, for a.e. $t \in [0,T]$, such that for all $X \in W^{3,1}$, one has

$$\int_0^T \|X_n'''(t) + AX_n''(t) + B_n(t)X_n'(t) + CX_n\| \, dt < \frac{1}{n}. \tag{2.14}$$

Let $\|B_n\|$ denote the norm of B_n. Then, by (2.13), there exists some $\beta \in L^1([0,T],\mathbb{R})$ such that

$$\|B_n(t)\| \leq \beta(t), \quad n = 1, 2, \dots \tag{2.15}$$

for a.e. $t \in [0,T]$, $n \in \mathbb{N}$. For example, one can take

$$\beta(t) \equiv \frac{1}{\|Y\|^2}\left[\|\langle M(t)Y,Y\rangle - \langle Y,Y\rangle\| + \|\langle N(t)Y,Y\rangle + \langle Y,Y\rangle\|\right].$$

Now, by the compact embedding of $W^{3,1}([0,T],\mathbb{R}^n)$ into $W^{2,1}([0,T],\mathbb{R}^n)$ and the continuous embedding of $W^{2,1}([0,T],\mathbb{R}^n)$ into $C^1([0,T],\mathbb{R}^n)$ imply that by going to subsequences if neccessary, we can assume that

$$X_n \to X \text{ in } C^1([0,T],\mathbb{R}^n), \quad X_n^{''} \to X^{''} \text{ in } L^\infty([0,T],\mathbb{R}^n) \subset L^1([0,T],\mathbb{R}^n). \tag{2.16}$$

Moreover, by (2.15), we deduce that

$$B_n \rightharpoonup B \text{ in } L^1([0,T],\mathbb{R}^{n^2}) \tag{2.17}$$

so that by (2.13),

$$\langle M(t)Y,Y\rangle \leq \langle B(t)Y,Y\rangle \leq \langle N(t)Y,Y\rangle \tag{2.18}$$

for a.e. $t \in [0,T]$.

On the other hand, for every $\Phi \in L^\infty([0,T],\mathbb{R}^n)$, we have by Schwarz inequality

$$\begin{aligned}
&\left\|\int_0^T \left\langle B_n(t)X_n^{'}(t) - B(t)X^{'}(t), \Phi(t)\right\rangle dt\right\| \\
&\leq \left\|\int_0^T \left\langle B_n(t)\big(X_n^{'}(t)-X^{'}(t)\big), \Phi(t)\right\rangle dt\right\| + \left\|\int_0^T \left\langle \big(B_n(t)-B(t)\big)X^{'}(t), \Phi(t)\right\rangle dt\right\| \\
&\leq \|\Phi\|_\infty \|\beta\|_{L^1}\|X_n^{'} - X^{'}\|_\infty + \left\|\int_0^T \left\langle \big(B_n(t) - B(t)\big)X^{'}(t), \Phi(t)\right\rangle dt\right\|.
\end{aligned} \tag{2.19}$$

The right hand side of (2.19) tends to zero by (2.16) and (2.17), and we deduce that

$$B_nX_n^{'} \rightharpoonup BX^{'} \text{ in } L^1([0,T],\mathbb{R}^n). \tag{2.20}$$

By (2.14), (2.16) and (2.20), it follows that

$$X_n^{'''} = -AX_n^{''} - B_n(\cdot)X_n^{'} - CX_n \rightharpoonup -AX^{''} - B(\cdot)X^{'} - CX \text{ in } L^1([0,T],\mathbb{R}^n). \tag{2.21}$$

Since the operator

$$\frac{d^3}{dt^3} : W^{3,1}([0,T],\mathbb{R}^n) \subset L^1([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n)$$

is weakly closed, this implies (by (2.16) and (2.21)) that $X \in W_T^{3,1}([0,T],\mathbb{R}^n)$, and $X^{'''} = -AX^{''} - B(\cdot)X^{'} - CX$, that is,

$$X^{'''}(t) + AX^{''}(t) + B(t)X^{'}(t) + CX(t) = 0, \tag{2.22}$$

for a.e. $t \in [0,T]$ and $X \in W^{3,1}([0,T],\mathbb{R}^n)$.

It follows from (2.9), (2.10), (2.18), (2.22) and Lemma 2.1 that $X \equiv 0$, that is, $X_n \to 0$ in $W^{3,1}([0,T],\mathbb{R}^n)$ as $n \to \infty$. But this clearly contradicts the initial assumption that $\|X_n\|_{W^{3,1}} = 1$ for all n, and the proof is complete. □

Lemma 2.3 *Let $D \in L^1\big([0,T],\mathbb{R}^{n^2}\big)$ be a nonsingular matrix such that $0 \le \lambda_i(D(t)) \le \omega^2$ a.e. on $[0,T]$, with the strict inequality holding on a subset of $[0,T]$ of positive measure. Then, there exists a constant $\eta = \eta(D) > 0$ such that for all $\widetilde{X} \in \widetilde{H}^1([0,T],\mathbb{R}^n)$, we have*

$$\frac{1}{T}\int_0^T \left[\left(\widetilde{X}^{\,'}(t)\right)^2 - \langle D(t)\widetilde{X}(t), \widetilde{X}(t)\rangle\right] dt \ge \eta\|\widetilde{X}\|_{H^1}^2 \tag{2.23}$$

Proof This is clearly the same as in the proof of Lemma 1 of [8] by setting $\lambda_i(D(t)) \equiv \Gamma_i(t)$, $i = 1,2,\ldots,n$, where $\Gamma_i \in L^1([0,T],\mathbb{R})$ satisfies $\Gamma_i(t) \le \omega^2$ a.e. on $[0,T]$, with the strict inequality holding on a subset of $[0,T]$ of positive measure, and replacing the period 2π by T. □

3 The main results

We now present our main results:

Theorem 3.1 *Let C be a nonsingular matrix. Suppose that G is L^1-Carathéodory and satisfies*

$$(\mathcal{G}_3)\qquad k^2\omega^2 \le \frac{\langle M(t)Y,Y\rangle}{\|Y\|^2} \le \liminf_{\|Y\|\to\infty}\frac{\langle G(t,Y),Y\rangle}{\|Y\|^2} \le \limsup_{\|Y\|\to\infty}\frac{\langle G(t,Y),Y\rangle}{\|Y\|^2} \le \frac{\langle N(t)Y,Y\rangle}{\|Y\|^2} \le (k+1)^2\omega^2$$

uniformly in $Y \in \mathbb{R}^n$ for a.e. $t \in [0,T]$, $k \in \mathbb{N}$ and $M, N \in L^1([0,T],\mathbb{R}^{n^2})$ are such that $k^2\omega^2\|Y\|^2 < \langle M(t)Y,Y\rangle$, $\langle N(t)Y,Y\rangle < (k+1)^2\omega^2\|Y\|^2$ on subsets of $[0,T]$ of positive measure. Then, for any arbitrary matrix A, the system (1.1)–(1.2) has at least one solution for every $P \in L^1([0,T],\mathbb{R}^n)$.

Proof Let $\epsilon > 0$ be as in Lemma 2.2. Then, by $(\mathcal{G}_3)$, we can fix a constant vector $\rho = \rho(\epsilon)$ with each $\rho_i > 0$ such that

$$\langle M(t)Y,Y\rangle - \epsilon\|Y\|^2 \le \langle G(t,Y),Y\rangle \le \langle N(t)Y,Y\rangle + \epsilon\|Y\|^2 \tag{3.1}$$

for a.e. $t \in [0,T]$ and all $Y \in \mathbb{R}^n$ with $|y_i| \ge \rho_i$.

Now define $\nu(t,Y) \equiv (\nu_i(t,Y))_{1\le i\le n} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ by

$$\nu_i(t,Y) = \begin{cases} y_i^{-1} g_i(t,Y), \text{ if } |y_i| \ge \rho_i; \\ y_i\rho_i^{-2} g_i(t,y_1,\ldots,y_{i-1},\rho_i,y_{i+1},\ldots,y_n) + (1-\frac{y_i}{\rho_i})\beta(t), \\ \text{if } 0 \le y_i < \rho_i; \\ y_i\rho_i^{-2} g_i(t,y_1,\ldots,y_{i-1},-\rho_i,y_{i+1},\ldots,y_n) + (1+\frac{y_i}{\rho_i})\beta(t), \\ \text{if } -\rho_i \le y_i < 0. \end{cases}$$

for a.e. $t \in [0,T]$, where β is given by

$$\beta(t) \equiv \frac{1}{\|Y\|^2}\left[\, \|\langle M(t)Y,Y\rangle - \langle Y,Y\rangle\| + \|\langle N(t)Y,Y\rangle + \langle Y,Y\rangle\| \,\right], \tag{3.2}$$

so that by construction and (3.1), we deduce that

$$\langle M(t)Y,Y\rangle - \epsilon\|Y\|^2 \le \langle \nu(t,Y),Y\rangle \le \langle N(t)Y,Y\rangle + \epsilon\|Y\|^2 \tag{3.3}$$

for a.e. $t \in [0,T]$ and $Y \in \mathbb{R}^n$.

The function $\tilde{G} \equiv (\tilde{g}_i(t,Y))_{1\le i\le n}[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ defined by $\tilde{g}_i(t,Y) = \nu_i(t,Y)y_i$ satisfies the Carathéodory conditions, by construction. Hence, setting $\Psi(t,Y) = G(t,Y) - \tilde{G}(t,Y)$, then $\Psi(t,Y)$ is also L^1-Carathéodory with

$$\|\Psi(t,Y)\| \le \sup_{|y_i|\le\rho_i} \|G(t,Y) - \tilde{G}(t,Y)\| \le \varphi(t) \tag{3.4}$$

for a.e. $t \in [0,T]$ and $Y \in \mathbb{R}^n$, for some $\varphi \in L^1([0,T],\mathbb{R})$ depending only on M, N and γ_r mentioned at the beginning in association with G. Then, the problem (1.1) is equivalent to

$$X^{'''}(t) + AX^{''}(t) + \tilde{G}(t,X^{'}(t)) + \Psi(t,X^{'}(t)) + CX(t) = P(t) \tag{3.5}$$

By the Leray–Schauder technique (see Mawhin [6]), the proof of the Theorem now follows by showing that there is a constant $K > 0$, independent of $\lambda \in (0,1)$, such that $\|X\|_{C^2} < K$, for all possible solutions X of the homotopy

$$X^{'''} + AX^{''} + (1-\lambda)N(t)X^{'} + \lambda\tilde{G}(t,X^{'}) + \lambda\Psi(t,X^{'}) + CX = \lambda P(t) \tag{3.6}$$

We observe from (3.3) that

$$\langle M(t)Y,Y\rangle - \epsilon\|Y\|^2 \le \langle\, (1-\lambda)N(t)Y + \lambda\tilde{G}(t,Y),Y\rangle \le \langle N(t)Y,Y\rangle + \epsilon\|Y\|^2 \tag{3.7}$$

for a.e. $t \in [0,T]$, $Y \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

Thus, we may set $(1-\lambda)N(t)X^{'} + \lambda\tilde{G}(t,X^{'}) \equiv B(t)X^{'}$, for a.e. $t \in [0,T]$, $X^{'} \in \mathbb{R}^n$ and $\lambda \in [0,1]$, where, by (3.7), $B(t)$ is such that

$$\langle M(t)X^{'},X^{'}\rangle - \epsilon\|X^{'}\|^2 \le \langle B(t)X^{'},X^{'}\rangle \le \langle N(t)X^{'},X^{'}\rangle + \epsilon\|X^{'}\|^2 \tag{3.8}$$

for a.e. $t \in [0,T]$, $X^{'} \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

Thus (3.6) becomes

$$0 \geq \|X^{'''} + AX^{''} + B(\cdot)X^{'} + CX\|_{L^1} - \|\Psi(\cdot, X^{'})\|_{L^1} - \|P(\cdot)\|_{L^1} \tag{3.9}$$

Using Lemma 2.2 and (3.4) finally gives

$$0 \geq \delta_0 \|X\|_{W^{3,1}} - \|\delta\|_{L^1} - \|P\|_{L^1} \tag{3.10}$$

which yields a constant $K_0 > 0$ such that $\|X\|_{W^{3,1}} \leq K_0$. Hence, we obtain the required constant $K > 0$ such that $\|X\|_{C^2} < K$, following a standard procedure just as in [2], and the conclusion follows. □

Remark 3.1 The result of Theorem 3.1 can be extended to nonlinear systems of the form

$$X^{'''} + \frac{d}{dt} \operatorname{grad} f(X^{'}) + G(t, X^{'}) + H(X) = P(t), \tag{3.11}$$

under suitable assumptions on G satisfying some requirements in respect of the first (possible) eigenvalue $\lambda = \omega^2$ of (2.1)–(1.2).

Here, $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2-function, $H : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies a sign condition, while G and P are as specified earlier.

Theorem 3.2 *Assume that G satisfies*

$$(\mathcal{G}_4) \qquad \lim_{\|Y\| \to +\infty} \frac{\langle G(t, Y), Y\rangle}{\|Y\|^2} \leq \frac{\langle N(t)Y, Y\rangle}{\|Y\|^2} \leq \omega^2$$

uniformly in $Y \in \mathbb{R}^n$ for a.e. $t \in [0, T]$, where $N \in L^1([0, T], \mathbb{R}^{n^2})$ is such that $\langle N(t)Y,\, Y\rangle < \omega^2 \|Y\|^2$ on subsets of $[0, T]$ of positive measure.

Moreover, suppose that H satisfies

$$(\mathcal{H}) \qquad \lim_{\|X\| \to +\infty} \operatorname{sgn}(X)\, H(X) = +\infty.$$

Then, (3.11)–(1.2) has at least one solution for every $P \in L^1([0, T], \mathbb{R}^n)$.

Proof As in the preceding proof, for each $\epsilon > 0$, there exists $\rho = \rho(\epsilon) > 0$ such that

$$\langle G(t, Y), Y\rangle \leq \langle N(t)Y, Y\rangle + \epsilon \|Y\|^2$$

for a.e. $t \in [0, T]$ and all $Y \in \mathbb{R}^n$ with $|y_i| \geq \rho_i$.

Then, define $\tilde{G}(t, Y)$ and $\Psi(t, Y)$ as before, so that the relations

$$\langle (1 - \lambda)N(t)Y + \lambda \tilde{G}(t, Y), Y\rangle \leq \langle N(t)Y, Y\rangle + \epsilon \|Y\|^2, \quad \lambda \in [0, 1]$$

and

$$\|\Psi(t, Y)\| \leq \varphi(t)$$

hold, for a.e. $t \in [0, T]$ and every $Y \in \mathbb{R}^n$.

It suffices to establish the neccessary (or appropriate) a-priori bounds for the λ-dependent family of systems

$$X^{'''} + \lambda \frac{d}{dt} \operatorname{grad} f(X^{'}) + (1-\lambda)N(t)X^{'} + \lambda\tilde{G}(t, X^{'}) + \lambda\Psi(t, X^{'}) \\ + (1-\lambda)CX + \lambda H(X) = \lambda P(t), \qquad (3.12)$$

for $\lambda \in [0, 1]$, where C is a fixed nonsingular and positive definite matrix.

Let X be a solution of (3.12)–(1.2). Taking the scalar product of (3.12) with $X^{'}(t)$ and integrating over $[0, T]$ using (1.2) gives

$$\int_0^T \|X^{''}\|^2\, dt = \int_0^T \langle (1-\lambda)N(t)X^{'} + \lambda\tilde{G}(t, X^{'}), X^{'}\rangle\, dt + \langle \Psi(\cdot, X^{'}) - P(\cdot), X^{'}\rangle_{L^2} \qquad (3.13)$$

That is, from above

$$\|X^{''}\|_{L^2}^2 \le \int_0^T \langle\, N(t)X^{'}(t), X^{'}(t)\,\rangle\, dt + \epsilon\|X^{'}\|_{L^2}^2 + (\|\varphi\|_{L^1} + \|P\|_{L^1})\|X^{'}\|_\infty \qquad (3.14)$$

Noting that by Lemma 2.3,

$$\|X^{''}\|_{L^2}^2 - \int_0^T \langle\, N(t)X^{'}(t), X^{'}(t)\rangle\, dt =$$

$$= \int_0^T ((X^{''}(t))^2 - \langle\, N(t)X^{'}(t), X^{'}(t)\rangle)\, dt \ge \eta\|X^{'}\|_{H^1}^2 = \frac{\eta}{T}\|X^{''}\|_{L^2}^2,$$

for some constant $\eta = \eta(\Gamma) > 0$, we obtain from (3.14)

$$\eta\|X^{''}\|_{L^2}^2 \le \frac{\epsilon T}{\omega^2}\|X^{''}\|_{L^2}^2 + (\|\varphi\|_{L^1} + \|P\|_{L^1})T^{\frac{3}{2}}\|X^{''}\|_{L^2} \qquad (3.15)$$

by the Wirtinger and other standard inequalities. Hence, taking $0 < \epsilon T < \omega^2\eta$, we deduce that

$$\|X^{''}\|_{L^2} \le c_1\ , \qquad (3.16)$$

for some $c_1 > 0$. Thus, we have

$$\|X^{'}\|_\infty \le \sqrt{T}\|X^{''}\|_{L^2} \le \sqrt{T}c_1 \qquad (3.17)$$

This implies that

$$\|X - X(t_0)\| \le T\|X^{'}\|_\infty \le T^{\frac{3}{2}}c_1 \qquad (3.18)$$

where $t_0 \in [0, T]$ is arbitrarily fixed.

Now observe that

$$\int_0^T (1-\lambda)N(t)X^{'} + \lambda\tilde{G}(t, X^{'})\, dt \le \int_0^T (\, N(t)X^{'} + \epsilon X^{'}\,)\, dt = 0 \qquad (3.19)$$

Then, taking the average of (3.12) on $[0,T]$, we obtain by the Mean Value Theorem,

$$\begin{aligned}\big\|(1-\lambda)X(t^\star)+\lambda C^{-1}H(X(t^\star))\,\big\| &= \\ =\Big\|(1-\lambda)\Big(\frac{1}{T}\int_0^T X(t)\,dt\Big)+\lambda\Big(\frac{1}{T}\int_0^T C^{-1}H(X(t))\,dt\Big)\Big\| \\ \leq \|C^{-1}\|\left(\frac{1}{T}\|\delta\|_{L^1}+\frac{1}{T}\|P\|_{L^1}\right) := c_2 \end{aligned} \tag{3.20}$$

for some $t^\star \in [0,T]$.

Now by hypothesis $(\mathcal{H})$, it follows that for any $k>0$, there exists a $q=q(k)>0$ such that

$$\|C^{-1}H(X)\| = \|\widetilde{H}(X)\| = \operatorname{sgn}(X)\widetilde{H}(X) > k, \tag{3.21}$$

for every $\|X\|>\max\{k,q\}$, and all positive definite C. Hence, for any $\lambda\in(0,1]$, we have

$$\big\|(1-\lambda)X+\lambda C^{-1}H(X)\,\big\| = \operatorname{sgn}(X)\big((1-\lambda)X+\lambda C^{-1}H(X)\big) \geq (1-\lambda)k+\lambda k = k \tag{3.22}$$

for every $\|X\|>\max\{k,q\}$. Thus, choosing $k>c_2$, it follows that

$$\|X(t^\star)\| \leq \max\{k,q\} := c_3 \tag{3.23}$$

Combining (3.18) and (3.23) with $t_0=t^\star$, we obtain

$$\|X\|_\infty \leq T^{\frac{3}{2}}c_1 + c_3 := c_4 \tag{3.24}$$

Lastly, integrating (3.12) and using the continuity of H and (3.24), we deduce the existence of a constant $c_5>0$, such that

$$\|X^{'''}\|_{L^1} \leq c_5, \tag{3.25}$$

so that

$$\|X^{''}\|_\infty \leq T\|X^{'''}\|_{L^1} = Tc_5 \tag{3.26}$$

Therefore, by (3.17), (3.24) and (3.26),

$$\|X\|_{C^2} = \|X\|_\infty + \|X^{'}\|_\infty + \|X^{''}\|_\infty \leq c_6, \tag{3.27}$$

for some $c_6>0$, and we are done. □

As pointed out earlier, Theorem 3.2 admits solutions for periodic systems associated with

$$X^{'''} + \frac{d}{dt}\operatorname{grad} f(X^{'}) + \frac{\omega^2}{2}(1+\sin t)X^{'} + H(X) = P(t). \tag{3.28}$$

Finally, we conclude this study with a uniqueness criterion for the system (1.1)–(1.2). The following result holds:

Theorem 3.3 *Let C be nonsingular and suppose that G satisfies, for some $k \in \mathbb{N}$,*

$$(\mathcal{G}_5) \qquad k^2\omega^2 \le \frac{\langle M(t)(Y_1 - Y_2), Y_1 - Y_2\rangle}{\|Y_1 - Y_2\|^2} \le \frac{\langle G(t,Y_1) - G(t,Y_2), Y_1 - Y_2\rangle}{\|Y_1 - Y_2\|^2}$$

$$\le \frac{\langle N(t)(Y_1 - Y_2), Y_1 - Y_2\rangle}{\|Y_1 - Y_2\|^2} \le (k+1)^2\omega^2,$$

or

$$(\mathcal{G}_6) \qquad \frac{\langle G(t,Y_1) - G(t,Y_2), Y_1 - Y_2\rangle}{\|Y_1 - Y_2\|^2} < \omega^2,$$

uniformly for a.e. $t \in [0,T]$ and $Y_1, Y_2 \in \mathbb{R}^n$ with $Y_1 \ne Y_2$.

Then, (1.1)–(1.2) has at most one solution.

Proof *Case (i)* G subject to $(\mathcal{G}_5)$: The PBVP satisfied by $V = Y_1 - Y_2$, for any two solutions Y_1, Y_2 of (1.1)–(1.2) is of the form

$$V^{'''}(t) + AV^{''}(t) + B^\star(t,V^{'})V^{'}(t) + CV(t) = 0, \tag{3.28}$$

with

$$V(0) - V(T) = V^{'}(0) - V^{'}(T) = V^{''}(0) - V^{''}(T) \tag{3.29}$$

where the matrix $B^\star \in L^1(0,T)$ is defined by

$$B^\star(t,V(t))V(t) = \begin{cases} G(t,V+Y_2) - G(t,Y_2), & \text{if } V \ne 0 \\ M(t), & \text{if } V = 0 \end{cases}$$

and by $(\mathcal{G}_5)$ satisfies

$$\lambda_i(M(t)) \le \lambda_i\left(B^\star(t,V(t))\right) \le \lambda_i(N(t))$$

uniformly in $V \in \mathbb{R}^n$ for a.e. $t \in [0,T]$.

Hence, using the arguments of Lemma 2.1, we see that $V \equiv 0$, and the uniqueness, subject to $(\mathcal{G}_5)$, is thus proved.

Case (ii) G subject to $(\mathcal{G}_6)$: We consider the PBVP (3.28)-(3.29) as before except that this time $B^\star$ is defined by

$$B^\star(t,V(t))V(t) = \begin{cases} G(t,V+Y_2) - G(t,Y_2), & \text{if } V \ne 0 \\ 0, & \text{if } V = 0 \end{cases}$$

so that by $(\mathcal{G}_6)$, $\lambda_i\left(B^\star(t,V(t))\right) < \omega^2$ uniformly in $V \in \mathbb{R}^n$ for $t \in [0,T]$.

Multiply now (3.28) scalarly by $V^{'}(t)$ and integrate over $[0,T]$ using (3.29) and we get

$$\int_0^T \|V^{''}(t)\|^2\,dt = \int_0^T \langle\, B^\star(t,V(t))V^{'}(t), V^{'}(t)\rangle\,dt \le \int_0^T \langle \widetilde{B}(t)V^{'}(t), V^{'}(t)\rangle\,dt, \tag{3.30}$$

where we set $\lambda_i(\widetilde{B}(t)) = \max\{0, \lambda_i(B^\star(t, V(t)))\}$ uniformly in V for a.e. $t \in [0, T]$.

Clearly then, $\widetilde{B}(t) \in L^1(0, T)$ is such that $0 \leq \lambda_i(\widetilde{B}(t)) < \omega^2$ for a.e. $t \in [0, T]$. Thus using Lemma 2.3 setting $\widetilde{X} = V^{'}$, (3.30) becomes

$$0 \geq \int_0^T \|V^{''}(t)\|^2 \, dt - \int_0^T \langle \widetilde{B}(t) V^{'}(t), V^{'}(t) \rangle \, dt \geq \eta \|V^{'}\|^2_{H^1} \tag{3.31}$$

from which we deduce that $V^{'} \equiv 0$, leading to $V \equiv 0$, and the proof is complete. □

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