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TWO ELEMENT DIRECT LIMIT CLASSES OF MONOUNARY ALGEBRAS

EMÍLIA HALUŠKOVÁ

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ABSTRACT. A class of algebras is said to be direct limit closed if it is closed with respect to direct limits. We describe all two element sets $S$ of monounary algebras such that $S$, together with all isomorphic copies of elements of $S$, is a direct limit closed class.

Direct limit classes of algebras, i.e. classes of algebras which are closed with respect to direct limits, were investigated in [3] and [6]. The class of all retracts of a finite algebra is a direct limit class, cf. [5].

The paper [3] contains a description of all monounary algebras $A$ such that \{A\} is a direct limit class.

The aim of the present paper is to describe all pairs $A, B$ of monounary algebras such that \{A, B\} is a direct limit class.

1. Preliminaries

For the notion of a direct limit, cf. e.g. Grätzer [1; §21].

Let $(P, \leq)$ be a directed partially ordered set, $P \neq \emptyset$. For each $p \in P$ let $A_p$ be an algebra of some fixed type. We assume that if $p, q \in P$, $p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements $p$ and $q$ in $P$ with $p < q$, there is defined a homomorphism $\varphi_{pq}$ of $A_p$ into $A_q$ such that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. For each $p \in P$ let $\varphi_{pp}$ be the identity on $A_p$. Then we say that \{P, $A_p, \varphi_{pq}$\} is the direct family.

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Assume that \( p,q \in P \) and \( x \in A_p \), \( y \in A_q \). Put \( x \equiv y \) if there exists \( s \in P \) with \( p \leq s \), \( q \leq s \) such that \( \varphi_{ps}(x) = \varphi_{qs}(y) \). For each \( z \in \bigcup_{p \in P} A_p \) put 
\[
\bar{z} = \left\{ t \in \bigcup_{p \in P} A_p : z \equiv t \right\}.
\]
Denote \( \bar{A} = \left\{ \bar{z} : z \in \bigcup_{p \in P} A_p \right\} \).

Let \( f \) be a \( n \)-ary operation from the type of algebras \( A_p \), \( p \in P \). Let \( x_j \in A_{p_j}, 1 \leq j \leq n \), and let \( s \) be an upper bound of \( p_j \). Define \( f(\bar{x}_1, \ldots, \bar{x}_n) = f(\varphi_{p_1s}(x_1), \ldots, \varphi_{p_ns}(x_n)) \). Then \( \bar{A} \) is an algebra which is said to be the direct limit of the direct family \( \{ P, A_p, \varphi_{pq} \} \).

We express this situation as follows
\[
(P, A_p, \varphi_{pq}) \rightarrow \bar{A}.
\]  
(1)

The operator \( L \) on classes of algebras was introduced in the textbook [1; §23]. By this definition, if \( \mathcal{K} \) is a class of algebras, then \( L(\mathcal{K}) \) is the class of all direct limits of algebras of \( \mathcal{K} \).

Let \( \mathcal{K} \) be a class of algebras. We denote by \( [\mathcal{K}] \) the class of all isomorphic copies of algebras of \( \mathcal{K} \). Further, we denote by \( L\mathcal{K} \) the class of all isomorphic copies of direct limits of algebras of \( \mathcal{K} \), i.e., \( L\mathcal{K} = [L(\mathcal{K})] \).

We put \( L^2\mathcal{K} = LL\mathcal{K}, \ L^3\mathcal{K} = LL^2\mathcal{K} \).

A class \( \mathcal{K} \) is called a direct limit class, if \( L[\mathcal{K}] = [\mathcal{K}] \).

For algebras \( A_1, \ldots, A_n \) we will use \( [A_1, \ldots, A_n] \) instead of \( \{[A_1, \ldots, A_n]\} \).

**Lemma 1.** Let \( A, B \) be algebras and \( L[A] = [A, B] \), \( L[B] = [B] \).

Then \( L[A, B] = [A, B] \).

**Proof.** Let (1) be valid and \( A_p \in [A, B] \) for every \( p \in P \). Put \( Q = \{ q \in P : A_q \cong B \} \). If \( Q \) is cofinal with \( P \), then \( \bar{A} \cong B \). If \( P - Q \) is cofinal with \( P \), then \( \bar{A} \cong A \) or \( \bar{A} \cong B \). \( \square \)

Let \( B \) be a subalgebra of \( A \). Assume that there exists a homomorphism \( \varphi \) of \( A \) onto \( B \) such that \( \varphi(b) = b \) for each \( b \in B \). Then \( B \) is said to be a retract of \( A \) and \( \varphi \) is called a retract mapping corresponding to \( B \).

In view of [6; Lemma 1.1] we have that \( L[A] \) contains all retracts of \( A \). We will often refer to this fact.

**Lemma 2.** Let \( A \) be an algebra and \( E \) be a retract of \( A \). If \( F \in L[E] \), then \( F \in L[A] \).

**Proof.** If \( F \cong E \), then the assertion is true.

Assume that \( F \) is not isomorphic to \( E \). Then there exists a direct limit family \( \{ P, A_p, \varphi_{pq} \} \) such that \( A_p \cong E \) for every \( p \in P \) and the direct limit \( \bar{A} \) of this
family is isomorphic to $F$. Suppose that $\psi_p$ is an isomorphism of $E$ onto $A_p$. According to [3; Lemma 7] the set $P$ is not upperbounded.

Let $p \in P$. Then there exists $A'_p$ such that $A'_p \cong A$ and $A_p \subseteq A'_p$. Further, let $\psi'_p$ be an isomorphism $A$ onto $A_p$ such that $\psi'_p(e) = \psi_p(e)$ for every $e \in E$.

Let $\varphi$ be a retract endomorphism of $A$ corresponding to $E$. Let $p, q \in P$, $p \leq q$. Put

$$\varphi'_{pq} = \psi_p^{-1} \circ \varphi_p \circ \varphi_{pq}.$$  

Then $\varphi'_{pq}(x) = \varphi_{pq}(x)$ for every $x \in A_p$ and $\varphi'_{pq}(A'_p) \subseteq A_q$.

The family $\{P, A'_p, \varphi'_{pq}\}$ is direct because $\varphi_{pq} \circ \varphi'_q \circ \varphi_p = \varphi_{pq}$. Assume that $\{P, A'_p, \varphi'_{pq}\} \rightarrow \overline{A'}$. For $z \in \bigcup_{p \in P} A'_p$ we denote by $\overline{z}'$ the corresponding element of $\overline{A'}$.

Let us define the mapping $\psi$ from $\overline{A}$ into $\overline{A'}$. Consider $p \in P$ and $x \in A_p$. Then $x \in \overline{A'}$. Put $\psi(x) = \overline{x}'$.

Assume that $p, q \in P$, $x \in A_p$, $y \in A_q$ and $\psi(x) = \psi(y)$. Then $\overline{x}' = \overline{y}'$. That means there exists $s \in P$ such that $p, q \leq s$ and $\varphi'_s(x) = \varphi'_s(y)$. Therefore $\varphi_{qs}(x) = \varphi_{qs}(y)$ and $\overline{x} = \overline{y}$.

Now assume that $p \in P$ and $a \in A'_p$. Let $q \in P$ be such that $p < q$. Then $\varphi'_{pq}(a) \in A_q \quad (\cong E)$. We obtain $\psi(\varphi'_{pq}(a)) = \overline{a}'$.

Finally let $p \in P$ and $x \in A_p$. Then $\psi(f(x)) = \psi(f(\overline{x})) = [f(x)]' = f(\overline{x})' = f(\psi(\overline{x}))$. We have proved that $\overline{A} \cong \overline{A'}$ and thus $F \in L[A]$. \hfill $\square$

For monounary algebras we will use the terminology as in [9].

Denote by $U$ the class of all monounary algebras. We will use the symbol $f$ for the operation in algebras of $U$.

Let $A, B \in U$ and $A_j \in U$ for every $j \in J$. Denote by $A + B$ and $\sum_{j \in J} A_j$, respectively a monounary algebra which is a disjoint union of $A, B$ and of $A_j$, $j \in J$, respectively.

The definition of a retract yields:

**Lemma 3.** Let $A \in U$. Let algebras $B_j$ be components of $A$ for all $j \in J$. If $B'$ is a retract of the algebra $\bigcup_{j \in J} B_j$, then the algebra $\left( A - \bigcup_{j \in J} B_j \right) + B'$ is a retract of $A$.

Retracts of monounary algebras was thoroughly studied by D. Studenovská, e.g. [7], [8].
In this paper we will often need to say that a subalgebra of $A$ is a retract of $A$. If it follows immediately from [7; Theorem 1.3], then we will not always refer to this fact.

Denote by $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{Z}$ the set of all positive integers, nonnegative integers and all integers, respectively.

Let $A \in \mathcal{U}$ and $R \subseteq A$. The set $R$ is said to be a chain of the algebra $A$, if one of the following conditions is satisfied:

1. $R = \{a_0, \ldots, a_n\}$, $n \in \mathcal{N}_0$, $a_i \neq a_j$ for $i \neq j$ and $f(a_i) = a_{i-1}$ for $i = 1, 2, \ldots, n$;

2. $R = \{a_i : i \in \mathcal{N}_0\}$, $a_i \neq a_j$ for $i \neq j$ and $f(a_i) = a_{i-1}$ for each $i \in \mathcal{N}$.

**NOTATION.** Let us denote by $\mathcal{N}$ the monounary algebra defined on the set $\mathcal{N}$ with the successor operation. Further, let $\mathcal{Z}$ be the monounary algebra defined on the set of all integers with the successor operation.

We denote

$\mathcal{T} = \{A \in \mathcal{U} : \text{every component of } A \text{ is a cycle and there are no components } B, C \text{ of } A \text{ such that } B \neq C \text{ and the length of } B \text{ divides the length of } C \}$;

$\mathcal{T}_1 = \{A \in \mathcal{U} : \text{there exists a chain } R \text{ of } A \text{ such that } A - R \in \mathcal{T} \text{ and } R \text{ fails to be a subalgebra of } A \}$;

$\mathcal{T}_2 = \{A \in \mathcal{U} : \text{there exist } B \in \mathcal{T} \text{ and } k, l \in \mathcal{N} \text{ such that } A = B + C,$

where $C$ is a cycle of length $l$, $B$ contains a cycle of length $k$ and $l$ is a multiple of $k$ \}$;

$\mathcal{T}_3 = \{A \in \mathcal{U} : \text{there exists } B \in \mathcal{T} \text{ such that } A = B + Z \}$;

$\mathcal{T}_4 = \{A \in \mathcal{U} : A \text{ is connected and there exists a chain } R \text{ of } A \text{ such that } A - R \cong Z \}$.

For monounary algebras we have that $L[A] = [A]$ if and only if $A \in \mathcal{T} \cup [Z]$. cf. [3; Theorem 1].

**NOTATION.** Let $A$ be a monounary algebra and let $\{B_j : j \in J\}$ be the set of all components of $A$. If $j \in J$ and $k \in \mathcal{N}$ are such that $B_j$ contains a cycle of the length $k$, then let $C_j$ be a cycle of the length $k$. If $j \in J$ is such that $B_j$ contains no cycle, then put $C_j \cong \mathcal{Z}$. We denote $A^\circ = \sum_{j \in J} C_j$.

Remark that if every component of $A$ has a cycle, then $A$ is isomorphic to a subalgebra of $A$.
The following result is proved in [2], cf. Lemma 4:

**LEMMA 4.** Let \( A \in \mathcal{U} \). Then \( A^\circ \in \mathcal{L}[A] \).

**DEFINITION.** Let \( A \in \mathcal{U} \). An element \( x \in A \) is called a *source* of \( A \) if \( f(y) \neq x \) is satisfied for all \( y \in A \). We denote by \( S \) the set of all sources of \( A \).

## 2. Algebras with \( A^\circ \in \mathcal{T} \)

In this section assume that \( A \) is a monounary algebra such that \( A \notin \mathcal{T} \) and \( A^\circ \in \mathcal{T} \). We will prove that we can obtain an algebra of the class \( \mathcal{T}_1 \) via direct limits from \( A \).

Let \( B \) be a subalgebra of \( A \). Then each component of \( B \) has a cycle in view of the fact that \( A^\circ \in \mathcal{T} \). We can suppose that \( B^\circ \subseteq B \).

Let \( \{B_j : j \in J\} \) be the set of all components of \( A \). Note that if \( \varphi \) is an endomorphism of \( A \), then \( \varphi(B_j) \subseteq B_j \) for all \( j \in J \) because by any homomorphism a cycle of the length \( k \) must be mapped into a cycle of the length \( l \) such that \( l \) divides \( k \) (cf. [10]). Further, there exists a component of \( A \) which is not a cycle.

**LEMMA 5.** Let (1) be valid and \( A_p \cong A \) for all \( p \in P \). Then \( (\overline{A})^\circ \cong A^\circ \).

**Proof.** In view of \( A^\circ \in \mathcal{T} \) it is sufficient to show that \( (\overline{A})^\circ \) is isomorphic to a subalgebra of \( A \) and \( A^\circ \) is isomorphic to a subalgebra of \( \overline{A} \).

Suppose that \( \psi_p \) is an isomorphism from \( A \) onto \( A_p \) for every \( p \in P \). Let \( C \) be a cycle of \( A \). We have \( \varphi_{pq} (\psi_p(C)) = \psi_q(C) \) for every \( p, q \in P, \ p \leq q \).

Thus \( \overline{A} \) possesses a cycle which is isomorphic to \( C \). Therefore \( \overline{A} \) possesses a subalgebra which is isomorphic to \( A^\circ \).

Assume that \( \overline{C} \) is a cycle of \( \overline{A} \) and \( k \) is the length of \( C \). Choose \( p \in P \), \( x \in A_p \) such that \( \overline{x} \in \overline{C} \). Then there exists \( q \in P \) such that \( p \leq q \) and \( \varphi_{pq}(f^k(x)) = \varphi_{pq}(x) \). We obtain that the algebra \( A_q \) has a cycle of the length \( k \) by \( A^\circ \in \mathcal{T} \). Thus \( \overline{C} \) is isomorphic to a subalgebra of \( A \) and \( A \) possesses a subalgebra which is isomorphic to \( (\overline{A})^\circ \).

\( \square \)

**NOTATION.** Let \( G \) be a component of \( A \) such that \( G \) is not a cycle.

The algebra \( G^\circ \) is a cycle. Let \( k \in \mathcal{N} \) be length of the cycle \( G^\circ \).

Choose \( a \in G^\circ \). For \( n = 1, 2, \ldots, k \) put
\[
a_n = f^n(a);
\]
\[
D_n = \{ x \in G - G^\circ : \text{there exists } m \in \mathcal{N} \text{ such that } f^m(x) = a_n, \ f^{m-1}(x) \notin G^\circ \};
\]
\[
N_n = \{ m \in \mathcal{N} : \text{there exists } x \in D_n \text{ such that } f^m(x) = a_n, \ f^{m-1}(x) \notin G^\circ \}.
\]
Further let
\[ N^{(M)} = \{ n \in \{1, \ldots, k\} : N_n \text{ has a maximal element} \}; \]
\[ N^{(E)} = \{ n \in \{1, \ldots, k\} : N_n = \emptyset \}. \]

We remark that \( G^o = \{a_1, \ldots, a_k\} \) and sets \( G^o, D_1, \ldots, D_k \) give a partition of \( G \). Moreover \( N^{(E)} \neq \{1, 2, \ldots, k\} \) is satisfied.

**Lemma 6.** Let \( N^{(M)} \cup N^{(E)} = \{1, \ldots, k\} \). Then \( L[A] \cap T_1 \neq \emptyset \).

**Proof.** Put \( r = \max\{\max N_n : n \in N^{(M)}\} \). Choose \( R \subseteq G - G^o \) such that \( R \) is a chain of length \( r \). Let \( D \) be a subalgebra of \( A \) such that \( D - R = A^o \). In view of [7; Theorem 1.3], we have that \( D \) is a retract of \( A \). Thus \( D \in L[A] \).

**Lemma 7.** Let \( n \in \{1, \ldots, k\} - (N^{(M)} \cup N^{(E)}) \) and \( D_n \) contain a chain of infinite length. Then \( L[A] \cap T_1 \neq \emptyset \).

**Proof.** Let \( R \subseteq D_n \) be a chain of infinite length. Let \( D \) be a subalgebra of \( A \) such that \( A^o = D - R \). Then \( D \in T_1 \). Moreover \( D \) is a retract of \( A \) and thus \( D \in L[A] \).

**Lemma 8.** Let \( n \in \{1, \ldots, k\} - (N^{(M)} \cup N^{(E)}) \) and \( D_n \) contain no chain of infinite length. Let \( t \in N \). Then there exists an algebra \( E_t \) such that

1. \( E_t \subseteq G^o \cup D_n \),
2. \( E_t \) is a retract of \( G \),
3. \( f^t(x) \not\in G^o \) for every \( x \in E_t \cap S \).

**Proof.** Recall that \( S \) is the set of all sources of \( A \). Consider \( T = \{ x \in D_n \cap S : f^t(x) \not\in G^o \} \). We have \( T \neq \emptyset \) by the assumption. Put \( E_t = \{ f^m(x) : m \in N, \ x \in T \} \).

**Corollary 1.** Let \( n \in \{1, \ldots, k\} - (N^{(M)} \cup N^{(E)}) \) and let the set \( D_n \) contain no infinite chain of \( A \). Further, let \( t \in N \) and let \( E_{t+1} \) be the algebra from Lemma 8. Then

1. \( (A - G) + E_{t+1} \) is a retract of \( A \).
2. There exists a mapping \( \varepsilon_t \) such that \( \varepsilon_t \) is a retract mapping of \( A \) corresponding to \( (A - G) + E_{t+1} \) and \( \varepsilon_t(D_n) \subseteq D_n \).

**Proof.** The claim (i) follows from Lemmas 8 and 3. The claim (ii) follows from the construction of all homomorphisms between two monounary algebras, cf. [10].
LEMMA 9. Let \( n \in \{1, \ldots, k\} - (N^M \cup N^E) \) and \( D_n \) contain no chain of infinite length. Then there exists an algebra \( D \in L[A] \) such that

1. \( D^o \in \mathcal{T} \);
2. \( D \) contains a chain of infinite length.

Proof. Let \( p \in \mathcal{N} \). Suppose that \( E_{p+1} \) is an algebra from the previous lemma and that \( \varepsilon_p \) is an endomorphism of \( A \) from the previous corollary (ii).

Assume that algebras \( A_p \) are pairwise disjoint and isomorphic to \( A \) for all \( p \in \mathcal{N} \). Let \( p \in \mathcal{N} \). Suppose that \( E_p \) is an isomorphism from \( A \) onto \( A_p \). We put \( \varphi_{pp} = \text{id}_{A_p} \). If \( p < q \), then we put

\[
\varphi_{pq} = \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \cdots \circ \varepsilon_{q-1} \circ \psi_q.
\]

The family \( \{N, A_p, \varphi_{pq}\} \) is direct. Denote by \( D \) its direct limit.

If \( u \in G^o \), then \( u \in (A - G) + E_{p+1} \) for all \( p \in \mathcal{N} \) according to Lemma 8a). Thus \( \varepsilon_p(u) = u \) by Corollary 1(ii). We obtain \( \varphi_{pq}(\psi_p(u)) = (\psi_p \circ \psi_{p+1} \circ \cdots \circ \psi_q)(u) = \psi_q(u) \).

We have \( D \in L[A] \). Assumptions of Lemma 5 are satisfied and thus \( D^o \cong A^o \in \mathcal{T} \).

Suppose that \( p \in \mathcal{N} \) and \( x \in \psi_p(D_n) \). We will show that there exist \( q \in \mathcal{N} \) and \( y \in \psi_q(D_n) \) such that \( f(y) = x \). Then the proof will be ready.

Let \( x \notin \psi_p(S) \). Then there exists \( y \in A_p \) such that \( f(y) = x \). Thus \( f(\bar{y}) = \bar{x} \).

Let \( x \in \psi_p(S) \). Consider \( q \in \mathcal{N} \) such that \( f^q(x) = \psi_q(a_n) \) and \( f^{q-1}(x) \notin \psi_q(G^o) \). Since \( a_n \in G^o \), we have

\[
f^q(\varphi_{pq}(x)) = \varphi_{pq}(f^q(x)) = \varphi_{pq}(\psi_p(a_n)) = \psi_q(a_n).
\]

Thus \( f^q(\psi_q^{-1}(\varphi_{pq}(x))) \in G^o \). Further, \( \psi_q^{-1}(\varphi_{pq}(x)) \in E_q \). That means \( \psi_q^{-1}(\varphi_{pq}(x)) \notin S \) according to Lemma 8c). Let \( z \in A \) be such that \( f(z) = \psi_q^{-1}(\varphi_{pq}(x)) \). Corollary 1(ii) and the definition of \( \varphi_{pq} \) yield that \( \psi_q^{-1}(\varphi_{pq}(x)) \in D_n \). Thus \( z \in D_n \). Put \( y = \psi_q(z) \). We have \( y \in \psi_q(D_n) \) and \( f(\bar{y}) = f(\bar{\psi_q(z)}) = \psi_q(f(z)) = \varphi_{pq}(x) = \bar{x} \).

PROPOSITION 1. If \( A \in \mathcal{U} - \mathcal{T} \) and \( A^o \in \mathcal{T} \), then \( L^2[A] \cap \mathcal{T}_1 \neq \emptyset \).

Proof. If either \( N^M \cup N^E = \{1, \ldots, k\} \) or there exists \( n \in \{1, \ldots, k\} \) such that \( D_n \) contains an infinite chain, then \( L[A] \cap \mathcal{T}_1 \neq \emptyset \) according to Lemmas 6 and 7. In the remaining case we take an algebra \( D \) from Lemma 9. This \( D \) satisfies all assumptions of Lemma 7 and thus \( L^2[A] \cap \mathcal{T}_1 \neq \emptyset \).
3. Connected algebras without cycles

In this section suppose that $A$ is a connected monounary algebra without a cycle and $A$ is not isomorphic to $Z$.

We will prove that we can obtain from $A$ an algebra of $\mathcal{T}_4$ or the algebra $N$ via direct limits.

We will analyse three cases:

(1) the algebra $A$ contains two distinct subalgebras isomorphic to $Z$;
(2) the algebra $A$ contains exactly one subalgebra isomorphic to $Z$;
(3) the algebra $A$ contains no subalgebra isomorphic to $Z$.


**Lemma 10.** Let $A$ contain two distinct subalgebras isomorphic to $Z$. Then $L[A] \cap \mathcal{T}_4 \neq \emptyset$.

**Proof.** Let $B$, $D$ be subalgebras of $A$, $B \neq D$, $B \cong Z$ and $D \cong Z$. Let $E$ be the subalgebra of $A$ which has underlying set $D \cup B$. Then $E \in \mathcal{T}_4$. The algebra $E$ is a retract of $A$ and thus $E \in L[A]$. □

3.2. Case 2.

We suppose that $A$ contains exactly one subalgebra isomorphic to $Z$. Let $B \cong Z$, $B = \{a_n : n \in \mathcal{Z}, f(a_n) = a_{n+1}\}$.

For every $z \in \mathcal{Z}$ we put

$$D_z = \{x \in A - B : \text{there exists } m \in \mathcal{N} \text{ such that } f^m(x) = a_z, f^{m-1}(x) \notin B\};$$

$$N_z = \{m \in \mathcal{N} : \text{there exists } x \in D_z \text{ such that } f^m(x) = a_z\}.$$

Further, let

$$Z^{(M)} = \{z \in \mathcal{Z} : N_z \text{ has a maximal element}\};$$

$$Z^{(E)} = \{n \in \mathcal{Z} : N_z = \emptyset\}.$$

We remark that sets $B$ and $D_z$ for all $z \in \mathcal{Z}$ give a partition of the set $A$.

**Lemma 11.** Let $Z^{(M)} \neq \emptyset$. Then $L[A] \cap \mathcal{T}_4 \neq \emptyset$.

**Proof.** Let $n \in Z^{(M)}$. Suppose that $R$ is a chain of $A$ such that $R$ contains max $N_n$ elements of $D_n$. Then $E = R \cup B$ is a retract of $A$. Thus $E \in L[A]$. Moreover $E \in \mathcal{T}_4$. □
LEMMA 12. Let $Z^{(M)} = \emptyset$. Then $L^2[A] \cap T \neq \emptyset$.

Proof. Consider $n \notin Z^{(E)}$. Such $n$ exists because $A$ is not isomorphic to $Z$.

We will prove three claims. The assertion follows from the third claim and Lemma 10.

CLAIM 1. Let $t \in N$. Then there exists an algebra $E_t$ such that

a) $E_t \subseteq D_n \cup B$,

b) $E_t$ is a retract of $A$, 

c) $f^t(x) \notin B$ for every $x \in E_t \cap S$.

Proof. Consider $T = \{x \in D_n \cap S : f^t(x) \notin B\}$. We have $T \neq \emptyset$ according to $n \notin Z^{(E)} \cup Z^{(M)}$. Put $E_t = \{f^m(x) : m \in N, x \in T\}$. □

CLAIM 2. Let $t \in N$ and $E_t$ be an algebra from the previous claim. Then there exists a mapping $\epsilon_t$ such that $\epsilon_t$ is a retract mapping of $A$ corresponding to $E_{t+1}$, $\epsilon_t(B) = B$ and $\epsilon_t(D_n) \subseteq D_n$.

Proof. It follows from the construction of all homomorphisms between two monounary algebras, cf. [10]. □

CLAIM 3. There exists an algebra $D \in L[A]$ such that

1. $D$ is a connected algebra;

2. $D$ contains two distinct subalgebras isomorphic to $Z$.

Proof. Let $p \in N$. Suppose that $E_{p+1}$ is an algebra from the first claim and that $\epsilon_p$ is a retract endomorphism of $A$ corresponding to $E_{p+1}$ from the second claim.

Assume that algebras $A_p$ are pairwise disjoint and isomorphic to $A$ for all $p \in N$. Let $p \in N$. Suppose that $\psi_p$ is an isomorphism of $A$ onto $A_p$. We put $\varphi_{pp} = \text{id}_{A_p}$. If $p < q$, then we put

$$
\varphi_{pq} = \psi_p^{-1} \circ \epsilon_p \circ \epsilon_{p+1} \circ \cdots \circ \epsilon_{q-1} \circ \psi_q.
$$

The family $\{N, A_p, \varphi_{pq}\}$ is direct. Denote by $D$ its direct limit.

We have $D \in L[A]$. In view of [3; Proposition 1] the algebra $D$ is connected. In view of [3; Lemma 10] the algebra $D$ has no cycle.

Let $p \in N$ and $E = \{\psi_p(a_k) : k \in Z\}$.

For every $k \in Z$ we have $f(\psi_p(a_{k-1})) = \psi_p(f(a_{k-1})) = \overline{\psi_p(a_k)}$. Thus $E$ is a subalgebra of $D$ isomorphic to $Z$. 

185
Suppose that \( x \in \psi_p(D_n) \). Then \( \bar{x} \notin E \) according to Claim 2 and the definition of \( \varphi_{pq} \). We will show that there exist \( q \in \mathcal{N} \) and \( y \in \psi_q(D_n) \) such that \( f(y) = \bar{x} \). Then the proof of this claim will finished.

Let \( x \notin \psi_p(S) \). Then there exists \( y \in A_p \) such that \( f(y) = x \). Thus \( f(\bar{y}) = \bar{x} \).

Let \( x \in \psi_p(S) \). Consider \( q \in \mathcal{N} \) such that \( f^q(x) = \psi_p(a_n) \), \( f^{q-1}(x) \notin \psi_p(B) \).

Since \( a_n \in E_t \) for every \( p < t \leq q \), we have

\[
\begin{align*}
  f^q(\varphi_{pq}(x)) &= \varphi_{pq}(f^q(x)) = \varphi_{pq}(\psi_p(a_n)) \\
  &= (\psi_p \circ \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \cdots \circ \varepsilon_{q-1} \circ \psi_q)(a_n) = \psi_q(a_n) \).
\end{align*}
\]

Thus \( f^q(\varphi_q^{-1}(\varphi_{pq}(x))) = a_n \in B \). The definition of \( \varphi_{pq} \) yields \( \psi_q^{-1}(\varphi_{pq}(x)) \in E_q \). That means \( \psi_q^{-1}(\varphi_{pq}(x)) \notin S \) according to Claim 1c). Let \( z \in A \) be such that \( f(z) = \psi_q^{-1}(\varphi_{pq}(x)) \). The Claim 2 and the definition of \( \varphi_{pq} \) imply that \( \psi_q^{-1}(\varphi_{pq}(x)) \in D_n \). Thus \( z \in D_n \). Put \( y = \psi_q(z) \). We have \( y \in \psi_q(D_n) \) and \( f(y) = \psi_q(f(z)) = \varphi_{pq}(x) = \bar{x} \).

**Lemma 13.** Let \( A \) contain exactly one subalgebra isomorphic to \( Z \). Then \( L^2[A] \cap T_4 \neq \emptyset \).

**Proof.** It follows from Lemmas 11 and 12.

**3.3. Case 3.**

In Lemmas 14–17 we will suppose that \( A \) contains no subalgebra isomorphic to \( Z \). Then \( S \neq \emptyset \).

**Notation.** Let \( a \in S \). Put \( B = \{ f^n(a) : n \in \mathcal{N}_0 \} \).

For \( n \in \mathcal{N} \) let us denote

\[
\begin{align*}
  a_n &= f^n(a) \\
  D_n &= \{ x \in A - B : \text{there exists } m \in \mathcal{N} \text{ such that } f^m(x) = a_n, f^{m-1}(x) \notin B \} \\
  N_n &= \{ m \in \mathcal{N} : \text{there exists } x \in D_n \text{ such that } f^m(x) = a_n \}.
\end{align*}
\]

Further, let

\[
N(M) = \{ n \in \mathcal{N} : N_n \text{ has a maximal element} \}.
\]

For \( n \in N(M) \) put \( j_n = \max N_n \).

Denote

\[
N(E) = \{ n \in \mathcal{N} : N_n = \emptyset \}
\]

and

\[
N(I) = \mathcal{N} - (N(M) \cup N(E)) \cdot
\]

We remark that \( B \) is a subalgebra of \( A \). Sets \( B \) and \( D_n \) for all \( n \in \mathcal{N} \) give a partition of the set \( A \).
LEMMA 14. Suppose that $N(M) \cup N(E) = \mathcal{N}$ and $\{j_n : n \in N(M)\}$ has a maximum. Then the algebra $N \in \text{L}[A]$.

Proof. Denote $j = \text{max}\{j_k : k \in N(M)\}$. Suppose that $n \in N(M)$ is such that $j_n = j$. Then there exists $x \in D_n$ such that $f^j(x) = a_n$, $f^{j-1}(x) \notin B$.

Let $j \geq n$. Put $D = \{f^m(x) : m \in N_0\}$. The algebra $D$ is a retract of $A$ and $D$ is isomorphic to $N$. Thus $N \in \text{L}[A]$.

If $j < n$, then $B$ is a retract of $A$. \hfill \Box

The proof of the following lemma will be similar to the proof of Lemma 9.

LEMMA 15. Let $N^{(I)} \neq \emptyset$. Then $\text{L}^3[A] \cap \mathcal{T}_4 \neq \emptyset$.

Proof. Let $n$ be the least number from $N^{(I)}$.

Since $A$ does not contain a subalgebra isomorphic to $Z$, the set $D_n \cap S$ is infinite.

We will prove three claims. The assertion follows from the third claim and Lemmas 10 and 13.

CLAIM 4. Let $t \in \mathcal{N}$. Then there exists an algebra $E_t$ such that

a) $E_t \subseteq D_n \cup \{a_k : k \geq n\}$,
b) $E_t$ is a retract of $A$,
c) $f^t(x) \notin B$ for every $x \in E_t \cap S$.

Proof. Consider $T = \{x \in D_n \cap S : f^t(x) \notin B\}$. We have $T \neq \emptyset$ according to $n \in N^{(I)}$. Put $E_t = \{f^m(x) : m \in \mathcal{N}, x \in T\}$. \hfill \Box

CLAIM 5. Let $t \in \mathcal{N}$ and $E_t$ be an algebra from the previous claim. Let $\varepsilon$ be a retract mapping corresponding to $E_t$. Then $\varepsilon(D_n) \subseteq D_n$.

Proof. Suppose that $x \in D_n$. Then there exists $m \in \mathcal{N}$ such that $f^m(x) = a_n$. We have $f^m(\varepsilon(x)) = \varepsilon(f^m(x)) = \varepsilon(a_n) = a_n$.

Therefore $\varepsilon(x) \in D_n$ according to a) in the previous claim. \hfill \Box

CLAIM 6. There exists an algebra $D \in \text{L}[A]$ such that

1. $D$ is a connected algebra without a cycle;
2. $D$ contains a subalgebra isomorphic to $Z$.

Proof. Let $p \in \mathcal{N}$. Suppose that $E_{p+1}$ is an algebra from the first claim and that $\varepsilon_p$ is a retract endomorphism of $A$ corresponding to $E_{p+1}$. \hfill 187
Assume that algebras $A_p$ are pairwise disjoint and isomorphic to $A$ for all $p \in \mathcal{N}$. Let $p \in \mathcal{N}$. Suppose that $\psi_p$ is an isomorphism from $A$ onto $A_p$. We put $\varphi_{pp} = \text{id}_{A_p}$. If $p < q$, then we put

$$
\varphi_{pq} = \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \ldots \circ \varepsilon_{q-1} \circ \psi_q.
$$

The family $\{\mathcal{N}, A_p, \varphi_{pq}\}$ is direct. Denote by $D$ its direct limit.

We have $D \in \mathbb{L}[A]$. In view of [3; Proposition 1], the algebra $D$ is connected. In view of [3; Lemma 10], the algebra $A$ contains no cycle.

Suppose that $p \in \mathcal{N}$ and $x \in \psi_p(D_n)$. The proof that there exists $q \in \mathcal{N}$ and $y \in \psi_q(D_n)$ such that $f(y) = \overline{x}$ is analogous to the end of the proof of Claim 3.

In the next notation and in Lemmas 16 and 17 we suppose that $N(M) \cup N(E) = \mathcal{N}$ and the set $\{j_n : n \in N(M)\}$ has no maximum.

**NOTATION.** We define a mapping $u$ of $\mathcal{N}$ into $N(M)$ by the following way: Let $u(1)$ be the least element of $N(M)$. By induction for $i \in \mathcal{N}$ let $u(i+1)$ be the least number such that $u(i+1) \in N(M)$, $u(i+1) > u(i)$ and $j_{u(i+1)} > j_{u(i)}$.

For $n \in \mathcal{N}$ let $e_n$ be an element of $D_{u(n)}$ such that

$$
f_{j_{u(n)}}(e_n) = a_{u(n)}.
$$

Remark that $e_n \in S$.

Let $i \in \mathcal{N}$. Define the mapping $\xi_i$ of $A$ into $A$ by the following way:

$$
\xi_i(x) = \begin{cases} 
  f_{u(i+1)-u(i)}(x) & \text{if } x \notin D_{u(i)}, \\
  f_{j_{u(i+1)}-m}(e_{i+1}) & \text{if } x \in D_{u(i)} \text{ and } m \in \mathcal{N}, \text{ such that } f^m(x) = a_{u(i)}.
\end{cases}
$$

**LEMMA 16.** Let $i \in \mathcal{N}$. Then the mapping $\xi_i$ is an endomorphism of $A$ such that

(a) $\xi_i(D_{u(i)}) \subseteq D_{u(i+1)}$;

(b) $\xi_i(B) \subseteq B$.

**Proof.** Let $x \in A$.

If either $f(x) \in D_{u(i)}$ and $x \notin D_{u(i)}$ or $x \notin D_{u(i)}$ and $f(x) \notin D_{u(i)}$, then it is easy to verify that $\xi_i(f(x)) = f(\xi_i(x))$. The case $x \notin D_{u(i)}$ and $f(x) \in D_{u(i)}$ cannot occur.

Suppose that $x \in D_{u(i)}$ and $f(x) \notin D_{u(i)}$. Then $f(x) = a_{u(i)} = f_{j_{u(i)}}(e_i)$.

Thus we have $\xi_i(x) = f_{j_{u(i)+1}-1}(e_{i+1})$ and

$$
\xi_i(f(x)) = f_{u(i+1)-u(i)}(f(x)) = f_{u(i+1)-u(i)}(a_{u(i)}) = f_{u(i+1)-u(i)}(f(u(i)(a))) = a_{u(i+1)} = f_{j_{u(i)+1}}(e_{i+1}) = f(\xi_i(x)).
$$

Assertions (a), (b) follow from the definition of $\xi_i$. 

LEMMA 17. There exists an algebra $F \in L[A]$ such that $F$ is connected, $F$ contains a subalgebra isomorphic to $Z$ and the algebra $F$ is not isomorphic to $Z$.

Proof. For $i \in \mathcal{N}$ let $A_i$ be pairwise disjoint algebras, which are isomorphic to $A$. Let $\psi_i$ be an isomorphism of the algebra $A$ onto $A_i$. Let $\varphi_{ii} = \text{id}_{A_i}$ and for $i < j$ let

$$\varphi_{ij} = \psi_i^{-1} \circ \xi_i \circ \xi_{i+1} \circ \cdots \circ \xi_{j-1} \circ \psi_j.$$ 

Then $\{\mathcal{N}, A_i, \varphi_{ij}\}$ is a direct family of algebras. Denote by $F$ its direct limit.

In view of [3; Proposition 1], the algebra $F$ is connected.

According to [3; Lemma 10], the algebra $F$ has no cycle.

Let $z \in F$. Choose $i \in \mathcal{N}$ and $y \in A_i$ such that $y \in z$. Then

$$\varphi_{i,i+1}(y) = \psi_{i+1}(\xi_i(\psi_i^{-1}(y))).$$

If $\psi_i^{-1}(y) \notin D_{u(i)}$, then for

$$x = \psi_{i+1}(f^{u(i+1)-u(i)-1}(\psi_i^{-1}(y)))$$

we have $x \in A_{i+1}$ and

$$f(x) = f\left(\psi_{i+1}\left(f^{u(i+1)-u(i)-1}(\psi_i^{-1}(y))\right)\right) = \psi_{i+1}\left(f^{u(i+1)-u(i)}(\psi_i^{-1}(y))\right) = \varphi_{i,i+1}(y) = z.$$

If $\psi_i^{-1}(y) \in D_{u(i)}$ and $m \in \mathcal{N}$ is such that $f^m(\psi_i^{-1}(y)) = a_{u(i)}$, then for

$$x = \psi_{i+1}(f^{j u(i+1)-m-1}(e_{i+1}))$$

we have $x \in A_{i+1}$ and

$$f(x) = f\left(\psi_{i+1}\left(f^{j u(i+1)-m-1}(e_{i+1})\right)\right) = \psi_{i+1}\left(f^{j u(i+1)-m}(e_{i+1})\right) = \psi_{i+1}\left(\xi_i(\psi_i^{-1}(y))\right) = \varphi_{i,i+1}(y) = z.$$

We conclude that the algebra $F$ contains a subalgebra isomorphic to $Z$.

Now we will prove that the operation of $F$ is not injective.

Let $w = f^{j u(2)-1}(e_2)$. Since $j_{u(2)} > 1$, we have $w \in D_{u(2)}$. Further,

$$f(\bar{\psi}_2(w)) = \bar{\psi}_2\left(f^{j u(2)}(e_2)\right) = \bar{\psi}_2(a_{u(2)}) = \bar{\psi}_2\left(f^{u(2)}(a)\right) = \bar{\psi}_2\left(f^{u(2)-1}(a)\right).$$

Let $k \in \mathcal{N}$, $k > 2$. In view of Lemma 16(a) we have

$$\varphi_{2k}(\bar{\psi}_2(w)) = (\xi_2 \circ \xi_3 \circ \cdots \circ \xi_{k-1} \circ \psi_k)(w) \in \bar{\psi}_k(D_{u(k)}).$$

In view of Lemma 16(b) we have

$$\varphi_{2k}\left(\psi_2\left(f^{u(2)-1}(a)\right)\right) = (\xi_2 \circ \xi_3 \circ \cdots \circ \xi_{k-1} \circ \psi_k)\left(f^{u(2)-1}(a)\right) \in \psi_k(B).$$

Since $B \cap D_{u(k)} = \emptyset$, we have $\varphi_{2k}(\bar{\psi}_2(w)) = \varphi_{2k}(\psi_2\left(f^{u(2)-1}(a)\right))$. Thus

$$\psi_2(w) \neq \bar{\psi}_2\left(f^{u(2)-1}(a)\right).$$
**Lemma 18.** Let $A$ do not contain a subalgebra isomorphic to $Z$. Then

$$L^3[A] \cap (T_4 \cup [N]) \neq \emptyset.$$ 

**Proof.** If $A$ satisfies assumptions of Lemma 14 or Lemma 15, then $L^3[A] \cap (T_4 \cup [N]) \neq \emptyset$.

The remaining case is that $N(M) \cup N(E) = N$ and the set $\{j_n : n \in N(M)\}$ has no maximum. Then an algebra $F$ from Lemma 17 satisfies either the assumptions of Lemma 10 or Lemma 13. That yields $L^2[F] \cap T_4 \neq \emptyset$. Thus $L^3[A] \cap T_4 \neq \emptyset$. \qed

We summarize the results of Lemmas 10, 13 and 18:

**Proposition 2.** If $A$ is a connected monounary algebra without cycle and $A$ is not isomorphic to $Z$, then

$$L^3[A] \cap (T_4 \cup [N]) \neq \emptyset.$$ 

**4. The main result**

In this section we describe all monounary algebras $A$, $B$ such that $L[A, B] = [A, B]$.

Next two theorems show that from every monounary algebra $A$ such that $A \notin T \cup [Z]$ we can obtain an algebra of the class $T_1 \cup T_2 \cup T_3 \cup T_4 \cup [N, Z + Z]$ via direct limits.

**Proposition 3.** Let $A$ be a monounary algebra such that $A \notin T \cup [Z]$. If every component of $A$ has a cycle, then

$$L^2[A] \cap (T_1 \cup T_2) \neq \emptyset.$$ 

**Proof.** Let $B$ be a subalgebra of $A$. We will suppose that $B^\circ$ is a subalgebra of $B$.

If $A^\circ \in T$, then $L^2[A] \cap T_1 \neq \emptyset$ according to Proposition 1.

Assume that $A^\circ \notin T$. Let $\{B_j : j \in J\}$ be the set of all components of $A$. Then $\{B_j^\circ : j \in J\}$ is the set of all components of $A^\circ$. Let $k(j)$ be the length of the cycle $B_j^\circ$ for every $j \in J$. There exists a subset $I$ of the set $J$ such that

1. if $i, j \in I$, then $k(i)$ does not divide $k(j)$;
2. if $j \in J - I$, then there exists $i \in I$ such that $k(i)$ divides $k(j)$.

Consider a set $I$ with these properties.

Let $E$ be an algebra which has the set of all components equal to $\{B_i \in I\}$. Then $E \in T$. Consider $m \in J - I$. Put $D = E + B_i$. The algebra $D$ is a retract of $A$ and $D \in T_2$. We have $L[A] \cap T_2 \neq \emptyset$. 

190
PROPOSITION 4. Let $A$ be a monounary algebra such that $A \not\in T \cup [Z]$. If $A$ contains a component without a cycle, then
\[ L^3[A] \cap (T_3 \cup T_4 \cup [N, Z + Z]) \neq \emptyset. \]

Proof. Let $A$ contain a cycle. Then $A^\circ$ possesses a cycle.

Consider an algebra $T$ such that $T$ is a retract of $A^\circ$ and $T \in \mathcal{T}$. Such $T$ exists in view of [3; Lemma 20]. Since $A$ contains a component without a cycle, the algebra $A^\circ$ contains a component $D$ isomorphic to $Z$. Denote $E = T + D$. We have $E \in T_3$ and $E$ is a retract of $A^\circ$. In view of Lemma 4 we obtain $L^2[A] \cap T_3 \neq \emptyset$.

Assume that $A$ has no cycle. If $A$ is not connected, then $Z + Z$ is isomorphic to a retract of $A^\circ$. We have $Z + Z \in L^2[A]$ according to Lemma 4. If $A$ is connected, then the class $L^3[A]$ contains an algebra from $T_4 \cup [N]$ according to Proposition 2.

For $A \in T_1 \cup T_2 \cup T_3$ we denote the algebra $A^*$ by the following way:

If $A \in T_1$ and $R$ is a chain of $A$ from the definition of $T_1$, then we put $A^* = A - R$.

If $A \in T_2$ and $B \in \mathcal{T}$ satisfies the conditions from the definition of $T_2$, then we put $A^* = B$.

If $A \in T_3$ and $B \in \mathcal{T}$ satisfies the conditions from the definition of $T_3$, then we put $A^* = B$.

Let us remark that $A^*$ is a retract of $A$. Thus $A^* \in L[A]$. Further, $A^* \neq A$.

PROPOSITION 5. If $A \in T_1 \cup T_2 \cup T_3$, then $L[A] = [A, A^*]$.

If $A \in T_1 \cup [Z + Z, N]$, then $L[A] = [A, Z]$.

Proof. It is a consequence of Lemma 4 and [3; Theorems 1, 2, 3].

COROLLARY 2. If $A \in T_1 \cup T_2 \cup T_3$, then $L[A, A^*] = [A, A^*]$. If $A \in T_4 \cup [N, Z + Z]$, then $L[A, Z] = [A, Z]$.

Proof. Let $A \in T_1 \cup T_2 \cup T_3$. Then $L[A] = [A, A^*]$. Since $A^* \in \mathcal{T}$, we have $L[A^*] = [A^*]$. Therefore $L[A, A^*] = [A, A^*]$ by Lemma 1.

Now let $A \in T_4 \cup [N, Z + Z]$. Then $L[A] = [A, Z]$. In view of Lemma 1 we have $L[A, Z] = [A, Z]$.

In Theorem 6 we will use the following notation. If $(p)$ is a condition for algebras $A$, $B$, then the symbol $(p')$ denotes the condition, which arise from $(p)$ in such a way that we change algebras $A$, $B$; further, the symbol $(p^*)$ denotes the condition which requires that either $(p)$ or $(p')$ is valid.
Consider conditions
(i)  $A, B \in \mathcal{T}$;
(ii) $A \in \mathcal{T}$, $B \cong Z$;
(iii) $A \cong N$, $B \cong Z$;
(iv) $A \cong Z$, $B \cong Z + Z$;
(v) $A \cong Z$, $B \in \mathcal{T}_4$;
(vi) $A \in \mathcal{T}$, $B \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, $B^* \cong A$;
(vii) $A \cong Z$, $B \cong Z$.

**Theorem 1.** Let $A, B \in \mathcal{U}$. Then

$$
\mathbf{L}[A, B] = [A, B]
$$

if and only if one of conditions (i), (ii*), (iii*), (iv*), (v*), (vi*), (vii) is valid.

**Proof.** If one of (i), (ii*), (vii) holds, then [3; Theorem 1] implies that $\mathbf{L}[A] = [A]$, $\mathbf{L}[B] = [B]$. Thus $\mathbf{L}[A, B] = [A, B]$ according to [3; Lemma 15].

If one of conditions (iii*) (vi*) holds, then $\mathbf{L}[A, B] = [A, B]$ according to Corollary 2.

Suppose that none of conditions (i), (ii*) (vi*), (vii) holds for $A, B$.

We have $[A, B] \subseteq \mathbf{L}[A, B] \subseteq \mathbf{L}^2[A, B] \subseteq \mathbf{L}^3[A, B]$ according [3; Lemma 12].

We will prove that the class $\mathbf{L}^3[A, B]$ contains an algebra which does not belong to $[A, B]$. Then $\mathbf{L}[A, B] \neq [A, B]$ will be proved.

We will discuss the following cases:

1. $A \in \mathcal{T}$;
2. $A \cong Z$;
3. $A \notin \mathcal{T} \cup [Z]$.

(1) In view of invalidity of (i) and (ii) we have $B \notin \mathcal{T} \cup [Z]$. Propositions 3 and 4 imply that the class $\mathbf{L}^3[B]$ contains an algebra $D$ such that

$$
D \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z].
$$

If $D$ is not isomorphic to $B$, then $D \in \mathbf{L}^3[A, B] - [A, B]$. Thus $D$ has the required property.

Let $B \cong D$. Then $B \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z]$.

Assume that there exists $i \in \{1, 2, 3\}$ such that $B \in \mathcal{T}_i$. Thus $B^* \in \mathcal{T}$ and $B^* \notin B$. Since the condition (vi) does not hold, the algebra $B^*$ is not isomorphic to $A$. We have $B^* \in \mathbf{L}[B] \subseteq \mathbf{L}[A, B]$. We conclude that $B^*$ has the required property.

If either $B \cong N$ or $B \cong Z + Z$ or $B \in \mathcal{T}_4$, then $Z \in \mathbf{L}[B] \subseteq \mathbf{L}[A, B]$ according to Lemma 4 and [4; Theorems 2, 3]. We have that $Z$ has the required property.
TWO ELEMENT DIRECT LIMIT CLASSES OF MONOUNARY ALGEBRAS

(2) Since (ii') fails to hold, we have $B \notin \mathcal{T}$. In view of the fact that (vii) is not valid, we obtain $B \notin [Z]$. Thus the algebra $B$ satisfies assumptions of either Proposition 3 or Proposition 4. Therefore the class $L^3[B]$ contains an algebra $D$ such that

$$D \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z].$$

If $D \not\subseteq B$, then the proof can be finished analogously as in the case (1).

Let $D \cong B$. Since (iii'), (iv), (v) do not hold, we have $B \notin \mathcal{T}_4 \cup [N, Z + Z]$. That means that there exists $i \in \{1, 2, 3\}$ such that $B \in \mathcal{T}_i$. The algebra $B^* \not\subseteq [A, B]$ and $B^* \in L[B] \subseteq L[A, B]$. Thus $B^*$ has the required property.

(3) The algebra $A$ satisfies assumptions either Proposition 3 or Proposition 4. Therefore $L^3[A]$ contains an algebra $D$ such that

$$D \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z].$$

If $D \not\subseteq A$ and $D \not\subseteq B$, then $D$ has the required property.

Let $D \not\subseteq A$ and $D \cong B$. If $B \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then $B^*$ has the required property. If $B \in \mathcal{T}_4 \cup [Z]$, then $Z$ has the required property.

Let $A \cong D$. If $A \in \mathcal{T}_i$ for $i = 1, 2, 3$, then $A^*$ is not isomorphic to $B$, because (vi) does not hold. Further, $A^* \not\subseteq A$ and $A^* \in L[A]$. Thus $A^*$ has the required property.

If $A \in \mathcal{T}_4 \cup [Z + Z]$, then $Z \in L[A]$, because $Z$ is a retract of $A$. If $A \cong N$, then $Z \in L[A]$ according to Lemma 4. The algebra $B$ is not isomorphic to $Z$ since no condition (iii), (iv'), or (v') is satisfied. □

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