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BOUNDDED DUALLY RESIDUATED LATTICE ORDERED MONOIDS AS A GENERALIZATION OF FUZZY STRUCTURES

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ABSTRACT. Dually residuated lattice ordered monoids (DRℓ-monoids) form a large class that contains among others all lattice ordered groups, fuzzy structures which need not be commutative, for instance, pseudo BL-algebras and GMV-algebras (= pseudo MV-algebras) and Brouwerian algebras. In the paper, two concepts of negation in bounded DRℓ-monoids are introduced and their properties are studied in general as well as in the case of the so-called good DRℓ-monoids. The sets of regular and dense elements of good DRℓ-monoids are described.

1. Introduction

Commutative dually residuated lattice ordered monoids (briefly: DRℓ-monoids) were introduced by Swamy in [18] as a common generalization of abelian lattice ordered groups and Brouwerian algebras. Moreover, the classes of MV-algebras and BL-algebras, i.e. algebraic counterparts of Łukasiewicz infinite valued and Hájek’s basic fuzzy logic introduced in [1] and [9], respectively, can be viewed as proper subclasses of the class of bounded commutative DRℓ-monoids. (In fact, we use the duals of BL-algebras.)

General DRℓ-monoids (i.e., the commutativity of the addition is not required) were introduced by Kovář in [11]. GMV-algebras introduced in [15] and, equivalently, pseudo MV-algebras introduced in [8] are non-commutative generalizations of MV-algebras. Further, pseudo BL-algebras introduced and studied in [4] and [5] and BL-algebras are in the same connection. By [16], GMV-algebras are an algebraic counterpart of a non-commutative logic between...
the Łukasiewicz logic and the bilinear logic (see [14]). Pseudo BL-algebras are by [10] an algebraic counterpart of Hájek’s pseudo basic logic. Analogously as in the commutative case, it was shown in [15] and [12] that GMV-algebras and duals of pseudo BL-algebras form proper subclasses of the class of bounded \( DR\ell \)-monoids.

In this paper we study bounded \( DR\ell \)-monoids as natural generalizations of GMV-algebras and pseudo BL-algebras introducing two, in general different, concepts of negation. All obtained results are applicable in the case of pseudo BL-algebras (and, consequently, of GMV-algebras). The particular case of negations in commutative \( DR\ell \)-monoids were studied in [17].

The basic concepts and results concerning \( MV \)-algebras, GMV-algebras, BL-algebras and pseudo BL-algebras can be found in [2], [6], [9] and [4], respectively.

2. Negations in bounded \( DR\ell \)-monoids

In this section we introduce notions of negations of elements in bounded \( DR\ell \)-monoids as generalizations of those in pseudo BL-algebras.

Firstly, let us recall the definition of a \( DR\ell \)-monoid.

**DEFINITION.** A dually residuated lattice ordered monoid (briefly: \( DR\ell \)-monoid) is an algebra \( M = (M; +, 0, \vee, \wedge, \rightarrow, \leftarrow) \) of signature \( \langle 2, 0, 2, 2, 2, 2 \rangle \) satisfying the following conditions:

\( M1 \) \((M; +, 0, \vee, \wedge)\) is a lattice ordered monoid, that means, \((M, +, 0)\) is a monoid, \((M, \vee, \wedge)\) is a lattice, and the operation + distributes from the left and from the right over the operations \( \vee \) and \( \wedge \).

\( M2 \) If \( \leq \) denotes the order on \( M \) induced by the lattice \((M, \vee, \wedge)\), then \( x \rightarrow y \) is the smallest \( s \in M \) such that \( s + y \geq x \) and \( x \leftarrow y \) is the smallest \( t \in M \) such that \( y + t \geq x \) for any \( x, y \in M \).

\( M3 \) \( M \) satisfies the identities

\[
((x \rightarrow y) \vee 0) + y \leq x \vee y, \quad y + ((x \leftarrow y) \vee 0) \leq x \vee y, \\
x \rightarrow x \geq 0, \quad x \leftarrow x \geq 0.
\]

In the paper, we will deal with bounded \( DR\ell \)-monoids. The least element in such a \( DR\ell \)-monoid is by [11] always 0. The greatest element will be denoted by 1 and bounded \( DR\ell \)-monoids will be considered as algebras \( M = (M, +, 0, 1, \vee, \wedge, \rightarrow, \leftarrow) \) of extended type \( \langle 2, 0, 0, 2, 2, 2 \rangle \).

When doing calculations, we use the following list of basic rules for bounded \( DR\ell \)-monoids.
LEMMA 1. ([11], [13]) In any bounded DRl-monoid $M$ we have for any $x, y, z \in M$:

1. $x \lor y = (x \rightarrow y) + y = y + (x \leftarrow y)$;
2. $x \rightarrow x = 0 = x \leftarrow x$, $x \rightarrow 0 = x = x \leftarrow 0$;
3. $x \leq y \implies x \rightarrow z \leq y \rightarrow z$, $x \leftarrow z \leq y \leftarrow z$;
4. $x \leq y \implies z \rightarrow x \geq z \rightarrow y$, $z \leftarrow x \geq z \leftarrow y$;
5. $x \rightarrow (y + z) = (x \rightarrow z) \rightarrow y$;
6. $x \leftarrow (y + z) = (x \leftarrow y) \leftarrow z$;
7. $x \rightarrow y \geq (z \rightarrow y) \leftarrow (z \rightarrow x)$;
8. $x \rightarrow y \geq (z \rightarrow y) \rightarrow (z \rightarrow x)$;
9. $x \leq y \iff x \rightarrow y = 0 \iff x \leftarrow y = 0$;
10. $x \rightarrow (y \land z) = (x \rightarrow y) \lor (x \rightarrow z)$, $x \leftarrow (y \land z) = (x \leftarrow y) \land (x \leftarrow z)$;
11. $x \rightarrow (y \leftarrow z) \leq (x \rightarrow y) + z$, $x \leftarrow (y \leftarrow z) \leq z + (x \leftarrow y)$;
12. $x \geq y \geq z \implies x \rightarrow z = (x \rightarrow y) + (y \rightarrow z)$, $x \leftarrow z = (y \leftarrow z) + (x \leftarrow y)$.

**DEFINITION.** Let $M = (M;+,0,1,\lor,\land,\rightarrow,\leftarrow)$ be a bounded DRl-monoid. For any $x \in M$ we set

$$-ix := 1 \leftarrow x, \quad \sim x := 1 \rightarrow x.$$ 

In the following lemma we will show the basic properties of the negations $\neg$ and $\sim$ in connection with the operations of bounded DRl-monoids.

**Lemma 2.** Let $M = (M;+,0,1,\lor,\land,\rightarrow,\leftarrow)$ be a bounded DRl-monoid and $x, y \in M$. Then

1. $\neg \neg 1 = 1 = \neg \sim 1$, $\neg \neg 0 = 0 = \neg \sim 0$;
2. $\neg \neg x \leq x$, $\neg \sim x \leq x$;
3. $\neg \neg x = \sim x$, $\neg \sim x = \neg x$;
4. $x + \sim x = 1$, $x + x = 1$;
5. $\neg x \leq y \leftrightarrow x + y = 1 \leftrightarrow \neg y \leq x$;
6. $y \leq \sim x \leq x$, $y \rightarrow \sim x \leq x$;
7. $\neg y \sim x \leq y \leftarrow x$, $\neg x \leftarrow y \leq y \leftarrow x$;
8. $\neg y \rightarrow x = \neg x \leftarrow y$, $x \leftarrow y = y \rightarrow \sim x$;
9. $x \leq y \implies \neg y \leq \sim x$, $\sim y \leq \sim x$;
10. $\sim x \rightarrow x = \neg x \leftarrow x$;
11. $\neg(x + y) = \sim x \rightarrow y$, $\neg(x + y) = \neg y \rightarrow x$;
12. $\neg(x \land y) = \sim x \lor \sim y$, $\neg(x \land y) = \neg x \lor \neg y$;
13. $\neg(x \lor y) \leq \sim x \lor \sim y$, $\neg(x \lor y) \leq \neg x \lor \neg y$;
14. $\neg(x \land y) \leq \sim x \land \sim y$, $\neg(x \land y) \leq \neg x \land \neg y$;
15. $\neg(x \rightarrow y) = \neg x \leftarrow y$, $\neg x \leftarrow \sim y = \neg x \rightarrow y$;
16. $\neg(x \rightarrow y) \leq \sim x + y$, $\sim(x \rightarrow y) \leq y + \sim x$;
17. $(x + y) \rightarrow y \leq x$, $(x + y) \leftarrow x \leq y$;
18. $y \rightarrow (y \leftarrow x) \leq x \land y$, $y \leftarrow (y \rightarrow x) \leq x \land y$. 

225
Proof.
(1) \( \sim \neg 1 = 1 \mapsto (1 \mapsto 1) = 1 \mapsto 0 = 1, \sim \neg 0 = 1 \mapsto (1 \mapsto 0) = 1 \mapsto 1 = 0. \) Analogously \( \sim \neg 1 = 1 \) and \( \sim \neg 0 = 0. \)

(2) We have \( \sim \neg x = 1 \mapsto (1 \mapsto x) \). By the definition of a \( DR\ell \)-monoid, 
\((1 \mapsto x) + (1 \mapsto (1 \mapsto x)) = 1, \) and at the same time \((1 \mapsto x) + x = 1 \lor x = 1, \) 
hence \( \sim \neg x \leq x. \) Analogously \( \sim \neg x \leq x. \)

(3) By (2), \( \neg \sim x \leq \sim x \) and \( \sim \sim x \leq x. \) Moreover, \( a \leq b \) implies \( 1 \mapsto a \geq 1 \mapsto b, \) i.e. \( \neg b \leq \neg a, \) and similarly, \( a \leq b \) implies \( \sim b \leq \sim a. \) Thus from \( \sim \neg x \leq x \) 
it follows that \( \neg x \leq \sim \sim x \) and \( \sim \sim x \leq x \) gives \( \sim x \leq \sim \sim x. \)

(4), (5) Immediately from the definition of a \( DR\ell \)-monoid.

(6) \( y \leq 1, \) hence by (4), \( y \leq \neg x + x, \) thus \( y \mapsto \neg x \leq x. \) Analogously the other inequality.

(7) By Lemma 1(8), \( \sim x \sim y = (1 \mapsto x) \mapsto (1 \mapsto y) \leq y \mapsto x. \) Analogously \( \sim x \sim y \leq y \mapsto x. \)

(8) We have \( \neg \sim y \leq y, \) hence \( \neg x \sim y \leq \neg x \sim \sim y, \) therefore by (7), 
\( \neg x \sim y \leq \sim y \sim x. \) Similarly \( \sim y \sim x \leq \neg x \sim y. \) The second assertion is dual.

(9) If \( x \leq y, \) then \( 1 \mapsto x \geq 1 \mapsto y, \) thus \( \neg y \leq \neg x. \) Analogously \( x \leq y \) 
implies \( \sim y \leq \sim x. \)

(10) By the definition of a \( DR\ell \)-monoid we have for any \( u \in M, \) \( \sim x \mapsto x \leq u \) 
iff \( \sim x \leq u + x, \) which holds iff \( x + (u + x) = 1, \) that means \((x + u) + x = 1. \) 
This is equivalent to \( \neg x \leq x + u \) and so to \( \neg x \mapsto x \leq u. \)

(11) By Lemma 1(6), (5), we have \( \sim x \sim y = (1 \mapsto x) \sim y = 1 \mapsto (x + y) = \sim(x + y) \) and \( \sim y \mapsto x = (1 \mapsto y) \mapsto x = 1 \mapsto (x + y) = \sim(x + y). \)

(12) By Lemma 1(10), \( \sim(x \land y) = 1 \mapsto (x \land y) = 1 \mapsto (1 \mapsto y) = \sim x \lor \sim y, \) 
and similarly, \( \sim(x \land y) = \sim x \lor \sim y. \)

(13) Follows from (9).

(14) \( x \land y \leq x, \) hence by (9) we obtain \( \sim \neg(x \land y) \leq \sim \neg x, \) and thus also 
\( \sim \neg(x \land y) \leq \sim \sim x \land \sim \sim y. \) Analogously the second inequality.

(15) By (8) and (3), \( \sim \neg x \mapsto \sim \neg y = \sim \sim y \sim x = \neg y \sim \neg x = \sim \sim x \sim y. \) 
Similarly the second inequality.

(16) By Lemma 1(11), \( 1 \mapsto (x \mapsto y) \leq (1 \mapsto x) + y, \) \( 1 \mapsto (x \mapsto y) \leq y + (1 \mapsto x). \)

(17) By the definition of a bounded \( DR\ell \)-monoid we have \((x + y) \mapsto y + y = (x + y) \lor y = x + y, \) hence \((x + y) \mapsto y \leq x. \) Similarly \( x + ((x + y) \mapsto x) = x \lor (x + y) = x + y, \) 
therefore \( (x + y) \mapsto x \leq y. \)

(18) By Lemma 1(11), \( y \mapsto (y \mapsto x) \leq (y \mapsto y) + x = 0 + x = x, \) and at the same time \( y \mapsto (y \mapsto x) \leq y, \) hence \( y \mapsto (y \mapsto x) \leq x \land y. \) Analogously \( y \mapsto (y \mapsto x) \leq x \land y. \)

\( \square \)
DEFINITION.

a) We say that a bounded $DR\ell$-monoid $M$ is *good* (or symmetric) if it satisfies the identity $\neg\neg x = \neg x$.

b) A bounded $DR\ell$-monoid is called *regular* if it satisfies the identity $\neg\neg x = x = \neg x$.

Note. We choose the name “good $DR\ell$-monoid” because it generalizes the notion of “good pseudo $BL$-algebra”, see e.g. [7].

LEMMA 3. Let $M$ be a good bounded $DR\ell$-monoid. Then for each $x, y \in M$ we have:

1. $\neg(\neg x + \neg y) = \neg(x + y)$;
2. $\neg(x + \neg x) = \neg(x + \neg x) = 1$;
3. $\neg(x + y) \leq \neg x + \neg y$;
4. $\neg(x + y) \leq \neg x + \neg y$;
5. $\neg(x \lor y) = \neg x \lor \neg y$;
6. $\neg x \lor y = \neg x \lor \neg y$, $\neg y \lor x = \neg y \lor \neg x$.

If, moreover, $M$ is regular, then

7. $y \lor x = \neg x \lor \neg y$, $y \lor x = \neg x \lor \neg y$;
8. $\neg(x + y) = \neg(x + y) = y \lor \neg x = x \lor \neg x$.

Proof.

1. Using Lemma 2(8), (11) we get $\neg(\neg x + \neg y) = \neg(x + y) = \neg x + \neg y = \neg x + \neg y = \neg(x + \neg y)$.

2. $x \lor \neg x \leq 1 \lor x = \neg x \lor x = \neg x$, hence $\neg x \leq \neg(x \lor \neg x)$, thus by Lemma 2(11), (2), $\neg(x \lor \neg x) = \neg(x \lor \neg x) = \neg(x \lor \neg x) \lor \neg x = \neg(x \lor \neg x) \lor \neg x = \neg x + \neg x = x \lor \neg x$, therefore by Lemma 2(4), $\neg(x \lor \neg x) = 1$. Analogously $\neg(x + \neg x) = 1$.

3. By Lemma 1 we have $\neg x \lor y = (1 \lor \neg x) \lor y = 1 \lor (y + (1 \lor x)) \leq 1 \lor (1 \lor (x \lor y)) = 1 \lor (x \lor y) = \neg(x \lor y)$.

Further, by Lemma 2(11), $\neg(x \lor y) = \neg(x + y) = \neg(x \lor y) = \neg(x \lor y)$, hence in our case we get $\neg(x \lor y) = \neg(x \lor y) = \neg(x \lor y) = \neg(x \lor y)$, and this is by Lemma 1 equal to $\neg((x \lor y) \lor (x \lor y)) = \neg(x \lor (x \lor y)) \leq \neg((x \lor y) \lor (x \lor y)) \leq \neg(x \lor y) \lor (x \lor y) \leq \neg(x \lor y) \lor (x \lor y) = \neg(x \lor y)$, hence by Lemma 2(4) $\neg(x + y) = \neg(x + y) = \neg(x + y)$.

Therefore by Lemma 2(15) we obtain $\neg(x \lor y) = \neg x \lor \neg y$. Analogously the second equality.

4. By Lemma 2(11), (15), $\neg(\neg x + \neg y) = \neg y \lor \neg x = \neg x \lor \neg y = \neg x \lor \neg y \lor x = \neg y \lor x = \neg x \lor \neg y \lor x = \neg y \lor x = \neg x + \neg y$, hence by Lemma 2(2) $\neg(x + y) = \neg x \lor \neg y$.
(5) \( \neg \neg x \leq \neg \neg (x \vee y) \) and \( \neg \neg y \leq \neg \neg (x \vee y) \). Hence \( \neg \neg x \vee \neg \neg y \leq \neg \neg (x \vee y) \).

Further, by (4) and (3), \( \neg \neg (x \vee y) = \neg \neg ((x \rightarrow y) + y) \leq \neg \neg (x \rightarrow y) + \neg \neg y = (\neg \neg x \rightarrow \neg \neg y) + \neg \neg y = \neg \neg x \vee \neg \neg y \).

(6) By Lemma 2(3), (11) and by equality (3), \( \neg x \neg \neg y = \neg \neg x \neg \neg y \Rightarrow \neg \neg (x \vee y) = \neg \neg (x + y) = \neg (x + y) = \neg x \neg y \). Analogously the other equality.

(7) By Lemma 2(7), \( y \rightarrow x = \neg \neg y \rightarrow \neg \neg x \leq \neg x \neg \neg y \leq \neg x \neg y \leq y \rightarrow x \).

(8) The first equality is proven in (1) for arbitrary good \( DRL \)-monoids. Further, by Lemma 2(11), \( \neg (\neg x + \neg y) = \neg \neg x \neg y = x \neg y \) and \( \neg (\neg x + \neg y) = \neg \neg x \neg x = y \rightarrow \neg x \).

Pseudo \( BL \)-algebras were introduced in [4] as a non-commutative generalization of Hájek’s \( BL \)-algebras ([9]). By [12], the duals of pseudo \( BL \)-algebras are special cases of bounded \( DRL \)-monoids which are characterized by the identities

\[
(x \rightarrow y) \land (y \leftarrow x) = (x \leftarrow y) \land (y \leftarrow x) = 0.
\]

**Lemma 4.** If \( M \) is a good dual pseudo \( BL \)-algebra, then \( M \) satisfies the identity

\[
\neg \neg (x + y) = \neg \neg x + \neg \neg y.
\]

**Proof.** Every dual pseudo \( BL \)-algebra satisfies, among others, the identity \( \neg \neg (x \vee y) = \neg \neg x \land \neg \neg y \). Hence by Lemmas 1, 2 and 3 we get \( \neg \neg (x + y) = \neg \neg (x \vee y) \lor \neg \neg x = \neg \neg x + (\neg \neg (x + y) \leftarrow \neg \neg x) = \neg \neg x + (\neg \neg (x + y) \leftarrow \neg \neg x) = \neg \neg x + (\neg \neg (y \rightarrow x) \leftarrow \neg \neg y) = \neg \neg x + (\neg \neg y \leftarrow x) = \neg \neg x + (\neg \neg y \leftarrow \neg \neg y) = \neg \neg x + (\neg \neg x \land \neg \neg y) = \neg \neg x + (\neg \neg x \land \neg \neg y) = (\neg \neg x + \neg \neg y) \land (\neg \neg x + \neg \neg y) = 1 \land (\neg \neg x + \neg \neg y) = \neg \neg x + \neg \neg y. \]

**Remark.** The class of bounded \( DRL \)-monoids satisfying the identities from Lemma 4 is essentially larger than the class of good dual pseudo \( BL \)-algebras. For instance, every Brouwerian algebra is a bounded (commutative) \( DRL \)-monoid that fulfills these identities.

**GMV-algebras** were introduced in [15] (equivalently as pseudo \( MV \)-algebras in [8]) as a non-commutative generalization of \( MV \)-algebras. If \( A = (A; \oplus, \neg, \neg, 0, 1) \) is a GMV-algebra, set \( x + y := x \oplus y \), \( x \circ y := \neg (\neg x \oplus \neg y) \), \( x \rightarrow y := \neg y \circ x \), \( x \leftarrow y := x \circ \neg y \), \( x \vee y := x \oplus (y \circ x) \) and \( x \wedge y := x \circ (y \circ x) \). Then \( M = M(A) = (A; +, 0, 1, \rightarrow, \leftarrow, \vee, \wedge) \) is a bounded \( DRL \)-monoid. (Recall that from this point of view, GMV-algebras form a proper subclass of the class of dual pseudo \( BL \)-algebras.)
BOUNDED DULLY RESIDUATED LATTICE ORDERED MONOIDS

By [15], DRℓ-monoids induced by GMV-algebras can be characterized by means of identities with negations. Namely, a bounded DRℓ-monoid \( M \) is induced by a GMV-algebra if and only if \( M \) satisfies the identities

\[
1 \dashv (1 \dashv x) = x = 1 \dashv (1 \dashv x), \\
1 \dashv ((1 \dashv x) + (1 \dashv y)) = 1 \dashv ((1 \dashv x) + (1 \dashv y)),
\]

that means

\[
\neg\neg x = x = \neg\neg x, \quad \neg(\neg x + \neg y) = \neg(\neg x + \neg y).
\]

We have proved in Lemma 3(1) that the last identity is satisfied in any good bounded DRℓ-monoid, therefore a good bounded DRℓ-monoid is induced by a GMV-algebra if and only if it is regular.

Let us show that the class of good dual pseudo BL-algebras is also a variety of bounded DRℓ-monoids that satisfies certain identities with negations.

**Proposition 5.** Let \( M \) be a bounded good DRℓ-monoid. Then the following conditions are equivalent.

1. \( \neg\neg(x \land y) = \neg\neg x \land \neg\neg y; \)
2. \( \neg(x \lor y) = \neg x \land \neg y, \neg(x \lor y) = \neg x \land \neg y; \)
3. \( \neg(x \lor y) + ((x \rightarrow y) \land (y \rightarrow x)) = \neg(x \lor y), \)
   \[
   ((x \rightarrow y) \land (y \rightarrow x)) + \neg(x \lor y) = \neg(x \lor y).
   \]

**Proof.**

(1) \( \implies \) (2): By Lemma 2(12) and Lemma 3(5), \( \neg x \land \neg y = \neg\neg(\neg x \land \neg y) = \neg(\neg x \lor \neg y) = \neg(\neg(x \lor y)) = \neg(x \lor y). \) Analogously \( \neg(x \lor y) = \neg x \land \neg y. \)

(2) \( \implies \) (1): Using Lemma 2(12), we have \( \neg\neg x \land \neg\neg y = \neg(x \lor y) = \neg(\neg\neg x \land \neg\neg y). \)

(2) \( \implies \) (3): By Lemma 1, \( \neg x = 1 \dashv x = (1 \dashv (x \lor y)) + ((x \lor y) \rightarrow x) = \neg(x \lor y) + (y \rightarrow x). \) Analogously \( \neg y = \neg(x \lor y) + (x \rightarrow y). \)

From this we get \( \neg(x \lor y) = \neg x \land \neg y = ((x \lor y) + (y \rightarrow x)) \land (\neg(x \lor y) + (x \rightarrow y)). \)

Similarly, by Lemma 1, \( \sim x = 1 \dashv x = ((x \lor y) \rightarrow x) + (1 \dashv (x \lor y)) \) and \( \sim y = 1 \dashv y = ((x \lor y) \rightarrow y) + (1 \dashv (x \lor y)), \) hence \( \sim(x \lor y) = ((x \rightarrow y) \land (y \rightarrow x)) + \sim(x \lor y). \)

(3) \( \implies \) (2): \( \neg x \land \neg y = ((x \lor y) + (y \rightarrow x)) \land (\neg(x \lor y) + (x \rightarrow y)) = \neg(x \lor y) + ((y \rightarrow x) \land (x \rightarrow y)) = \neg(x \lor y). \)

Similarly \( \sim x \land \sim y = \sim(x \lor y). \)

Let us recall that the duals of pseudo BL-algebras are exactly the bounded DRℓ-monoids satisfying the equalities

\[
(x \rightarrow y) \land (y \rightarrow x) = 0, \quad (x \leftarrow y) \land (y \leftarrow x) = 0.
\]

229
**Corollary 6.** Every good dual pseudo BL-algebra satisfies all the identities from the preceding proposition.

### 3. Regular and dense elements

Let $M$ be a bounded $DR\ell$-monoid and $x \in M$. Then $x$ is called a regular element in $M$ if $\neg\neg x = x = \neg x$.

Denote by $R(M)$ the set of all regular elements in $M$.

**Proposition 7.** If a bounded $DR\ell$-monoid $M$ is good, then $R(M)$ is a subalgebra of the reduct $(M; 0, 1, \lor, \rightarrow, \neg)$.

**Proof.** It follows from Lemma 2(1) and Lemma 3(3), (5). □

As a consequence of preceding propositions we get the following theorem.

**Theorem 8.**

(a) If $M$ is a bounded good $DR\ell$-monoid satisfying the identity $\neg\neg(x + y) = \neg\neg x + \neg\neg y$, then $R(M)$ is a subalgebra of $(M; +, 0, 1, \lor, \rightarrow, \neg)$ and the mapping $x \mapsto \neg\neg x$ is a retract of $(M; +, 0, 1, \lor, \rightarrow, \neg)$ onto $(R(M); +, 0, 1, \lor, \rightarrow, \neg)$.

(b) If $M$ is a good dual BL-algebra, then $R(M)$ is a subalgebra of $M$.

**Theorem 9.** If a bounded good $DR\ell$-monoid $M$ satisfies the identity $\neg\neg(x + y) = \neg\neg x + \neg\neg y$, then $R(M) = (R(M); +, 0, 1, \lor, \land, \rightarrow, \neg)$, where $y \land_{R(M)} z = \neg\neg(y \lor z)$ for any $y, z \in R(M)$, is a $DR\ell$-monoid induced by a GMV-algebra.

**Proof.** From Lemma 2(2) and from the fact that operations $\rightarrow$ and $\neg$ are antitone in the second variable it follows that $\neg\neg$ is an interior operator on the lattice $(M; \lor, \land)$. Hence $\neg\neg x$ is the greatest element in $R(M)$ which is contained in $x \in M$. Furthermore, $(R(M); \leq)$ is a lattice and for any $y, z \in R(M)$ it holds that

$$y \lor_{R(M)} z = y \lor z, \quad y \land_{R(M)} z = \neg\neg(y \land z).$$

Let $w, y, z \in R(M)$. Then

$$w + (y \land_{R(M)} z) = w + \neg\neg(y \land z) = \neg\neg w + \neg\neg(y \land z) = \neg\neg((w + y) \land (w + z)) = (w + y) \land_{R(M)} (w + z).$$

Similarly we can prove the distributivity from the right. Moreover, if $y, z \in R(M)$, then

$$y \rightarrow_{R(M)} z \quad \text{and} \quad y \leftarrow_{R(M)} z.$$
BOUNDED DUALLY RESIDUATED LATTICE ORDERED MONOIDS

exist and
\[ y \rightarrow_{R(M)} z = y \leftarrow z \quad \text{and} \quad y \leftarrow_{R(M)} z = y \rightarrow z. \]

Thus \((R(M); +, 0, 1, \lor, \land, \rightarrow, \leftarrow)\) is a bounded DR\ell-monoid. Since it is regular, it is induced by a GMV-algebra.

Let \( M \) be a bounded DR\ell-monoid. Then an element \( x \in M \) is called dense if \( \neg\neg x = \neg x = 0 \). Denote by \( D(M) \) the set of all dense elements in \( M \).

Let us recall the notions of an ideal and a normal ideal of \( M \). Let again \( M \) be a bounded DR\ell-monoid and \( \emptyset \neq I \subseteq M \). Then \( I \) is called an ideal of \( M \) if

(a) \( x, y \in I \implies x + y \in I \);
(b) \( x \in I, \ z \in M, \ z \leq x \implies z \in I \).

An ideal \( I \) is called normal if for any \( x, y \in M \),

(c) \( x \rightarrow y \in I \iff x \leftarrow y \in I \).

By [13], normal ideals of \( M \) are in a one-to-one correspondence with congruences on \( M \). Namely, let \( I \) be a normal ideal of \( M \). Then \( \Theta(I) \), the congruence on \( M \) induced by \( I \), is determined as follows: If \( x, y \in M \), then

\[ \langle x, y \rangle \in \Theta(I) \iff (x \rightarrow y) \lor (y \rightarrow x) \in I \]

(which is equivalent to \((x \leftarrow y) \lor (y \leftarrow x) \in I)\).

Conversely, let \( \Theta \) be a congruence on \( M \). Then \( I(\Theta) = [0]_{\Theta} = \{ x \in M : \langle x, 0 \rangle \in \Theta \} \) is the normal ideal of \( M \) corresponding to \( \Theta \).

**THEOREM 10.** If \( M \) is a bounded good DR\ell-monoid, then \( D(M) \) is a normal ideal of \( M \) and \( M/D(M) \cong R(M) \).

**Proof.** Let \( x, y \in D(M) \). Then by Lemma 3(4), \( \neg\neg(x + y) \leq \neg\neg x + \neg\neg y = 0 \), thus \( x + y \in D(M) \). If \( x \in D(M) \), \( z \in M \) and \( z \leq x \), then \( \neg\neg z \leq \neg\neg x = 0 \), hence \( z \in D(M) \). Therefore \( D(M) \) is an ideal of \( M \).

Further, if \( x, y \in M \), then \( x \rightarrow y \in D(M) \) iff \( \neg\neg(x \rightarrow y) = 0 \) iff (by Lemmas 3(3) and 1) \( \neg\neg x \leftarrow \neg\neg y = 0 \), hence again by Lemma 3(3) \( \neg\neg(x \leftarrow y) = 0 \), i.e. iff \( x \leftarrow y \in D(M) \). Therefore the ideal \( D(M) \) is normal.

Let us consider the congruence \( \Theta(D(M)) \) induced by \( D(M) \). That means, if \( x, y \in M \), then \( \langle x, y \rangle \in \Theta(D(M)) \) iff \( (x \rightarrow y) \lor (y \rightarrow x) \in D(M) \), hence iff \( \neg\neg((x \rightarrow y) \lor (y \rightarrow x)) = 0 \), hence by Lemma 3(5) \( \neg\neg(x \rightarrow y) \lor \neg\neg(y \rightarrow x) = 0 \), and by Lemma 3(3) \( (\neg\neg x \rightarrow \neg\neg y) \lor (\neg\neg y \rightarrow \neg\neg x) = 0 \), and this holds iff \( \neg\neg x \rightarrow \neg\neg y = 0 = \neg\neg y \rightarrow \neg\neg x \). By Lemma 1 it is equivalent to \( \neg\neg x \leq \neg\neg y \leq \neg\neg x \), i.e. with \( \neg\neg x = \neg\neg y \).

Therefore \( M/D(M) \cong R(M) \). □
Remark. In an analogous theorem in [17], for a commutative bounded $DRl$-monoid $M$ it was, moreover, supposed that $M$ satisfies the identity $\neg(x + y) = \neg x + \neg y$. The proof of Theorem 10 shows that the mentioned assumption was superfluous.

A $DRl$-monoid $M$ is called (congruence) simple if $M$ is non-trivial and has no proper congruence different from the identity.

**Theorem 11.** If $M$ is a bounded good $DRl$-monoid satisfying the identity $\neg(x + y) = \neg x + \neg y$, then $M$ is simple if and only if it is induced by a simple GMV-algebra.

**Proof.** By Theorem 10, $D(M)$ is a normal ideal in $M$ for any bounded good $DRl$-monoid $M$. Let $M$ satisfy the identity $\neg(x + y) = \neg x + \neg y$ and let $M$ be simple. Then $M$ has a unique proper normal ideal, hence $D(M) = \{0\}$. Therefore, by Theorem 9, $M$ is induced by a GMV-algebra. □

Let $M$ be a bounded $DRl$-monoid and $I$ be a normal ideal in $M$. Then $I$ is called a GMV-ideal if the $DRl$-monoid $M/Q(I)$ is induced by a GMV-algebra.

**Theorem 12.** Let $M$ be a bounded good $DRl$-monoid satisfying the identity $\neg(x + y) = \neg x + \neg y$ and $I$ be a normal ideal in $M$. Then the following conditions are equivalent.

1. $I$ is a GMV-ideal.
2. $\neg x \in I$ for each $x \in M$.
3. $\neg x \in I \implies x \in I$ for each $x \in M$.
4. $D(M) \subseteq I$.

**Proof.**

1. $\iff$ 2.: Since $M$ is good, $M/Q(I)$ is induced by a GMV-algebra if and only if $(x, \neg x) \in \Theta(I)$ for each $x \in M$, i.e. if and only if $(x \rightarrow \neg x) \lor (\neg x \rightarrow x) \in I$ for each $x \in M$, and this is equivalent to $x \rightarrow \neg x \in I$ for each $x \in M$.

2. $\implies$ 3.: Let $x \rightarrow \neg x \in I$ and $\neg x \in I$. Then $x = x \lor \neg x = (x \rightarrow \neg x) + \neg x \in I$.

3. $\implies$ 4.: Let $y \in D(M)$. Then $\neg y = 0 \in I$, hence $y \in I$. Therefore $D(M) \subseteq I$.

4. $\implies$ 1.: Let $D(M) \subseteq I$. Then $M/Q(I)$ is isomorphic to a subalgebra of $M/Q(D(M))$ which is induced by a GMV-algebra. □

**Theorem 13.** Let $M$ be a bounded good $DRl$-monoid satisfying the identity $\neg(x + y) = \neg x + \neg y$. If $I$ is a maximal ideal in $M$ and is normal, then $I$ is a GMV-ideal.

**Proof.** Let $I$ be a maximal and normal ideal in $M$. Then $M/I$ is a simple $DRl$-monoid, thus by Theorem 11 we have that $I$ is a GMV-ideal. □
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