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OSCILLATORY PROPERTIES OF SOLUTIONS OF SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

PAVEL ŠOLTÉS

Oscillatory properties of solutions of second order differential equations with argument delay are investigated by numerous authors. As a rule the methods applicable to ordinary differential equations are generalized and used to find criteria of oscillatoriness and non-oscillatoriness of solutions of differential equations with argument delay as in [1], [2], [4] and [5], where properties of solutions of a differential equation

$$y''(t)+p(t)y(\varrho(t))=0 \tag{1}$$

with $p(t)\geqq 0$, $\varrho(t)\leqq t$, $\varrho(t)\to\infty$ for $t\to\infty$ are investigated.

The first part of the present paper deals with generalizations of certain results of [3], where equation (1) is investigated for $p(t)<0$; the second part deals with generalizations of certain results of [1].

I.

Consider the nonlinear differential equation

$$y''(t)+p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t)))=0, \tag{2}$$

where

1. $p(t)<0$ and continuous for every $t\geqq t_0$;
2. $f(x)$ is continuous and nondecreasing in $x\in(-\infty,\infty)$, $xf(x)>0$ for $x\neq 0$;
3. $h(y)>0$ and continuous for every $y\in(-\infty,\infty)$;
4. for every $t\geqq t_0$ $\varrho_i(t)$ is continuous, $\varrho_i(t)\to\infty$ for $t\to\infty$, $i=1,2$, $\varrho_1(t)<t$, $\varrho_2(t)\leqq t$.

We restrict our consideration to those solutions $y(t)$ of (2) which exist on some ray $\langle t_0,\infty)$ and satisfy

$$\sup\{|y(s)|: t\leqq s<\infty\}>0$$

for any $t \in \langle t_0, \infty)$. Such a solution is said to be oscillatory if the set of zeros of $y(t)$ is not bounded from the right. Otherwise the solution $y(t)$ is said to be nonoscillatory.

Then we have

Theorem 1. *Suppose that*

$$\int^{\infty} tp(t)\,\mathrm{d}t = -\infty. \tag{3}$$

Then for every nonoscillatory solution $y(t)$ of (2) either

$$|y(t)| \to +\infty \quad \text{for} \quad t \to \infty$$

or

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0.$$

Proof. Let $y(t)$ be a nonoscillatory solution of (2). Then there exists $t_1 \geqq t_0$ such that $y(t) \neq 0$ for all $t \geqq t_1$. Then

$$y''(t) = -p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t))) > 0$$

for every $t \geqq t_2 \geqq t_1$ where $t_2 > 0$ is such that $y(\varrho_1(t)) > 0$ for $t \geqq t_2$. Evidently $y'(t)$ is increasing; we have to investigate the following cases:

1° $y'(t) < 0$ for every $t \geqq t_2$.

2° There exists $t_3 \geqq t_2$ such that for $t \geqq t_3$ $y'(t) > 0$.

If case 2° obtains, then for $t \geqq t_3$ $y'(t) \geqq y'(t_3)$, which means that

$$y(t) \geqq y(t_3) + y'(t_3)(t - t_3)$$

and therefore $\lim_{t\to\infty} y(t) = +\infty$.

Suppose that 1° obtains. Define

$$y(\infty) = \lim_{t\to\infty} y(t), \quad y'(\infty) = \lim_{t\to\infty} y'(t).$$

Evidently $y(\infty) \geqq 0$, $y'(\infty) \leqq 0$. Suppose that $y(\infty) > 0$. Then (2) yields

$$\begin{aligned} ty'(t) - y(t) &= t_2 y'(t_2) - y(t_2) - \int_{t_2}^{t} sp(s)f(y(\varrho_1(s)))h(y'(\varrho_2(s)))\,\mathrm{d}s \geqq \\ &\geqq k_0 - f(y(\infty))\int_{t_2}^{t} sp(s)h(y'(\varrho_2(s)))\,\mathrm{d}s, \end{aligned} \tag{4}$$

where $k_0 = t_2 y'(t_2) - y(t_2)$.

Since $h(y)$ is continuous, there exists $\alpha \in \langle y'(\varrho_2(t_2)), 0\rangle$ such that for $t \geqq t_2$

$$h(\alpha) \leqq h(y'(\varrho_2(t)))$$

and therefore (4) implies

$$ty'(t) \geqq k_0 - h(\alpha)f(y(\infty))\int_{t_2}^{t} sp(s)\,ds \to +\infty$$

for $t \to \infty$, which contradicts the fact that $y'(t) < 0$ for all $t \geqq t_2$. Therefore $y(\infty) = 0$.

Let $y'(\infty) < 0$. Since $y''(t) > 0$, we have $y'(t) < y'(\infty)$ so that for $t \geqq t_2$

$$y(t) < y(t_2) + y'(\infty)(t - t_2),$$

which again contradicts the fact that $y(t) > 0$.

Assuming that $y(t) < 0$ the proof is analogous.

Remark 1. For $f(x) \equiv x$, $\varrho(t) \equiv t - \tau$ with $\tau > 0$ a constant, $h(x) \equiv 1$ we obtain — as a special case of Theorem 1 — Lemma 2.1 of [3].

In the sequel we shall assume that in addition

α) $f(x)$ and $h(y)$ are differentiable and for each x, $y \in (-\infty, \infty)$ $f'(x) \geqq 0$, $h'(y)y \geqq 0$;

β) for each $t \geqq t_0$ and $i = 1, 2$ $\varrho_i(t)$ is differentiable and $\varrho_i'(t) \geqq 0$.

Then we have

Theorem 2. *Let $p(t)$ be differentiable, $p'(t) \geqq 0$ for $t \geqq t_0$ and let there exist a number $k > 0$ such that*

$$\lim_{y \to 0} \frac{f(y)}{y} > k. \tag{5}$$

If for every $t \geqq t_0$

$$p(t) \leqq -\frac{2}{kh(0)(\varrho_1(t) - t)^2}, \tag{6}$$

then any bounded solution $y(t)$ of (2) is oscillatory.

Proof. Let $y(t)$ be an arbitrary bounded and nonoscillatory solution of (2). From (6) we get

$$tp(t) \leqq -\frac{2}{kh(0)}\frac{t}{(\varrho_1(t) - t)^2} \leqq -\frac{2}{kh(0)}\frac{1}{t},$$

and therefore, from the proof of Theorem 1 we have

$$y(t)y'(t) < 0$$

for $t \geqq t_2 \geqq t_0$, $y(\infty) = y'(\infty) = 0$.

Suppose, e.g., that $y(\varrho_1(t)) > 0$ for $t \geqq t_2$. Then by the proof of Theorem 1 we have: $y'(t) < 0$, $y''(t) > 0$ for $t \geqq t_2$.

Consider the function $y'(s)$, $s \geqq t_2$. Since $y''(s) > 0$ and

$$\begin{aligned}(y'(s))'' = &-p'(s)f(y(\varrho_1(s)))h(y'(\varrho_2(s))) - \\ &- p(s)f'(y(\varrho_1(s)))h(y'(\varrho_2(s)))y'(\varrho_1(s))\varrho_1'(s) - \\ &- p(s)f(y(\varrho_1(s)))h'(y'(\varrho_2(s)))y''(\varrho_2(s))\varrho_2'(s) < 0\end{aligned}$$

for $s \geqq t_2$, $y'(s)$ is concave. If we construct a tangent to the curve $y'(s)$ at an arbitrary point $(t, y'(t))$ with $t \geqq t_2$, we have

$$y'(t)-p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t)))(s-t) \geqq y'(s) \quad \text{for} \quad s, t \geqq t_2$$

and integrating with respect to s from $\varrho_1(t)$ to $t > \varrho_1(t)$ we obtain

$$\begin{aligned}(t-\varrho_1(t))y'(t)+p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t)))\frac{(\varrho_1(t)-t)^2}{2} \geqq \\ \geqq y(t)-y(\varrho_1(t)).\end{aligned} \tag{7}$$

Since $y'(\varrho_2(t))<0$ and $h'(y)y \geqq 0$, $h'(y'(\varrho_2(t)) \leqq 0$, which means that

$$h(y'(\varrho_2(t)) \geqq h(0).$$

Using (5) we see that there exists $t_3 \geqq t_2$ such that for every $t \geqq t_3$

$$\frac{f(y(\varrho_1(t)))}{y(\varrho_1(t))} \geqq k$$

and therefore from (7) we have

$$(t-\varrho_1(t))y'(t)+\left[\frac{1}{2}kh(0)p(t)(\varrho_1(t)-t)^2+1\right]y(\varrho_1(t)) \geqq y(t).$$

This leads to a contradiction with assumption (6).

If we assume that $y(t)<0$ is a bounded solution of (2), the proof is analogous.

Remark 2. For $f(x) \equiv x$, $h(y) \equiv 1$, $\varrho_1(t)=t-\tau$, $\tau>0$ a constant, we obtain Theorem 2.1 or 2.2 of [3] as a special case.

Corollary. *Suppose that $\varrho(t)$ is continuous on $\langle t_0, \infty)$, $\varrho(t)<t$, $\lim_{t\to\infty} \varrho(t)=\infty$. It is a consequence of Theorem 2 that if for every $t \geqq t_0 \geqq 0$*

$$p(t) \leqq -\frac{2}{(\varrho(t)-t)^2}, \quad p' \geqq 0,$$

then every bounded solution $y(t)$ of the equation

$$y''(t)+p(t)y^{\alpha}(\varrho(t))=0,$$

with $\alpha=\frac{n}{m}$, where n, m are odd natural numbers and $\alpha \in (0, 1\rangle$, is oscillatory.

Remark 3. The condition (6) for the oscillatoriness of a bounded solution $y(t)$ of (2) is necessary. For example the equation

$$y''(t)-\frac{3}{\sqrt[5]{2}}t^{-(14/5)}y^{3/5}\left(\frac{1}{2}t\right)=0$$

does not satisfy (6) and has a nonoscillatory bounded solution $y(t)=t^{-2}$.

II.

The next part of the present paper is concerned with the investigation of oscillatoriness of the differential equation

$$(r(t)y'(t))' + p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t))) = 0, \tag{8}$$

where $h(y)$ satisfies assumption 3 from part **I.** Moreover, suppose that
a) $p(t) \geqq 0$, $r(t) > 0$ and continuous for all $t \geqq t_0$;
b) $f(x)$ is continuous and $f(x)x > 0$ for each $x \neq 0$;
c) $\varrho_i(t)$ is continuous for every $t \geqq t_0$, $\varrho_i(t) \to \infty$ for $t \to \infty$, $\varrho_i(t) \leqq t$, $i = 1, 2$.
Then we have

Theorem 3. *Let $f(x)$ be nondecreasing on $(-\infty, \infty)$ and $h(y)$ nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$. If*

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} = \int^{\infty} p(s)\, \mathrm{d}s = +\infty, \tag{9}$$

then any solution $y(t)$ of (8) is oscillatory.

Proof. Suppose that (8) has a nonoscillatory solution, e.g. that $y(t) > 0$, $y(\varrho_1(t)) > 0$ for all $t \geqq t_1 \geqq t_0$. From equation (8) we get

$$(r(t)y'(t))' \leqq 0 \tag{10}$$

and therefore one of the following two cases must hold:
1° There exists $t_2 \geqq t_1$ such that $y'(t_2) < 0$.
2° For each $t \geqq t_1$ $y'(t) \geqq 0$.
If 1° holds, then $y'(t) < 0$ for each $t \geqq t_2$ and relation (10) yields

$$r(t)y'(t) \leqq r(t_2)y'(t_2) < 0.$$

Using (9) we see that $y(t) \to -\infty$ for $t \to \infty$, which contradicts the positivity of $y(t)$ for $t \geqq t_2$.

2° Let $y(t) > 0$, $y'(t) \geqq 0$ and let t_1 be such that $y'(\varrho_2(t)) \geqq 0$ for each $t \geqq t_1$. Considering the hypotheses of the theorem, equation (8) yields

$$(r(t)y'(t))' + f(y(\varrho_1(t_1)))h(0)p(t) \leqq 0$$

and therefore $r(t)y'(t) \to -\infty$ for $t \to \infty$, which is again a contradiction. The proof that equation (8) has no nonoscillatory solution $y(t) < 0$ is analogous.

Theorem 4. *Suppose that for each $t \geqq t_0$ $r(t)$ is differentiable, $r'(t) \geqq 0$ and $f(x)$ is nondecreasing on $(-\infty, \infty)$. If (9) holds, then any solution $y(t)$ of (8) is oscillatory.*

Proof. Suppose that there exists a nonoscillatory solution $y(t)$ and such that $y(t) > 0$, $y(\varrho_1(t)) > 0$ for $t \geqq t_1 \geqq 0$. Analogously as for Theorem 3 we show that $y'(t) \geqq 0$, $y'(\varrho_2(t)) \geqq 0$ for $t \geqq t_1$. However, from equation (8) we see that $y''(t) \leqq 0$

so that $y'(t)$ is bounded and there exists a number $\alpha \in \langle 0, y'(\varrho_2(t_1))\rangle$ such that for $t \geqq t_1$

$$h(\alpha) \leqq h(y'(\varrho_2(t))).$$

Since for all $t \geqq t_1$ $y(\varrho_1(t)) \geqq y(\varrho_1(t_1))$, equation (8) yields

$$(r(t)y'(t))' + h(\alpha)f(y(\varrho_1(t_1)))p(t) \leqq 0,$$

which leads to a contradiction.

The proof is analogous if we assume the existence of $y(t) < 0$.

Remark 4. Evidently if $r(t) \equiv 1$, $f(x) \equiv x$, $h(y) \equiv 1$, $\varrho_1(t) = t - \tau(t)$ with $0 \leqq \tau(t) \leqq m$ in equation (8), Theorem 1 of [1] is a consequence of Theorems 3 and 4.

Theorem 5. *Suppose that for every* $t \geqq t_0$

$$p(t) \geqq p_0 > 0$$

where p_0 *is a constant, and that moreover* $r(t)$ *is differentiable and* $\varrho_1(t)$ *twice differentiable; suppose further that for every* $t \geqq t_0$

$$r'(t) \geqq 0, \quad \varrho_1'(t) \geqq 0, \quad (r(t)\varrho_1'(t))' \leqq 0.$$

If (9) *holds and*

$$\lim_{|y| \to \infty} F(y) = \lim_{|y| \to \infty} \int_0^y f(s)\,\mathrm{d}s = +\infty, \tag{11}$$

then any solution $y(t)$ *of* (8) *is oscillatory.*

Proof. Suppose that $y(t) > 0$, $y(\varrho_1(t)) > 0$ for every $t \geqq t_1 \geqq t_0$. Then $y'(t) \geqq 0$, $y''(t) \leqq 0$ for every $t \geqq t_1$. Suppose that t_1 is such that besides this $y'(\varrho_2(t)) \geqq 0$, $y'(\varrho_1(t)) \geqq 0$ for $t \geqq t_1$. Multiplying (8) by $y'(\varrho_1(t))\varrho_1'(t)$, we obtain after some rearrangements

$$r(t)y''(t)y'(\varrho_1(t))\varrho_1'(t) + h(\alpha)p(t)\frac{\mathrm{d}}{\mathrm{d}t}F(y(\varrho_1(t))) \leqq 0, \tag{12}$$

or

$$y'(\varrho_1(t_1))r(t)\varrho_1'(t)y''(t) + h(\alpha)p_0\frac{\mathrm{d}}{\mathrm{d}t}F(y(\varrho_1(t))) \leqq 0.$$

Integrating this from t_1 to $t \geqq t_1$ we obtain

$$\begin{aligned} &y'(\varrho_1(t_1))r(t)\varrho_1'(t)y'(t) - \\ &-y'(\varrho_1(t_1))\int_{t_1}^{t}[r(s)\varrho_1'(s)]'y'(s)\,\mathrm{d}s + h(\alpha)p_0F(y(\varrho_1(t))) \lesseqgtr \\ &\leqq y'(\varrho_1(t_1))r(t_1)\varrho_1'(t_1)y'(t_1) + h(\alpha)p_0F(y\varrho_1(t_1))) = K_0, \end{aligned} \tag{13}$$

where $\alpha \in \langle 0, y'(\varrho_2(t_1))\rangle$ is a number such that for every $t \geqq t_1$

$$h(\alpha) \leqq h(y'(\varrho_2(t))).$$

From (13) we see that

$$F(y(\varrho_1(t)))\leqq\frac{K_0}{h(\alpha)p_0}$$

for every $t\geqq t_1$. From the last inequality and (11) we get that $y(t)$ is bounded on $\langle t_1, \infty)$.

Suppose now that $\beta\in\langle y(t_1), \lim_{t\to\infty} y(t)\rangle$ is a number such that for every $t\geqq t_1$

$$f(y(\varrho_1(t)))\geqq f(\beta).$$

Then (8) yields

$$(r(t)y'(t))'+f(\beta)h(\alpha)p(t)\leqq 0$$

and therefore $r(t)y'(t)\to-\infty$ as $t\to\infty$, which contradicts the positivity of $y'(t)$ for $t\geqq t_1$.

For $y(t)<0$ the proof is analogous.

Theorem 6. *The hypotheses of this theorem are the same as those for Theorem 5 except that instead of*

$$r'(t)\varrho_1'(t)+r(t)\varrho_1''(t)\leqq 0$$

we suppose that $0<r(t)\leqq r_0$, r_0 — *const. and* $\varrho_1'(t)$ *is nonincreasing for* $t\geqq 0$. *Then any solution* $y(t)$ *of* (8) *is oscillatory.*

Proof. The theorem is proved analogously as for Theorem 5. From (12) we get

$$r_0\varrho_1'(t_1)y'(\varrho_1(t_1))[y'(t)-y'(t_1)]+p_0h(\alpha)F(y(\varrho_1(t)))\leqq$$
$$\leqq p_0h(\alpha)F(y(\varrho_1(t_1))),$$

from which the boundedness of $y(t)$ can be proved.

Theorem 7. *Suppose that for every* $t\geqq t_0$ $r(t)\geqq r_0>0$, *where* r_0 – *const. and that* $f(x)$ *is nondecreasing on* $(-\infty, \infty)$. *If* (9) *holds, then every solution of* (8) *is oscillatory.*

Proof. Suppose that (8) has a nonoscillatory solution $y(t)$, e.g. that $y(t)>0$, $y(\varrho_1(t))>0$ for all $t\geqq t_1\geqq t_0$. Analogously as for Theorem 3 we show that $y'(t)\geqq 0$, $y'(\varrho_2(t))\geqq 0$ for $t\geqq t_1$. Integrating (10) from t_1 to $t\geqq t_1$ we get

$$r(t)y'(t)\leqq r(t_1)y'(t_1)$$

and therefore for each $t\geqq t_1$

$$0\leqq y'(t)\leqq\frac{r(t_1)y'(t_1)}{r(t)}\leqq\frac{r(t_1)y'(t_1)}{r_0}$$

so that $y'(t)$ is bounded on $\langle t_1, \infty)$.

The rest of the proof is analogous to the proof of Theorem 4.

Theorem 8. *Suppose that for every* $t \geqq t_0$ $r(t) \geqq r_0 > 0$, *where* r_0 *is a constant and that for every* $\delta > 0$

$$\inf_{\delta \leq |x| < \infty} \frac{f(x)}{x} > 0.$$

If (9) *holds, then every solution* $y(t)$ *of* (8) *is oscillatory.*

Proof. Evidently if (8) has a nonoscillatory solution, e.g. that $y(t) > 0$, then there exists $t_1 \geqq t_0$ such that

$$y(t) > 0, \quad y(\varrho_1(t)) > 0, \quad 0 \leqq y'(t) \leqq K < \infty.$$

Then (8) yields

$$(r(t)y'(t))' + h(\alpha)p(t)f(y(\varrho_1(t))) \leqq 0$$

and therefore

$$\frac{(r(t)y'(t))'}{y(\varrho_1(t))} + h(\alpha)p(t)\frac{f(y(\varrho_1(t)))}{y(\varrho_1(t))} \leqq 0. \tag{14}$$

Since for $t \geqq t_1$ $(r(t)y'(t))' \leqq 0$, we have from (14) that

$$\frac{1}{y(\varrho_1(t_1))}(r(t)y'(t))' + h(\alpha)p(t) \inf_{y(\varrho_1(t_1)) \leqq x < \infty} \frac{f(x)}{x} \leqq 0.$$

Integrating this from t_1 to $t \geqq t_1$, we prove that $r(t)y'(t) \to -\infty$ as $t \to \infty$, which contradicts the positivity of $y'(t)$ for $t \geqq t_1$.

The proof is analogous if we assume the existence of a nonoscillatory solution $y(t)$ of (8) such that $y(t) < 0$.

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ОСЦИЛЛЯЦИОННЫЕ СВОЙСТВА РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 2-ОГО ПОРIДКА С ЗАПАЗДЫВАНИЕМ

Павел Шолтес

Резюме

В работе приведены достаточные условия для осцилляции решений дифференциального уравнения с запаздыванием вида

$$(r(t)y'(t))' + p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t))) = 0.$$