Emília Halušková

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Mathematica Slovaca, Vol. 52 (2002), No. 2, 177--194

Persistent URL: http://dml.cz/dmlcz/132952

Math. Slovaca, **52** (2002), No. 2, 177–194

Mathematica Slovaca

TWO ELEMENT DIRECT LIMIT CLASSES OF MONOUNARY ALGEBRAS

EMÍLIA HALUŠKOVÁ

(*Communicated by Tibor Katriňák*)

ABSTRACT. A class of algebras is said to be direct limit closed if it is closed with respect to direct limits. We describe all two element sets $\mathcal{S}$ of monounary algebras such that $\mathcal{S}$, together with all isomorphic copies of elements of $\mathcal{S}$, is a direct limit closed class.

Direct limit classes of algebras, i.e. classes of algebras which are closed with respect to direct limits, were investigated in [3] and [6]. The class of all retracts of a finite algebra is a direct limit class, cf. [5].

The paper [3] contains a description of all monounary algebras A such that $\{A\}$ is a direct limit class.

The aim of the present paper is to describe all pairs A, B of monounary algebras such that $\{A, B\}$ is a direct limit class.

1. Preliminaries

For the notion of a direct limit, cf. e.g. Grätzer [1; §21].

Let $\langle P, \leq \rangle$ be a directed partially ordered set, $P \neq \emptyset$. For each $p \in P$ let A_p be an algebra of some fixed type. We assume that if $p, q \in P$, $p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$, there is defined a homomorphism φ_{pq} of A_p into A_q such that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. For each $p \in P$ let φ_{pp} be the identity on A_p. Then we say that $\{P, A_p, \varphi_{pq}\}$ is the direct family.

2000 Mathematics Subject Classification: Primary 08A60.
Keywords: direct limit, retract, monounary algebra.

Supported by grants VEGA 2/5125/98 and VEGA 1/7468/20.

Assume that $p, q \in P$ and $x \in A_p$, $y \in A_q$. Put $x \equiv y$ if there exists $s \in P$ with $p \leq s$, $q \leq s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. For each $z \in \bigcup\limits_{p\in P} A_p$ put $\overline{z} = \Big\{ t \in \bigcup\limits_{p\in P} A_p : \ z \equiv t \Big\}$. Denote $\overline{A} = \Big\{ \overline{z} : \ z \in \bigcup\limits_{p\in P} A_p \Big\}$.

Let f be a n-ary operation from the type of algebras A_p, $p \in P$. Let $x_j \in A_{p_j}$, $1 \leq j \leq n$, and let s be an upper bound of p_j. Define $f(\overline{x}_1, \dots, \overline{x}_n) = \overline{f(\varphi_{p_1 s}(x_1), \dots, \varphi_{p_n s}(x_n))}$. Then $\overline{A}$ is an algebra which is said to be the *direct limit of the direct family* $\{P, A_p, \varphi_{pq}\}$.

We express this situation as follows

$$\{P, A_p, \varphi_{pq}\} \longrightarrow \overline{A}. \tag{1}$$

The operator $\underset{\rightarrow}{\mathbf{L}}$ on classes of algebras was introduced in the textbook [1; §23]. By this definition, if $\mathcal{K}$ is a class of algebras, then $\underset{\rightarrow}{\mathbf{L}}(\mathcal{K})$ is the class of all direct limits of algebras of $\mathcal{K}$.

Let $\mathcal{K}$ be a class of algebras. We denote by $[\mathcal{K}]$ the class of all isomorphic copies of algebras of $\mathcal{K}$. Further, we denote by $\mathbf{L}\mathcal{K}$ the class of all isomorphic copies of direct limits of algebras of $\mathcal{K}$, i.e., $\mathbf{L}\mathcal{K} = \big[\underset{\rightarrow}{\mathbf{L}}(\mathcal{K})\big]$.

We put $\mathbf{L}^2\mathcal{K} = \mathbf{L}\mathbf{L}\mathcal{K}$, $\mathbf{L}^3\mathcal{K} = \mathbf{L}\mathbf{L}^2\mathcal{K}$.

A class $\mathcal{K}$ is called a *direct limit class*, if $\mathbf{L}[\mathcal{K}] = [\mathcal{K}]$.

For algebras $A_1, \dots, A_n$ we will use $[A_1, \dots, A_n]$ instead of $\big[\{A_1, \dots, A_n\}\big]$.

Lemma 1. *Let A, B be algebras and $\mathbf{L}[A] = [A, B]$, $\mathbf{L}[B] = [B]$. Then $\mathbf{L}[A, B] = [A, B]$.*

Proof. Let (1) be valid and $A_p \in [A, B]$ for every $p \in P$. Put $Q = \{q \in P : A_q \cong B\}$. If Q is cofinal with P, then $\overline{A} \cong B$. If $P - Q$ is cofinal with P, then $\overline{A} \cong A$ or $\overline{A} \cong B$. □

Let B be a subalgebra of A. Assume that there exists a homomorphism φ of A onto B such that $\varphi(b) = b$ for each $b \in B$. Then B is said to be a *retract* of A and φ is called a *retract mapping corresponding to* B.

In view of [6; Lemma 1.1] we have that $\mathbf{L}[A]$ contains all retracts of A. We will often refer to this fact.

Lemma 2. *Let A be an algebra and E be a retract of A. If $F \in \mathbf{L}[E]$, then $F \in \mathbf{L}[A]$.*

Proof. If $F \cong E$, then the assertion is true.

Assume that F is not isomorphic to E. Then there exists a direct limit family $\{P, A_p, \varphi_{pq}\}$ such that $A_p \cong E$ for every $p \in P$ and the direct limit $\overline{A}$ of this family is isomorphic to F. Suppose that ψ_p is an isomorphism of E onto A_p. According to [3; Lemma 7] the set P is not upperbounded.

Let $p \in P$. Then there exists A'_p such that $A'_p \cong A$ and $A_p \subseteq A'_p$. Further, let ψ'_p be an isomorphism A onto A_p such that $\psi'_p(e) = \psi_p(e)$ for every $e \in E$.

Let φ be a retract endomorphism of A corresponding to E. Let $p, q \in P$, $p \leq q$. Put

$$\varphi'_{pq} = {\psi'_p}^{-1} \circ \varphi \circ \psi_p \circ \varphi_{pq} .$$

Then $\varphi'_{pq}(x) = \varphi_{pq}(x)$ for every $x \in A_p$ and $\varphi'_{pq}(A'_p) \subseteq A_q$.

The family $\{P, A'_p, \varphi'_{pq}\}$ is direct because $\varphi_{pq} \circ {\psi'_q}^{-1} \circ \varphi \circ \psi_q = \varphi_{pq}$. Assume that $\{P, A'_p, \varphi'_{pq}\} \longrightarrow \overline{A}'$. For $z \in \bigcup\limits_{p \in P} A'_p$ we denote by $\overline{z}'$ the corresponding element of $\overline{A}'$.

Let us define the mapping ψ from $\overline{A}$ into $\overline{A}'$. Consider $p \in P$ and $x \in A_p$. Then $x \in \overline{A}'$. Put $\psi(\overline{x}) = \overline{x}'$.

Assume that $p, q \in P$, $x \in A_p$, $y \in A_q$ and $\psi(\overline{x}) = \psi(\overline{y})$. Then $\overline{x}' = \overline{y}'$. That means there exists $s \in P$ such that $p, q \leq s$ and $\varphi'_{ps}(x) = \varphi'_{qs}(y)$. Therefore $\varphi_{ps}(x) = \varphi_{qs}(y)$ and $\overline{x} = \overline{y}$.

Now assume that $p \in P$ and $a \in A'_p$. Let $q \in P$ be such that $p < q$. Then $\varphi'_{pq}(a) \in A_q$ $(\cong E)$. We obtain $\psi\bigl(\overline{\varphi'_{pq}(a)}\bigr) = \overline{a}'$.

Finally let $p \in P$ and $x \in A_p$. Then $\psi(f(\overline{x})) = \psi\bigl(\overline{f(x)}\bigr) = \overline{[f(x)]}' = f(\overline{x}') = f(\psi(\overline{x}))$.

We have proved that $\overline{A} \cong \overline{A'}$ and thus $F \in \mathbf{L}[A]$. □

For monounary algebras we will use the terminology as in [9].

Denote by $\mathcal{U}$ the class of all monounary algebras. We will use the symbol f for the operation in algebras of $\mathcal{U}$.

Let $A, B \in \mathcal{U}$ and $A_j \in \mathcal{U}$ for every $j \in J$. Denote by $A + B$ and $\sum\limits_{j \in J} A_j$, respectively a monounary algebra which is a disjoint union of A, B and of A_j, $j \in J$, respectively.

The definition of a retract yields:

LEMMA 3. *Let $A \in \mathcal{U}$. Let algebras B_j be components of A for all $j \in J$. If B' is a retract of the algebra $\bigcup\limits_{j \in J} B_j$, then the algebra $\Bigl(A - \bigcup\limits_{j \in J} B_j\Bigr) + B'$ is a retract of A.*

Retracts of monounary algebras was thoroughly studied by D. Studenovská, e.g. [7], [8].

In this paper we will often need to say that a subalgebra of A is a retract of A. If it follows immediately from [7; Theorem 1.3], then we will not always refer to this fact.

Denote by $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{Z}$ the set of all positive integers, nonnegative integers and all integers, respectively.

Let $A \in \mathcal{U}$ and $R \subset A$. The set R is said to be a *chain* of the algebra A, if one of the following conditions is satisfied:

(1) $R = \{a_0, \dots, a_n\}$, $n \in \mathcal{N}_0$, $a_i \neq a_j$ for $i \neq j$ and $f(a_i) = a_{i-1}$ for $i = 1, 2, \dots, n$;

(2) $R = \{a_i : i \in \mathcal{N}_0\}$, $a_i \neq a_j$ for $i \neq j$ and $f(a_i) = a_{i-1}$ for each $i \in \mathcal{N}$.

Notation. Let us denote by N the monounary algebra defined on the set $\mathcal{N}$ with the successor operation. Further, let Z be the monounary algebra defined on the set of all integers with the successor operation.

We denote

$$\begin{aligned}\mathcal{T} = \{A \in \mathcal{U} :\ & \text{every component of } A \text{ is a cycle and} \\ & \text{there are no components } B, C \text{ of } A \text{ such that } B \neq C \text{ and} \\ & \text{the length of } B \text{ divides the length of } C\};\end{aligned}$$

$$\begin{aligned}\mathcal{T}_1 = \{A \in \mathcal{U} :\ & \text{there exists a chain } R \text{ of } A \text{ such that} \\ & A - R \in \mathcal{T} \text{ and } R \text{ fails to be a subalgebra of } A\};\end{aligned}$$

$$\begin{aligned}\mathcal{T}_2 = \{A \in \mathcal{U} :\ & \text{there exist } B \in \mathcal{T} \text{ and } k, l \in \mathcal{N} \text{ such that } A = B + C, \\ & \text{where } C \text{ is a cycle of length } l,\ B \text{ contains a cycle of length } k \\ & \text{and } l \text{ is a multiple of } k\};\end{aligned}$$

$$\mathcal{T}_3 = \{A \in \mathcal{U} : \text{there exists } B \in \mathcal{T} \text{ such that } A = B + Z\};$$

$$\begin{aligned}\mathcal{T}_4 = \{A \in \mathcal{U} :\ & A \text{ is connected and there exists a chain } R \text{ of } A \\ & \text{such that } A - R \cong Z\}.\end{aligned}$$

For monounary algebras we have that $\mathbf{L}[A] = [A]$ if and only if $A \in \mathcal{T} \cup [Z]$, cf. [3; Theorem 1].

Notation. Let A be a monounary algebra and let $\{B_j : j \in J\}$ be the set of all components of A. If $j \in J$ and $k \in \mathcal{N}$ are such that B_j contains a cycle of the length k, then let C_j be a cycle of the length k. If $j \in J$ is such that B_j contains no cycle, then put $C_j \cong Z$. We denote $A^\diamond = \sum_{j \in J} C_j$.

Remark that if every component of A has a cycle, then $A^\diamond$ is isomorphic to a subalgebra of A.

The following result is proved in [2], cf. Lemma 4:

LEMMA 4. *Let* $A \in \mathcal{U}$. *Then* $A^\diamond \in \mathbf{L}[A]$.

DEFINITION. Let $A \in \mathcal{U}$. An element $x \in A$ is called a *source* of A if $f(y) \neq x$ is satisfied for all $y \in A$. We denote by S the set of all sources of A.

2. Algebras with $A^\diamond \in \mathcal{T}$

In this section assume that A is a monounary algebra such that $A \notin \mathcal{T}$ and $A^\diamond \in \mathcal{T}$. We will prove that we can obtain an algebra of the class $\mathcal{T}_1$ via direct limits from A.

Let B be a subalgebra of A. Then each component of B has a cycle in view of the fact that $A^\diamond \in \mathcal{T}$. We can suppose that $B^\diamond \subseteq B$.

Let $\{B_j : j \in J\}$ be the set of all components of A. Note that if φ is an endomorphism of A, then $\varphi(B_j) \subseteq B_j$ for all $j \in J$ because by any homomorphism a cycle of the length k must be mapped into a cycle of the length l such that l divides k (cf. [10]). Further, there exists a component of A which is not a cycle.

LEMMA 5. *Let* (1) *be valid and* $A_p \cong A$ *for all* $p \in P$. *Then* $(\overline{A})^\diamond \cong A^\diamond$.

P r o o f . In view of $A^\diamond \in \mathcal{T}$ it is sufficient to show that $(\overline{A})^\diamond$ is isomorphic to a subalgebra of A and $A^\diamond$ is isomorphic to a subalgebra of $\overline{A}$.

Suppose that ψ_p is an isomorphism from A onto A_p for every $p \in P$. Let C be a cycle of A. We have $\varphi_{pq}(\psi_p(C)) = \psi_q(C)$ for every $p, q \in P$, $p \leq q$. Thus $\overline{A}$ possesses a cycle which is isomorphic to C. Therefore $\overline{A}$ possesses a subalgebra which is isomorphic to $A^\diamond$.

Assume that $\overline{C}$ is a cycle of $\overline{A}$ and k is the length of C. Choose $p \in P$, $x \in A_p$ such that $\overline{x} \in \overline{C}$. Then there exists $q \in P$ such that $p \leq q$ and $\varphi_{pq}(f^k(x)) = \varphi_{pq}(x)$. We obtain that the algebra A_q has a cycle of the length k by $A^\diamond \in \mathcal{T}$. Thus $\overline{C}$ is isomorphic to a subalgebra of A and A possesses a subalgebra which is isomorphic to $(\overline{A})^\diamond$. □

NOTATION. Let G be a component of A such that G is not a cycle.

The algebra $G^\diamond$ is a cycle. Let $k \in \mathcal{N}$ be length of the cycle $G^\diamond$.

Choose $a \in G^\diamond$. For $n = 1, 2, \dots, k$ put

$a_n = f^n(a)$;

$$D_n = \{x \in G - G^\diamond : \text{ there exists } m \in \mathcal{N} \text{ such that } f^m(x) = a_n,\ f^{m-1}(x) \notin G^\diamond\};$$

$$N_n = \{m \in \mathcal{N} : \text{ there exists } x \in D_n \text{ such that } f^m(x) = a_n,\ f^{m-1}(x) \notin G^\diamond\}.$$

Further let

$$N^{(M)} = \{n \in \{1, \dots, k\} : \ N_n \text{ has a maximal element}\};$$
$$N^{(E)} = \{n \in \{1, \dots, k\} : \ N_n = \emptyset\}.$$

We remark that $G^\diamond = \{a_1, \dots, a_k\}$ and sets $G^\diamond, D_1, \dots, D_k$ give a partition of G. Moreover $N^{(E)} \neq \{1, 2, \dots, k\}$ is satisfied.

LEMMA 6. *Let $N^{(M)} \cup N^{(E)} = \{1, \dots, k\}$. Then $\mathbf{L}[A] \cap \mathcal{T}_1 \neq \emptyset$.*

Proof. Put $r = \max\{\max N_n : \ n \in N^{(M)}\}$. Choose $R \subseteq G - G^\diamond$ such that R is a chain of length r. Let D be a subalgebra of A such that $D - R = A^\diamond$. In view of [7; Theorem 1.3], we have that D is a retract of A. Thus $D \in \mathbf{L}[A]$. □

LEMMA 7. *Let $n \in \{1, \dots, k\} - (N^{(M)} \cup N^{(E)})$ and D_n contain a chain of infinite length. Then $\mathbf{L}[A] \cap \mathcal{T}_1 \neq \emptyset$.*

Proof. Let $R \subseteq D_n$ be a chain of infinite length. Let D be a subalgebra of A such that $A^\diamond = D - R$. Then $D \in \mathcal{T}_1$. Moreover D is a retract of A and thus $D \in \mathbf{L}[A]$. □

LEMMA 8. *Let $n \in \{1, \dots, k\} - (N^{(M)} \cup N^{(E)})$ and D_n contain no chain of infinite length. Let $t \in \mathcal{N}$. Then there exists an algebra E_t such that*

a) $E_t \subseteq G^\diamond \cup D_n$,
b) *E_t is a retract of G,*
c) *$f^t(x) \notin G^\diamond$ for every $x \in E_t \cap S$.*

Proof. Recall that S is the set of all sources of A. Consider $T = \{x \in D_n \cap S : \ f^t(x) \notin G^\diamond\}$. We have $T \neq \emptyset$ by the assumption. Put $E_t = \{f^m(x) : m \in \mathcal{N}, \ x \in T\}$. □

COROLLARY 1. *Let $n \in \{1, \dots, k\} - (N^{(M)} \cup N^{(E)})$ and let the set D_n contain no infinite chain of A. Further, let $t \in \mathcal{N}$ and let E_{t+1} be the algebra from Lemma 8. Then*

(i) *$(A - G) + E_{t+1}$ is a retract of A.*
(ii) *There exists a mapping ε_t such that ε_t is a retract mapping of A corresponding to $(A - G) + E_{t+1}$ and $\varepsilon_t(D_n) \subseteq D_n$.*

Proof. The claim (i) follows from Lemmas 8 and 3. The claim (ii) follows from the construction of all homomorphisms between two monounary algebras, cf. [10]. □

LEMMA 9. *Let* $n \in \{1,\dots,k\} - \big(N^{(M)} \cup N^{(E)}\big)$ *and* D_n *contain no chain of infinite length. Then there exists an algebra* $D \in \mathbf{L}[A]$ *such that*

1. $D^\diamond \in \mathcal{T}$;
2. *D contains a chain of infinite length.*

P r o o f. Let $p \in \mathcal{N}$. Suppose that E_{p+1} is an algebra from the previous lemma and that ε_p is an endomorphism of A from the previous corollary (ii).

Assume that algebras A_p are pairwise disjoint and isomorphic to A for all $p \in \mathcal{N}$. Let $p \in \mathcal{N}$. Suppose that ψ_p is an isomorphism from A onto A_p. We put $\varphi_{pp} = \mathrm{id}_{A_p}$. If $p < q$, then we put

$$\varphi_{pq} = \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \dots \circ \varepsilon_{q-1} \circ \psi_q .$$

The family $\{\mathcal{N}, A_p, \varphi_{pq}\}$ is direct. Denote by D its direct limit.

If $u \in G^\diamond$, then $u \in (A-G) + E_{p+1}$ for all $p \in \mathcal{N}$ according to Lemma 8a). Thus $\varepsilon_p(u) = u$ by Corollary 1(ii). We obtain $\varphi_{pq}(\psi_p(u)) = \big(\psi_p \circ \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \dots \circ \varepsilon_{q-1} \circ \psi_q\big)(u) = \psi_q(u)$.

We have $D \in \mathbf{L}[A]$. Assumptions of Lemma 5 are satisfied and thus $D^\diamond \cong A^\diamond \in \mathcal{T}$.

Suppose that $p \in \mathcal{N}$ and $x \in \psi_p(D_n)$. We will show that there exist $q \in \mathcal{N}$ and $y \in \psi_q(D_n)$ such that $f(\overline{y}) = \overline{x}$. Then the proof will be ready.

Let $x \notin \psi_p(S)$. Then there exists $y \in A_p$ such that $f(y) = x$. Thus $f(\overline{y}) = \overline{x}$.

Let $x \in \psi_p(S)$. Consider $q \in \mathcal{N}$ such that $f^q(x) = \psi_p(a_n)$ and $f^{q-1}(x) \notin \psi_p(G^\diamond)$. Since $a_n \in G^\diamond$, we have

$$f^q\big(\varphi_{pq}(x)\big) = \varphi_{pq}\big(f^q(x)\big) = \varphi_{pq}\big(\psi_p(a_n)\big) = \psi_q(a_n) .$$

Thus $f^q\big(\psi_q^{-1}(\varphi_{pq}(x))\big) \in G^\diamond$. Further, $\psi_q^{-1}\big(\varphi_{pq}(x)\big) \in E_q$. That means $\psi_q^{-1}\big(\varphi_{pq}(x)\big) \notin S$ according to Lemma 8c). Let $z \in A$ be such that $f(z) = \psi_q^{-1}\big(\varphi_{pq}(x)\big)$. Corollary 1(ii) and the definition of φ_{pq} yield that $\psi_q^{-1}\big(\varphi_{pq}(x)\big) \in D_n$. Thus $z \in D_n$. Put $y = \psi_q(z)$. We have $y \in \psi_q(D_n)$ and $f(\overline{y}) = f\big(\overline{\psi_q(z)}\big) = \overline{\psi_q\big(f(z)\big)} = \overline{\varphi_{pq}(x)} = \overline{x}$. □

PROPOSITION 1. *If* $A \in \mathcal{U} - \mathcal{T}$ *and* $A^\diamond \in \mathcal{T}$, *then* $\mathbf{L}^2[A] \cap \mathcal{T}_1 \neq \emptyset$.

P r o o f. If either $N^{(M)} \cup N^{(E)} = \{1,\dots,k\}$ or there exists $n \in \{1,\dots,k\}$ such that D_n contains an infinite chain, then $\mathbf{L}[A] \cap \mathcal{T}_1 \neq \emptyset$ according to Lemmas 6 and 7. In the remaining case we take an algebra D from Lemma 9. This D satisfies all assumptions of Lemma 7 and thus $\mathbf{L}^2[A] \cap \mathcal{T}_1 \neq \emptyset$. □

3. Connected algebras without cycles

In this section suppose that A is a connected monounary algebra without a cycle and A is not isomorphic to Z.

We will prove that we can obtain from A an algebra of $\mathcal{T}_4$ or the algebra N via direct limits.

We will analyse three cases:

(1) the algebra A contains two distinct subalgebras isomorphic to Z;
(2) the algebra A contains exactly one subalgebra isomorphic to Z;
(3) the algebra A contains no subalgebra isomorphic to Z.

3.1. Case 1.

LEMMA 10. *Let A contain two distinct subalgebras isomorphic to Z. Then* $\mathbf{L}[A] \cap \mathcal{T}_4 \neq \emptyset$.

P r o o f. Let B, D be subalgebras of A, $B \neq D$, $B \cong Z$ and $D \cong Z$. Let E be the subalgebra of A which has underlying set $D \cup B$. Then $E \in \mathcal{T}_1$. The algebra E is a retract of A and thus $E \in \mathbf{L}[A]$. □

3.2. Case 2.

We suppose that A contains exactly one subalgebra isomorphic to Z. Let $B \cong Z$, $B = \{a_n : n \in \mathcal{Z},\ f(a_n) = a_{n+1}\}$.

For every $z \in \mathcal{Z}$ we put

$$D_z = \{x \in A - B : \text{ there exists } m \in \mathcal{N} \text{ such that } f^m(x) = a_z,\ f^{m-1}(x) \notin B\};$$
$$N_z = \{m \in \mathcal{N} : \text{ there exists } x \in D_z \text{ such that } f^m(x) = a_z\}.$$

Further, let

$$Z^{(M)} = \{z \in \mathcal{Z} : N_z \text{ has a maximal element}\};$$
$$Z^{(E)} = \{n \in \mathcal{Z} : N_z = \emptyset\}.$$

We remark that sets B and D_z for all $z \in \mathcal{Z}$ give a partition of the set A.

LEMMA 11. *Let $Z^{(M)} \neq \emptyset$. Then* $\mathbf{L}[A] \cap \mathcal{T}_4 \neq \emptyset$.

P r o o f. Let $n \in Z^{(M)}$. Suppose that R is a chain of A such that R contains $\max N_n$ elements of D_n. Then $E = R \cup B$ is a retract of A. Thus $E \in \mathbf{L}[A]$. Moreover $E \in \mathcal{T}_1$. □

LEMMA 12. *Let* $Z^{(M)} = \emptyset$. *Then* $\mathbf{L}^2[A] \cap \mathcal{T}_4 \neq \emptyset$.

Proof. Consider $n \notin Z^{(E)}$. Such n exists because A is not isomorphic to Z.

We will prove tree claims. The assertion follows from the third claim and Lemma 10.

CLAIM 1. *Let* $t \in \mathcal{N}$. *Then there exists an algebra* E_t *such that*

a) $E_t \subseteq D_n \cup B$,
b) E_t *is a retract of* A,
c) $f^t(x) \notin B$ *for every* $x \in E_t \cap S$.

Proof. Consider $T = \{x \in D_n \cap S : f^t(x) \notin B\}$. We have $T \neq \emptyset$ according to $n \notin Z^{(E)} \cup Z^{(M)}$. Put $E_t = \{f^m(x) : m \in \mathcal{N},\ x \in T\}$. □

CLAIM 2. *Let* $t \in \mathcal{N}$ *and* E_t *be an algebra from the previous claim. Then there exists a mapping* ε_t *such that* ε_t *is a retract mapping of* A *corresponding to* E_{t+1}, $\varepsilon_t(B) = B$ *and* $\varepsilon_t(D_n) \subseteq D_n$.

Proof. It follows from the construction of all homomorphisms between two monounary algebras, cf. [10]. □

CLAIM 3. *There exists an algebra* $D \in \mathbf{L}[A]$ *such that*

1. *D is a connected algebra;*
2. *D contains two distinct subalgebras isomorphic to Z.*

Proof. Let $p \in \mathcal{N}$. Suppose that E_{p+1} is an algebra from the first claim and that ε_p is a retract endomorphism of A corresponding to E_{p+1} from the second claim.

Assume that algebras A_p are pairwise disjoint and isomorphic to A for all $p \in \mathcal{N}$. Let $p \in \mathcal{N}$. Suppose that ψ_p is an isomorphism of A onto A_p. We put $\varphi_{pp} = \mathrm{id}_{A_p}$. If $p < q$, then we put

$$\varphi_{pq} = \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \cdots \circ \varepsilon_{q-1} \circ \psi_q .$$

The family $\{\mathcal{N}, A_p, \varphi_{pq}\}$ is direct. Denote by D its direct limit.

We have $D \in \mathbf{L}[A]$. In view of [3; Proposition 1] the algebra D is connected. In view of [3; Lemma 10] the algebra D has no cycle.

Let $p \in \mathcal{N}$ and $E = \{\overline{\psi_p(a_k)} : k \in \mathcal{Z}\}$.

For every $k \in \mathcal{Z}$ we have $f(\overline{\psi_p(a_{k-1})}) = \overline{\psi_p(f(a_{k-1}))} = \overline{\psi_p(a_k)}$. Thus E is a subalgebra of D isomorphic to Z.

Suppose that $x \in \psi_p(D_n)$. Then $\overline{x} \notin E$ according to Claim 2 and the definition of φ_{pq}. We will show that there exist $q \in \mathcal{N}$ and $y \in \psi_q(D_n)$ such that $f(\overline{y}) = \overline{x}$. Then the proof of this claim will finished.

Let $x \notin \psi_p(S)$. Then there exists $y \in A_p$ such that $f(y) = x$. Thus $f(\overline{y}) = \overline{x}$.

Let $x \in \psi_p(S)$. Consider $q \in \mathcal{N}$ such that $f^q(x) = \psi_p(a_n)$, $f^{q-1}(x) \notin \psi_p(B)$. Since $a_n \in E_t$ for every $p < t \leq q$, we have

$$f^q(\varphi_{pq}(x)) = \varphi_{pq}(f^q(x)) = \varphi_{pq}(\psi_p(a_n))$$
$$= (\psi_p \circ \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \cdots \circ \varepsilon_{q-1} \circ \psi_q)(a_n) = \psi_q(a_n).$$

Thus $f^q(\psi_q^{-1}(\varphi_{pq}(x))) = a_n \in B$. The definition of φ_{pq} yields $\psi_q^{-1}(\varphi_{pq}(x)) \in E_q$. That means $\psi_q^{-1}(\varphi_{pq}(x)) \notin S$ according to Claim 1c). Let $z \in A$ be such that $f(z) = \psi_q^{-1}(\varphi_{pq}(x))$. The Claim 2 and the definition of φ_{pq} imply that $\psi_q^{-1}(\varphi_{pq}(x)) \in D_n$. Thus $z \in D_n$. Put $y = \psi_q(z)$. We have $y \in \psi_q(D_n)$ and $f(\overline{y}) = f(\overline{\psi_q(z)}) = \overline{\psi_q(f(z))} = \overline{\varphi_{pq}(x)} = \overline{x}$. □

LEMMA 13. *Let A contain exactly one subalgebra isomorphic to Z. Then*

$$\mathbf{L}^2[A] \cap \mathcal{T}_4 \neq \emptyset.$$

P r o o f. It follows from Lemmas 11 and 12. □

3.3. Case 3.

In Lemmas 14 – 17 we will suppose that A contains no subalgebra isomorphic to Z. Then $S \neq \emptyset$.

NOTATION. Let $a \in S$. Put $B = \{f^n(a) : n \in \mathcal{N}_0\}$.

For $n \in \mathcal{N}$ let us denote

$$a_n = f^n(a);$$
$$D_n = \{x \in A - B : \text{ there exists } m \in \mathcal{N} \text{ such that } f^m(x) = a_n,\ f^{m-1}(x) \notin B\};$$
$$N_n = \{m \in \mathcal{N} : \text{ there exists } x \in D_n \text{ such that } f^m(x) = a_n\}.$$

Further, let

$$N^{(M)} = \{n \in \mathcal{N} : N_n \text{ has a maximal element}\}.$$

For $n \in N^{(M)}$ put $j_n = \max N_n$.

Denote

$$N^{(E)} = \{n \in \mathcal{N} : N_n = \emptyset\}$$

and

$$N^{(I)} = \mathcal{N} - (N^{(M)} \cup N^{(E)}).$$

We remark that B is a subalgebra of A. Sets B and D_n for all $n \in \mathcal{N}$ give a partition of the set A.

LEMMA 14. *Suppose that $N^{(M)} \cup N^{(E)} = \mathcal{N}$ and $\{j_n : n \in N^{(M)}\}$ has a maximum. Then the algebra $N \in \mathbf{L}[A]$.*

Proof. Denote $j = \max\{j_k : k \in N^{(M)}\}$. Suppose that $n \in N^{(M)}$ is such that $j_n = j$. Then there exists $x \in D_n$ such that $f^j(x) = a_n$, $f^{j-1}(x) \notin B$.

Let $j \geq n$. Put $D = \{f^m(x) : m \in \mathcal{N}_0\}$. The algebra D is a retract of A and D is isomorphic to N. Thus $N \in \mathbf{L}[A]$.

If $j < n$, then B is a retract of A. □

The proof of the following lemma will be similar to the proof of Lemma 9.

LEMMA 15. *Let $N^{(I)} \neq \emptyset$. Then $\mathbf{L}^3[A] \cap \mathcal{T}_4 \neq \emptyset$.*

Proof. Let n be the least number from $N^{(I)}$.

Since A does not contain a subalgebra isomorphic to Z, the set $D_n \cap S$ is infinite.

We will prove tree claims. The assertion follows from the third claim and Lemmas 10 and 13.

CLAIM 4. *Let $t \in \mathcal{N}$. Then there exists an algebra E_t such that*

a) *$E_t \subseteq D_n \cup \{a_k : k \geq n\}$,*
b) *E_t is a retract of A,*
c) *$f^t(x) \notin B$ for every $x \in E_t \cap S$.*

Proof. Consider $T = \{x \in D_n \cap S : f^t(x) \notin B\}$. We have $T \neq \emptyset$ according to $n \in N^{(I)}$. Put $E_t = \{f^m(x) : m \in \mathcal{N},\ x \in T\}$. □

CLAIM 5. *Let $t \in \mathcal{N}$ and E_t be an algebra from the previous claim. Let ε be a retract mapping corresponding to E_t. Then $\varepsilon(D_n) \subseteq D_n$.*

Proof. Suppose that $x \in D_n$. Then there exists $m \in \mathcal{N}$ such that $f^m(x) = a_n$. We have

$$f^m(\varepsilon(x)) = \varepsilon(f^m(x)) = \varepsilon(a_n) = a_n .$$

Therefore $\varepsilon(x) \in D_n$ according to a) in the previous claim. □

CLAIM 6. *There exists an algebra $D \in \mathbf{L}[A]$ such that*

1. *D is a connected algebra without a cycle;*
2. *D contains a subalgebra isomorphic to Z.*

Proof. Let $p \in \mathcal{N}$. Suppose that E_{p+1} is an algebra from the first claim and that ε_p is a retract endomorphism of A corresponding to E_{p+1}.

Assume that algebras A_p are pairwise disjoint and isomorphic to A for all $p \in \mathcal{N}$. Let $p \in \mathcal{N}$. Suppose that ψ_p is an isomorphism from A onto A_p. We put $\varphi_{pp} = \mathrm{id}_{A_p}$. If $p < q$, then we put

$$\varphi_{pq} = \psi_p^{-1} \circ \varepsilon_p \circ \varepsilon_{p+1} \circ \cdots \circ \varepsilon_{q-1} \circ \psi_q .$$

The family $\{\mathcal{N}, A_p, \varphi_{pq}\}$ is direct. Denote by D its direct limit.

We have $D \in \mathbf{L}[A]$. In view of [3; Proposition 1], the algebra D is connected. In view of [3; Lemma 10], the algebra A contains no cycle.

Suppose that $p \in \mathcal{N}$ and $x \in \psi_p(D_n)$. The proof that there exists $q \in \mathcal{N}$ and $y \in \psi_q(D_n)$ such that $f(\overline{y}) = \overline{x}$ is analogous to the end of the proof of Claim 3. □

In the next notation and in Lemmas 16 and 17 we suppose that $N^{(M)} \cup N^{(E)} = \mathcal{N}$ and the set $\{j_n : n \in N^{(M)}\}$ has no maximum.

NOTATION. We define a mapping u of $\mathcal{N}$ into $N^{(M)}$ by the following way: Let $u(1)$ be the least element of $N^{(M)}$. By induction for $i \in \mathcal{N}$ let $u(i+1)$ be the least number such that $u(i+1) \in N^{(M)}$, $u(i+1) > u(i)$ and $j_{u(i+1)} > j_{u(i)}$.

For $n \in \mathcal{N}$ let e_n be an element of $D_{u(n)}$ such that

$$f^{j_{u(n)}}(e_n) = a_{u(n)} .$$

Remark that $e_n \in S$.

Let $i \in \mathcal{N}$. Define the mapping ξ_i of A into A by the following way:

$$\xi_i(x) = \begin{cases} f^{u(i+1)-u(i)}(x) & \text{if } x \notin D_{u(i)}, \\ f^{j_{u(i+1)}-m}(e_{i+1}) & \text{if } x \in D_{u(i)} \text{ and } \\ & m \in \mathcal{N} \text{ is such that } f^m(x) = a_{u(i)} . \end{cases}$$

LEMMA 16. *Let $i \in \mathcal{N}$. Then the mapping ξ_i is an endomorphism of A such that*

(a) $\xi_i(D_{u(i)}) \subseteq D_{u(i+1)}$;

(b) $\xi_i(B) \subseteq B$.

Proof. Let $x \in A$.

If either $f(x) \in D_{u(i)}$ and $x \in D_{u(i)}$ or $x \notin D_{u(i)}$ and $f(x) \notin D_{u(i)}$, then it is easy to verify that $\xi_i(f(x)) = f(\xi_i(x))$. The case $x \notin D_{u(i)}$ and $f(x) \in D_{u(i)}$ cannot occur.

Suppose that $x \in D_{u(i)}$ and $f(x) \notin D_{u(i)}$. Then $f(x) = a_{u(i)} = f^{j_{u(i)}}(e_i)$. Thus we have $\xi_i(x) = f^{j_{u(i+1)}-1}(e_{i+1})$ and

$$\xi_i(f(x)) = f^{u(i+1)-u(i)}(f(x)) = f^{u(i+1)-u(i)}(a_{u(i)}) = f^{u(i+1)-u(i)}(f^{u(i)}(a))$$
$$= a_{u(i+1)} = f^{j_{u(i+1)}}(e_{i+1}) = f(f^{j_{u(i+1)}-1}(e_{i+1})) = f(\xi_i(x)) .$$

Assertions (a), (b) follow from the definition of ξ_i. □

LEMMA 17. *There exists an algebra $F \in \mathbf{L}[A]$ such that F is connected, F contains a subalgebra isomorphic to Z and the algebra F is not isomorphic to Z.*

Proof. For $i \in \mathcal{N}$ let A_i be pairwise disjoint algebras, which are isomorphic to A. Let ψ_i be an isomorphism of the algebra A onto A_i. Let $\varphi_{ii} = \mathrm{id}_{A_i}$ and for $i < j$ let

$$\varphi_{ij} = \psi_i^{-1} \circ \xi_i \circ \xi_{i+1} \circ \cdots \circ \xi_{j-1} \circ \psi_j .$$

Then $\{\mathcal{N}, A_i, \varphi_{ij}\}$ is a direct family of algebras. Denote by F its direct limit.

In view of [3; Proposition 1], the algebra F is connected.

According to [3; Lemma 10], the algebra F has no cycle.

Let $z \in F$. Choose $i \in \mathcal{N}$ and $y \in A_i$ such that $y \in z$. Then

$$\varphi_{i,i+1}(y) = \psi_{i+1}(\xi_i(\psi_i^{-1}(y))) .$$

If $\psi_i^{-1}(y) \notin D_{u(i)}$, then for

$$x = \psi_{i+1}\left(f^{u(i+1)-u(i)-1}(\psi_i^{-1}(y))\right)$$

we have $x \in A_{i+1}$ and

$$\begin{aligned} f(\overline{x}) &= f\left(\overline{\psi_{i+1}(f^{u(i+1)-u(i)-1}(\psi_i^{-1}(y)))}\right) \\ &= \overline{\psi_{i+1}(f^{u(i+1)-u(i)}(\psi_i^{-1}(y)))} = \overline{\varphi_{i,i+1}(y)} = z . \end{aligned}$$

If $\psi_i^{-1}(y) \in D_{u(i)}$ and $m \in \mathcal{N}$ is such that $f^m(\psi_i^{-1}(y)) = a_{u(i)}$, then for

$$x = \psi_{i+1}\left(f^{j_{u(i+1)}-m-1}(e_{i+1})\right)$$

we have $x \in A_{i+1}$ and

$$\begin{aligned} f(\overline{x}) &= f\left(\overline{\psi_{i+1}(f^{j_{u(i+1)}-m-1}(e_{i+1}))}\right) \\ &= \overline{\psi_{i+1}(f^{j_{u(i+1)}-m}(e_{i+1}))} = \overline{\psi_{i+1}(\xi_i(\psi_i^{-1}(y)))} = \overline{\varphi_{i,i+1}(y)} = z . \end{aligned}$$

We conclude that the algebra F contains a subalgebra isomorphic to Z.

Now we will prove that the operation of F is not injective.

Let $w = f^{j_{u(2)}-1}(e_2)$. Since $j_{u(2)} > 1$, we have $w \in D_{u(2)}$. Further,

$$f\left(\overline{\psi_2(w)}\right) = \overline{\psi_2(f^{j_{u(2)}}(e_2))} = \overline{\psi_2(a_{u(2)})} = \overline{\psi_2(f^{u(2)}(a))} = f\left(\overline{\psi_2(f^{u(2)-1}(a))}\right) .$$

Let $k \in \mathcal{N}$, $k > 2$. In view of Lemma 16(a) we have

$$\varphi_{2k}(\psi_2(w)) = (\xi_2 \circ \xi_3 \circ \cdots \circ \xi_{k-1} \circ \psi_k)(w) \in \psi_k(D_{u(k)}) .$$

In view of Lemma 16(b) we have

$$\varphi_{2k}(\psi_2(f^{u(2)-1}(a))) = (\xi_2 \circ \xi_3 \circ \cdots \circ \xi_{k-1} \circ \psi_k)(f^{u(2)-1}(a)) \in \psi_k(B) .$$

Since $B \cap D_{u(k)} = \emptyset$, we have $\varphi_{2k}(\psi_1(w)) = \varphi_{2k}(\psi_2(f^{u(2)-1}(a)))$. Thus

$$\overline{\psi_2(w)} \neq \overline{\psi_2(f^{u(2)-1}(a))} .$$

□

LEMMA 18. *Let A do not contain a subalgebra isomorphic to Z. Then*

$$\mathbf{L}^3[A] \cap (\mathcal{T}_4 \cup [N]) \neq \emptyset .$$

P r o o f . If A satisfies assumptions of Lemma 14 or Lemma 15, then $\mathbf{L}^3[A] \cap (\mathcal{T}_4 \cup [N]) \neq \emptyset$.

The remaining case is that $N^{(M)} \cup N^{(E)} = \mathcal{N}$ and the set $\{j_n : n \in N^{(M)}\}$ has no maximum. Then an algebra F from Lemma 17 satisfies either the assumptions of Lemma 10 or Lemma 13. That yields $\mathbf{L}^2[F] \cap \mathcal{T}_4 \neq \emptyset$. Thus $\mathbf{L}^3[A] \cap \mathcal{T}_4 \neq \emptyset$. □

We summarize the results of Lemmas 10, 13 and 18:

PROPOSITION 2. *If A is a connected monounary algebra without cycle and A is not isomorphic to Z, then*

$$\mathbf{L}^3[A] \cap (\mathcal{T}_4 \cup [N]) \neq \emptyset .$$

4. The main result

In this section we describe all monounary algebras A, B such that $\mathbf{L}[A,B] = [A,B]$.

Next two theorems show that from every monounary algebra A such that $A \notin \mathcal{T} \cup [Z]$ we can obtain an algebra of the class $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z+Z]$ via direct limits.

PROPOSITION 3. *Let A be a monounary algebra such that $A \notin \mathcal{T} \cup [Z]$. If every component of A has a cycle, then*

$$\mathbf{L}^2[A] \cap (\mathcal{T}_1 \cup \mathcal{T}_2) \neq \emptyset .$$

P r o o f . Let B be a subalgebra of A. We will suppose that $B^\diamond$ is a subalgebra of B.

If $A^\diamond \in \mathcal{T}$, then $\mathbf{L}^2[A] \cap \mathcal{T}_1 \neq \emptyset$ according to Proposition 1.

Assume that $A^\diamond \notin \mathcal{T}$. Let $\{B_j : j \in J\}$ be the set of all components of A. Then $\{B_j^\diamond : j \in J\}$ is the set of all components of $A^\diamond$. Let $k(j)$ be the length of the cycle $B_j^\diamond$ for every $j \in J$. There exists a subset I of the set J such that

(1) if $i, j \in I$, then $k(i)$ does not divide $k(j)$;

(2) if $j \in J - I$, then there exists $i \in I$ such that $k(i)$ divides $k(j)$.

Consider a set I with these properties.

Let E be an algebra which has the set of all components equal to $\{B_i : i \in I\}$. Then $E \in \mathcal{T}$. Consider $m \in J - I$. Put $D = E + B_m$. The algebra D is a retract of A and $D \in \mathcal{T}_2$. We have $\mathbf{L}[A] \cap \mathcal{T}_2 \neq \emptyset$.

PROPOSITION 4. *Let A be a monounary algebra such that $A \notin \mathcal{T} \cup [Z]$. If A contains a component without a cycle, then*

$$\mathbf{L}^3[A] \cap (\mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z+Z]) \neq \emptyset .$$

P r o o f . Let A contain a cycle. Then $A^\diamond$ possesses a cycle.

Consider an algebra T such that T is a retract of $A^\diamond$ and $T \in \mathcal{T}$. Such T exists in view of [3; Lemma 20]. Since A contains a component without a cycle, the algebra $A^\diamond$ contains a component D isomorphic to Z. Denote $E = T + D$. We have $E \in \mathcal{T}_3$ and E is a retract of $A^\diamond$. In view of Lemma 4 we obtain $\mathbf{L}^2[A] \cap \mathcal{T}_3 \neq \emptyset$.

Assume that A has no cycle. If A is not connected, then $Z+Z$ is isomorphic to a retract of $A^\diamond$. We have $Z+Z \in \mathbf{L}^2[A]$ according to Lemma 4. If A is connected, then the class $\mathbf{L}^3[A]$ contains an algebra from $\mathcal{T}_4 \cup [N]$ according to Proposition 2. □

For $A \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ we denote the algebra A^* by the following way:

If $A \in \mathcal{T}_1$ and R is a chain of A from the definition of $\mathcal{T}_1$, then we put $A^* = A - R$.

If $A \in \mathcal{T}_2$ and $B \in \mathcal{T}$ satisfies the conditions from the definition of $\mathcal{T}_2$, then we put $A^* = B$.

If $A \in \mathcal{T}_3$ and $B \in \mathcal{T}$ satisfies the conditions from the definition of $\mathcal{T}_3$, then we put $A^* = B$.

Let us remark that A^* is a retract of A. Thus $A^* \in \mathbf{L}[A]$. Further, $A^* \not\cong A$.

PROPOSITION 5. *If $A \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then $\mathbf{L}[A] = [A, A^*]$.*
If $A \in \mathcal{T}_4 \cup [Z+Z, N]$, then $\mathbf{L}[A] = [A, Z]$.

P r o o f . It is a consequence of Lemma 4 and [3; Theorems 1, 2, 3]. □

COROLLARY 2. *If $A \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then $\mathbf{L}[A, A^*] = [A, A^*]$. If $A \in \mathcal{T}_4 \cup [N, Z+Z]$, then $\mathbf{L}[A, Z] = [A, Z]$.*

P r o o f . Let $A \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Then $\mathbf{L}[A] = [A, A^*]$. Since $A^* \in \mathcal{T}$, we have $\mathbf{L}[A^*] = [A^*]$. Therefore $\mathbf{L}[A, A^*] = [A, A^*]$ by Lemma 1.

Now let $A \in \mathcal{T}_4 \cup [N, Z+Z]$. Then $\mathbf{L}[A] = [A, Z]$. In view of Lemma 1 we have $\mathbf{L}[A, Z] = [A, Z]$. □

In Theorem 6 we will use the following notation. If (p) is a condition for algebras A, B, then the symbol (p') denotes the condition, which arise from (p) in such a way that we change algebras A, B; further, the symbol (p*) denotes the condition which requires that either (p) or (p') is valid.

Consider conditions

(i) $A, B \in \mathcal{T}$;
(ii) $A \in \mathcal{T}$, $B \cong Z$;
(iii) $A \cong N$, $B \cong Z$;
(iv) $A \cong Z$, $B \cong Z + Z$;
(v) $A \cong Z$, $B \in \mathcal{T}_4$;
(vi) $A \in \mathcal{T}$, $B \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, $B^* \cong A$;
(vii) $A \cong Z$, $B \cong Z$.

THEOREM 1. *Let $A, B \in \mathcal{U}$. Then*

$$\mathbf{L}[A, B] = [A, B]$$

if and only if one of conditions (i), (ii*), (iii*), (iv*), (v*), (vi*), (vii) *is valid.*

P r o o f. If one of (i), (ii*), (vii) holds, then [3; Theorem 1] implies that $\mathbf{L}[A] = [A]$, $\mathbf{L}[B] = [B]$. Thus $\mathbf{L}[A, B] = [A, B]$ according to [3; Lemma 15].

If one of conditions (iii*) (vi*) holds, then $\mathbf{L}[A, B] = [A, B]$ according to Corollary 2.

Suppose that none of conditions (i), (ii*) (vi*), (vii) holds for A, B.

We have $[A, B] \subseteq \mathbf{L}[A, B] \subseteq \mathbf{L}^2[A, B] \subseteq \mathbf{L}^3[A, B]$ according [3; Lemma 12]. We will prove that the class $\mathbf{L}^3[A, B]$ contains an algebra which does not belong to $[A, B]$. Then $\mathbf{L}[A, B] \neq [A, B]$ will be proved.

We will discuss the following cases:

(1) $A \in \mathcal{T}$;
(2) $A \cong Z$;
(3) $A \notin \mathcal{T} \cup [Z]$.

(1) In view of invalidity of (i) and (ii) we have $B \notin \mathcal{T} \cup [Z]$. Propositions 3 and 4 imply that the class $\mathbf{L}^3[B]$ contains an algebra D such that

$$D \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z].$$

If D is not isomorphic to B, then $D \in \mathbf{L}^3[A, B] - [A, B]$. Thus D has the required property.

Let $B \cong D$. Then $B \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z]$.

Assume that there exists $i \in \{1, 2, 3\}$ such that $B \in \mathcal{T}_i$. Thus $B^* \in \mathcal{T}$ and $B^* \ncong B$. Since the condition (vi) does not hold, the algebra B^* is not isomorphic to A. We have $B^* \in \mathbf{L}[B] \subseteq \mathbf{L}[A, B]$. We conclude that B^* has the required property.

If either $B \cong N$ or $B \cong Z + Z$ or $B \in \mathcal{T}_4$, then $Z \in \mathbf{L}[B] \subseteq \mathbf{L}[A, B]$ according to Lemma 4 and [4; Theorems 2, 3]. We have that Z has the required property.

(2) Since (ii') fails to hold, we have $B \notin \mathcal{T}$. In view of the fact that (vii) is not valid, we obtain $B \notin [Z]$. Thus the algebra B satisfies assumptions of either Proposition 3 or Proposition 4. Therefore the class $\mathbf{L}^3[B]$ contains an algebra D such that

$$D \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z].$$

If $D \ncong B$, then the proof can be finished analogously as in the case (1).

Let $D \cong B$. Since (iii'), (iv), (v) do not hold, we have $B \notin \mathcal{T}_4 \cup [N, Z + Z]$. That means that there exists $i \in \{1, 2, 3\}$ such that $B \in \mathcal{T}_i$. The algebra $B^* \notin [A, B]$ and $B^* \in \mathbf{L}[B] \subseteq \mathbf{L}[A, B]$. Thus B^* has the required property.

(3) The algebra A satisfies assumptions either Proposition 3 or Proposition 4. Therefore $\mathbf{L}^3[A]$ contains an algebra D such that

$$D \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [N, Z + Z].$$

If $D \ncong A$ and $D \ncong B$, then D has the required property.

Let $D \ncong A$ and $D \cong B$. If $B \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then B^* has the required property. If $B \in \mathcal{T}_4 \cup [Z]$, then Z has the required property.

Let $A \cong D$. If $A \in \mathcal{T}_i$ for $i = 1, 2, 3$, then A^* is not isomorphic to B, because (vi) does not hold. Further, $A^* \ncong A$ and $A^* \in \mathbf{L}[A]$. Thus A^* has the required property.

If $A \in \mathcal{T}_4 \cup [Z + Z]$, then $Z \in \mathbf{L}[A]$, because Z is a retract of A. If $A \cong N$, then $Z \in \mathbf{L}[A]$ according to Lemma 4. The algebra B is not isomorphic to Z since no condition (iii), (iv'), or (v') is satisfied. □

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Received November 23, 2000
Revised August 28, 2001

Matematický ústav SAV
Grešákova 6
SK-040 01 Košice
SLOVAKIA
E-mail: ehaluska@mail.saske.sk