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BOUNDED DUALLY RESIDUATED LATTICE ORDERED MONOIDS AS A GENERALIZATION OF FUZZY STRUCTURES

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ABSTRACT. Dually residuated lattice ordered monoids ($DR\ell$-monoids) form a large class that contains among others all lattice ordered groups, fuzzy structures which need not be commutative, for instance, pseudo BL-algebras and GMV-algebras (= pseudo MV-algebras) and Brouwerian algebras. In the paper, two concepts of negation in bounded $DR\ell$-monoids are introduced and their properties are studied in general as well as in the case of the so-called good $DR\ell$-monoids. The sets of regular and dense elements of good $DR\ell$-monoids are described.

1. Introduction

Commutative dually residuated lattice ordered monoids (briefly: $DR\ell$-monoids) were introduced by Swamy in [18] as a common generalization of abelian lattice ordered groups and Brouwerian algebras. Moreover, the classes of MV-algebras and BL-algebras, i.e. algebraic counterparts of Łukasiewicz infinite valued and Hájek's basic fuzzy logic introduced in [1] and [9], respectively, can be viewed as proper subclasses of the class of bounded commutative $DR\ell$-monoids. (In fact, we use the duals of BL-algebras.)

General $DR\ell$-monoids (i.e., the commutativity of the addition is not required) were introduced by Kovář in [11]. GMV-algebras introduced in [15] and, equivalently, pseudo MV-algebras introduced in [8] are non-commutative generalizations of MV-algebras. Further, pseudo BL-algebras introduced and studied in [4] and [5] and BL-algebras are in the same connection. By [16], GMV-algebras are an algebraic counterpart of a non-commutative logic between

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the Łukasiewicz logic and the bilinear logic (see [14]). Pseudo BL-algebras are by [10] an algebraic counterpart of Hájek's pseudo basic logic. Analogously as in the commutative case, it was shown in [15] and [12] that GMV-algebras and duals of pseudo BL-algebras form proper subclasses of the class of bounded $DR\ell$-monoids.

In this paper we study bounded $DR\ell$-monoids as natural generalizations of GMV-algebras and pseudo BL-algebras introducing two, in general different, concepts of negation. All obtained results are applicable in the case of pseudo BL-algebras (and, consequently, of GMV-algebras). The particular case of negations in commutative $DR\ell$-monoids were studied in [17].

The basic concepts and results concerning MV-algebras, GMV-algebras, BL-algebras and pseudo BL-algebras can be found in [2], [6], [9] and [4], respectively.

2. Negations in bounded $DR\ell$-monoids

In this section we introduce notions of negations of elements in bounded $DR\ell$-monoids as generalizations of those in pseudo BL-algebras.

Firstly, let us recall the definition of a $DR\ell$-monoid.

DEFINITION. A *dually residuated lattice ordered monoid* (briefly: *$DR\ell$-monoid*) is an algebra $M = (M; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2,0,2,2,2,2\rangle$ satisfying the following conditions:

(M1) $(M; +, 0, \vee, \wedge)$ is a lattice ordered monoid, that means, $(M, +, 0)$ is a monoid, $(M, \vee, \wedge)$ is a lattice, and the operation $+$ distributes from the left and from the right over the operations $\vee$ and $\wedge$.

(M2) If $\leq$ denotes the order on M induced by the lattice $(M, \vee, \wedge)$, then $x \rightharpoonup y$ is the smallest $s \in M$ such that $s + y \geq x$ and $x \leftharpoondown y$ is the smallest $t \in M$ such that $y + t \geq x$ for any $x, y \in M$.

(M3) M satisfies the identities

$$\big((x \rightharpoonup y) \vee 0\big) + y \leq x \vee y\,, \qquad y + \big((x \leftharpoondown y) \vee 0\big) \leq x \vee y\,,$$
$$x \rightharpoonup x \geq 0\,, \qquad x \leftharpoondown x \geq 0\,.$$

In the paper, we will deal with bounded $DR\ell$-monoids. The least element in such a $DR\ell$-monoid is by [11] always 0. The greatest element will be denoted by 1 and bounded $DR\ell$-monoids will be considered as algebras $M = (M, +, 0, 1, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of extended type $\langle 2,0,0,2,2,2,2\rangle$.

When doing calculations, we use the following list of basic rules for bounded $DR\ell$-monoids.

Lemma 1. ([11], [13]) *In any bounded $DR\ell$-monoid M we have for any $x, y, z \in M$:*

(1) $x \vee y = (x \rightharpoonup y) + y = y + (x \leftharpoondown y)$;
(2) $x \rightharpoonup x = 0 = x \leftharpoondown x$, $x \rightharpoonup 0 = x = x \leftharpoondown 0$;
(3) $x \leq y \implies x \rightharpoonup z \leq y \rightharpoonup z$, $x \leftharpoondown z \leq y \leftharpoondown z$;
(4) $x \leq y \implies z \rightharpoonup x \geq z \rightharpoonup y$, $z \leftharpoondown x \geq z \leftharpoondown y$;
(5) $x \rightharpoonup (y + z) = (x \rightharpoonup z) \rightharpoonup y$;
(6) $x \leftharpoondown (y + z) = (x \leftharpoondown y) \leftharpoondown z$;
(7) $x \rightharpoonup y \geq (z \rightharpoonup y) \leftharpoondown (z \rightharpoonup x)$;
(8) $x \leftharpoondown y \geq (z \leftharpoondown y) \rightharpoonup (z \leftharpoondown x)$;
(9) $x \leq y \iff x \rightharpoonup y = 0 \iff x \leftharpoondown y = 0$;
(10) $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \vee (x \rightharpoonup z)$, $x \leftharpoondown (y \wedge z) = (x \leftharpoondown y) \vee (x \leftharpoondown z)$;
(11) $x \rightharpoonup (y \leftharpoondown z) \leq (x \rightharpoonup y) + z$, $x \leftharpoondown (y \rightharpoonup z) \leq z + (x \leftharpoondown y)$;
(12) $x \geq y \geq z \implies x \rightharpoonup z = (x \rightharpoonup y) + (y \rightharpoonup z)$, $x \leftharpoondown z = (y \leftharpoondown z) + (x \leftharpoondown y)$;

Definition. Let $M = (M; +, 0, 1, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ be a bounded $DR\ell$-monoid. For any $x \in M$ we set

$$\neg x := 1 \rightharpoonup x, \qquad {\sim} x := 1 \leftharpoondown x.$$

In the following lemma we will show the basic properties of the negations $\neg$ and $\sim$ in connection with the operations of bounded $DR\ell$-monoids.

Lemma 2. *Let $M = (M; +, 0, 1, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ be a bounded $DR\ell$-monoid and $x, y \in M$. Then*

(1) ${\sim}\neg 1 = 1 = \neg{\sim} 1$, ${\sim}\neg 0 = 0 = \neg{\sim} 0$;
(2) ${\sim}\neg x \leq x$, $\neg{\sim} x \leq x$;
(3) ${\sim}\neg{\sim} x = {\sim} x$, $\neg{\sim}\neg x = \neg x$;
(4) $x + {\sim} x = 1$, $\neg x + x = 1$;
(5) ${\sim} x \leq y \iff x + y = 1 \iff \neg y \leq x$;
(6) $y \leftharpoondown \neg x \leq x$, $y \rightharpoonup {\sim} x \leq x$;
(7) ${\sim} x \rightharpoonup {\sim} y \leq y \leftharpoondown x$, $\neg x \leftharpoondown \neg y \leq y \rightharpoonup x$;
(8) ${\sim} y \rightharpoonup x = \neg x \leftharpoondown y$, $x \leftharpoondown \neg y = y \rightharpoonup {\sim} x$;
(9) $x \leq y \implies \neg y \leq \neg x$, ${\sim} y \leq {\sim} x$;
(10) ${\sim} x \rightharpoonup x = \neg x \leftharpoondown x$;
(11) ${\sim}(x + y) = {\sim} x \leftharpoondown y$, $\neg(x + y) = \neg y \rightharpoonup x$;
(12) ${\sim}(x \wedge y) = {\sim} x \vee {\sim} y$, $\neg(x \wedge y) = \neg x \vee \neg y$;
(13) ${\sim}(x \vee y) \leq {\sim} x \wedge {\sim} y$, $\neg(x \vee y) \leq \neg x \wedge \neg y$;
(14) ${\sim}\neg(x \wedge y) \leq {\sim}\neg x \wedge {\sim}\neg y$, $\neg{\sim}(x \wedge y) \leq \neg{\sim} x \wedge \neg{\sim} y$;
(15) ${\sim}\neg x \rightharpoonup {\sim}\neg y = {\sim}\neg x \rightharpoonup y$, $\neg{\sim} x \leftharpoondown \neg{\sim} y = \neg{\sim} x \leftharpoondown y$;
(16) $\neg(x \leftharpoondown y) \leq \neg x + y$, ${\sim}(x \rightharpoonup y) \leq y + {\sim} x$;
(17) $(x + y) \rightharpoonup y \leq x$, $(x + y) \leftharpoondown x \leq y$;
(18) $y \rightharpoonup (y \leftharpoondown x) \leq x \wedge y$, $y \leftharpoondown (y \rightharpoonup x) \leq x \wedge y$.

P r o o f .

(1) $\sim\neg 1 = 1 \leftharpoondown (1 \rightharpoonup 1) = 1 \leftharpoondown 0 = 1$, $\sim\neg 0 = 1 \leftharpoondown (1 \rightharpoonup 0) = 1 \leftharpoondown 1 = 0$. Analogously $\neg\sim 1 = 1$ and $\neg\sim 0 = 0$.

(2) We have $\sim\neg x = 1 \leftharpoondown (1 \rightharpoonup x)$. By the definition of a $DR\ell$-monoid, $(1 \rightharpoonup x) + \big(1 \leftharpoondown (1 \rightharpoonup x)\big) = 1$, and at the same time $(1 \rightharpoonup x) + x = 1 \vee x = 1$, hence $\sim\neg x \leq x$. Analogously $\neg\sim x \leq x$.

(3) By (2), $\sim\neg\sim x \leq \sim x$ and $\neg\sim\neg x \leq \neg x$. Moreover, $a \leq b$ implies $1 \rightharpoonup a \geq 1 \rightharpoonup b$, i.e. $\neg b \leq \neg a$, and similarly, $a \leq b$ implies $\sim b \leq \sim a$. Thus from $\sim\neg x \leq x$ it follows that $\neg x \leq \neg\sim\neg x$ and $\neg\sim x \leq x$ gives $\sim x \leq \sim\neg\sim x$.

(4), (5) Immediately from the definition of a $DR\ell$-monoid.

(6) $y \leq 1$, hence by (4), $y \leq \neg x + x$, thus $y \leftharpoondown \neg x \leq x$. Analogously the other inequality.

(7) By Lemma 1(8), $\sim x \rightharpoonup \sim y = (1 \leftharpoondown x) \rightharpoonup (1 \leftharpoondown y) \leq y \leftharpoondown x$. Analogously $\neg x \leftharpoondown \neg y \leq y \rightharpoonup x$.

(8) We have $\neg\sim y \leq y$, hence $\neg x \leftharpoondown y \leq \neg x \leftharpoondown \neg\sim y$, therefore by (7), $\neg x \leftharpoondown y \leq \sim y \rightharpoonup x$. Similarly $\sim y \rightharpoonup x \leq \neg x \leftharpoondown y$. The second assertion is dual.

(9) If $x \leq y$, then $1 \rightharpoonup x \geq 1 \rightharpoonup y$, thus $\neg y \leq \neg x$. Analogously $x \leq y$ implies $\sim y \leq \sim x$.

(10) By the definition of a $DR\ell$-monoid we have for any $u \in M$, $\sim x \rightharpoonup x \leq u$ iff $\sim x \leq u + x$, which holds iff $x + (u + x) = 1$, that means $(x + u) + x = 1$. This is equivalent to $\neg x \leq x + u$ and so to $\neg x \leftharpoondown x \leq u$.

(11) By Lemma 1(6), (5), we have $\sim x \leftharpoondown y = (1 \leftharpoondown x) \leftharpoondown y = 1 \leftharpoondown (x + y) = \sim(x + y)$ and $\neg y \rightharpoonup x = (1 \rightharpoonup y) \rightharpoonup x = 1 \rightharpoonup (x + y) = \neg(x + y)$.

(12) By Lemma 1(10), $\sim(x \wedge y) = 1 \leftharpoondown (x \wedge y) = (1 \leftharpoondown x) \vee (1 \leftharpoondown y) = \sim x \vee \sim y$, and similarly, $\neg(x \wedge y) = \neg x \vee \neg y$.

(13) Follows from (9).

(14) $x \wedge y \leq x$, hence by (9) we obtain $\sim\neg(x \wedge y) \leq \sim\neg x$, and thus also $\sim\neg(x \wedge y) \leq \sim\neg x \wedge \sim\neg y$. Analogously the second inequality.

(15) By (8) and (3), $\sim\neg x \rightharpoonup \sim\neg y = \neg\sim\neg y \leftharpoondown \neg x = \neg y \leftharpoondown \neg x = \sim\neg x \rightharpoonup y$. Similarly the second inequality.

(16) By Lemma 1(11), $1 \rightharpoonup (x \leftharpoondown y) \leq (1 \rightharpoonup x) + y$, $1 \leftharpoondown (x \rightharpoonup y) \leq y + (1 \leftharpoondown x)$.

(17) By the definition of a bounded $DR\ell$-monoid we have $\big((x+y) \rightharpoonup y\big) + y = (x + y) \vee y = x + y$, hence $(x + y) \rightharpoonup y \leq x$. Similarly $x + \big((x + y) \leftharpoondown x\big) = x \vee (x + y) = x + y$, therefore $(x + y) \leftharpoondown x \leq y$.

(18) By Lemma 1(11), $y \rightharpoonup (y \leftharpoondown x) \leq (y \rightharpoonup y) + x = 0 + x = x$, and at the same time $y \rightharpoonup (y \leftharpoondown x) \leq y$, hence $y \rightharpoonup (y \leftharpoondown x) \leq x \wedge y$. Analogously $y \leftharpoondown (y \rightharpoonup x) \leq x \wedge y$. □

Definition.

a) We say that a bounded $DR\ell$-monoid M is *good* (or *symmetric*) if it satisfies the identity $\neg{\sim}x = {\sim}\neg x$.

b) A bounded $DR\ell$-monoid is called *regular* if it satisfies the identity $\neg{\sim}x = x = {\sim}\neg x$.

Note. We choose the name "good $DR\ell$-monoid" because it generalizes the notion of "good pseudo BL-algebra", see e.g. [7].

Lemma 3. *Let M be a good bounded $DR\ell$-monoid. Then for each $x, y \in M$ we have:*

(1) ${\sim}(\neg x + \neg y) = \neg({\sim}x + {\sim}y)$;

(2) $\neg(x \leftharpoondown {\sim}\neg x) = {\sim}(x \rightharpoonup {\sim}\neg x) = 1$;

(3) $\neg{\sim}(x \rightharpoonup y) = \neg{\sim}x \rightharpoonup \neg{\sim}y$, ${\sim}\neg(x \leftharpoondown y) = {\sim}\neg x \leftharpoondown {\sim}\neg y$;

(4) $\neg{\sim}(x + y) \leq \neg{\sim}x + \neg{\sim}y$;

(5) $\neg{\sim}(x \vee y) = \neg{\sim}x \vee \neg{\sim}y$;

(6) ${\sim}x \leftharpoondown y = {\sim}x \leftharpoondown {\sim}\neg y$, $\neg y \rightharpoonup x = \neg y \rightharpoonup \neg{\sim}x$.

If, moreover, M is regular, then

(7) $y \leftharpoondown x = {\sim}x \rightharpoonup {\sim}y$, $y \rightharpoonup x = \neg x \leftharpoondown \neg y$;

(8) ${\sim}(\neg x + \neg y) = \neg({\sim}x + {\sim}y) = y \rightharpoonup {\sim}x = x \leftharpoondown \neg y$.

Proof.

(1) Using Lemma 2(8), (11) we get $\neg({\sim}x + {\sim}y) = \neg{\sim}y \rightharpoonup {\sim}x = {\sim}\neg y \rightharpoonup {\sim}x = \neg{\sim}x \leftharpoondown \neg y = {\sim}\neg x \leftharpoondown \neg y = {\sim}(\neg x + \neg y)$.

(2) $x \leftharpoondown \neg{\sim}x \leq 1 \leftharpoondown \neg{\sim}x = {\sim}\neg{\sim}x = {\sim}x$, hence $\neg{\sim}x \leq \neg(x \leftharpoondown \neg{\sim}x)$, thus by Lemma 2(11), (2), $\neg(x \leftharpoondown \neg{\sim}x) = \neg(x \leftharpoondown \neg{\sim}x) \vee \neg{\sim}x = (\neg(x \leftharpoondown \neg{\sim}x) \rightharpoonup \neg{\sim}x) + \neg{\sim}x = \neg\big(\neg{\sim}x + (x \leftharpoondown \neg{\sim}x)\big) + \neg{\sim}x = \neg(\neg{\sim}x \vee x) + \neg{\sim}x = \neg x + \neg{\sim}x = \neg x + {\sim}\neg x$, therefore by Lemma 2(4), $\neg(x \leftharpoondown \neg{\sim}x) = 1$. Analogously ${\sim}(x \rightharpoonup {\sim}\neg x) = 1$.

(3) By Lemma 1 we have $\neg{\sim}x \rightharpoonup y = (1 \rightharpoonup {\sim}x) \rightharpoonup y = 1 \rightharpoonup \big(y + (1 \leftharpoondown x)\big) \leq 1 \rightharpoonup \big(1 \leftharpoondown (x \rightharpoonup y)\big) = 1 \rightharpoonup {\sim}(x \rightharpoonup y) = \neg{\sim}(x \rightharpoonup y)$.

Further, by Lemma 2(11), $\neg{\sim}(\neg{\sim}x \rightharpoonup y) = \neg{\sim}\big(\neg(y + {\sim}x)\big) = \neg(y + {\sim}x) = \neg{\sim}x \rightharpoonup y$, hence in our case we get $\neg{\sim}(x \rightharpoonup y) \rightharpoonup (\neg{\sim}x \rightharpoonup y) = \neg{\sim}\big(\neg{\sim}(x \rightharpoonup y) \rightharpoonup (\neg{\sim}x \rightharpoonup y)\big) \leq \neg{\sim}\big((x \rightharpoonup y) \rightharpoonup (\neg{\sim}x \rightharpoonup y)\big)$, and this is by Lemma 1 equal to $\neg{\sim}\big(x \rightharpoonup \big((\neg{\sim}x \rightharpoonup y) + y\big)\big) = \neg{\sim}\big(x \rightharpoonup (\neg{\sim}x \vee y)\big) \leq \neg{\sim}\big((x \rightharpoonup \neg{\sim}x) \wedge (x \rightharpoonup y)\big) \leq \neg{\sim}(x \rightharpoonup \neg{\sim}x) = \neg 1 = 0$, thus $\neg{\sim}(x \rightharpoonup y) \leq \neg{\sim}x \rightharpoonup y$.

Therefore by Lemma 2(15) we obtain $\neg{\sim}(x \rightharpoonup y) = \neg{\sim}x \rightharpoonup \neg{\sim}y$. Analogously the second equality.

(4) By Lemma 2(11), (15), $\neg(\neg{\sim}x + \neg{\sim}y) = \neg\neg{\sim}y \rightharpoonup \neg{\sim}x = \neg{\sim}\neg y \rightharpoonup \neg{\sim}x = \neg{\sim}\neg y \rightharpoonup x = \neg y \rightharpoonup x = \neg(x + y)$, hence by Lemma 2(2) $\neg{\sim}(x + y) = {\sim}\neg(x + y) = {\sim}\neg(\neg{\sim}x + \neg{\sim}y) \leq \neg{\sim}x + \neg{\sim}y$.

(5) $\neg{\sim}x \le \neg{\sim}(x \vee y)$ and $\neg{\sim}y \le \neg{\sim}(x \vee y)$, hence $\neg{\sim}x \vee \neg{\sim}y \le \neg{\sim}(x \vee y)$.

Further, by (4) and (3), $\neg{\sim}(x \vee y) = \neg{\sim}\big((x \rightharpoonup y)+y\big) \le \neg{\sim}(x \rightharpoonup y)+\neg{\sim}y = (\neg{\sim}x \rightharpoonup \neg{\sim}y) + \neg{\sim}y = \neg{\sim}x \vee \neg{\sim}y$.

(6) By Lemma 2(3), (11) and by equality (3), ${\sim}x \leftharpoondown {\sim}\neg y = {\sim}\neg{\sim}x \leftharpoondown {\sim}\neg y = {\sim}\neg({\sim}x \leftharpoondown y) = {\sim}\neg{\sim}(x+y) = {\sim}(x+y) = {\sim}x \leftharpoondown y$. Analogously the other equality.

(7) By Lemma 2(7), $y \leftharpoondown x = \neg{\sim}y \leftharpoondown \neg{\sim}x \le {\sim}x \rightharpoonup {\sim}y \le y \leftharpoondown x$ and $y \rightharpoonup x = {\sim}\neg y \rightharpoonup {\sim}\neg x \le \neg x \leftharpoondown \neg y \le y \rightharpoonup x$.

(8) The first equality is proven in (1) for arbitrary good $DR\ell$-monoids. Further, by Lemma 2(11), ${\sim}(\neg x + \neg y) = {\sim}\neg x \leftharpoondown \neg y = x \leftharpoondown \neg y$ and $\neg({\sim}x + {\sim}y) = \neg{\sim}y \rightharpoonup {\sim}x = y \rightharpoonup {\sim}x$. □

Pseudo BL-algebras were introduced in [4] as a non-commutative generalization of Hájek's BL-algebras ([9]). By [12], the duals of pseudo BL-algebras are special cases of bounded $DR\ell$-monoids which are characterized by the identities

$$(x \rightharpoonup y) \wedge (y \rightharpoonup x) = (x \leftharpoondown y) \wedge (y \leftharpoondown x) = 0\,.$$

LEMMA 4. *If M is a good dual pseudo BL-algebra, then M satisfies the identity*

$$\neg{\sim}(x+y) = \neg{\sim}x + \neg{\sim}y\,.$$

P r o o f. Every dual pseudo BL-algebra satisfies, among others, the identity ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$. Hence by Lemmas 1, 2 and 3 we get $\neg{\sim}(x+y) = \neg{\sim}(x+y) \vee \neg{\sim}x = \neg{\sim}x + \big(\neg{\sim}(x+y) \leftharpoondown \neg{\sim}x\big) = \neg{\sim}x + \big({\sim}\neg(x+y) \leftharpoondown \neg{\sim}x\big) = \neg{\sim}x + \big({\sim}(\neg y \rightharpoonup x) \leftharpoondown \neg{\sim}x\big) = \neg{\sim}x + \big({\sim}(\neg y \rightharpoonup \neg{\sim}x) \leftharpoondown \neg{\sim}x\big) = \neg{\sim}x + {\sim}\big((\neg y \rightharpoonup \neg{\sim}x) + \neg{\sim}x\big) = \neg{\sim}x + {\sim}(\neg y \vee \neg{\sim}x) = \neg{\sim}x + {\sim}(\neg{\sim}x \vee \neg y) = \neg{\sim}x + ({\sim}x \wedge {\sim}\neg y) = \neg{\sim}x + ({\sim}x \wedge \neg{\sim}y) = (\neg{\sim}x + {\sim}x) \wedge (\neg{\sim}x + \neg{\sim}y) = 1 \wedge (\neg{\sim}x + \neg{\sim}y) = \neg{\sim}x + \neg{\sim}y$. □

Remark. The class of bounded $DR\ell$-monoids satisfying the identities from Lemma 4 is essentially larger than the class of good dual pseudo BL-algebras. For instance, every Brouwerian algebra is a bounded (commutative) $DR\ell$-monoid that fulfils these identities.

GMV-algebras were introduced in [15] (equivalently as *pseudo MV-algebras* in [8]) as a non-commutative generalization of MV-algebras. If $A = (A; \oplus, \neg, {\sim}, 0, 1)$ is a GMV-algebra, set $x + y := x \oplus y$, $x \odot y := {\sim}(\neg x \oplus \neg y)$, $x \rightharpoonup y := \neg y \odot x$, $x \leftharpoondown y := x \odot {\sim}y$, $x \vee y := x \oplus (y \odot {\sim}x)$ and $x \wedge y := x \odot (y \oplus {\sim}x)$. Then $M = M(A) = (A; +, 0, 1, \rightharpoonup, \leftharpoondown, \vee, \wedge)$ is a bounded $DR\ell$-monoid. (Recall that from this point of view, GMV-algebras form a proper subclass of the class of dual pseudo BL-algebras.)

By [15], $DR\ell$-monoids induced by GMV-algebras can be characterized by means of identities with negations. Namely, a bounded $DR\ell$-monoid M is induced by a GMV-algebra if and only if M satisfies the identities

$$1 \rightharpoonup (1 \leftharpoondown x) = x = 1 \leftharpoondown (1 \rightharpoonup x),$$
$$1 \rightharpoonup \big((1 \leftharpoondown x) + (1 \leftharpoondown y)\big) = 1 \leftharpoondown \big((1 \rightharpoonup x) + (1 \rightharpoonup y)\big),$$

that means

$$\neg{\sim}x = x = {\sim}\neg x, \qquad \neg({\sim}x + {\sim}y) = {\sim}(\neg x + \neg y).$$

We have proved in Lemma 3(1) that the last identity is satisfied in any good bounded $DR\ell$-monoid, therefore a good bounded $DR\ell$-monoid is induced by a GMV-algebra if and only if it is regular.

Let us show that the class of good dual pseudo BL-algebras is also a variety of bounded $DR\ell$-monoids that satisfies certain identities with negations.

PROPOSITION 5. *Let M be a bounded good $DR\ell$-monoid. Then the following conditions are equivalent.*

(1) $\neg{\sim}(x \wedge y) = \neg{\sim}x \wedge \neg{\sim}y$;
(2) $\neg(x \vee y) = \neg x \wedge \neg y$, ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$;
(3) $\neg(x \vee y) + \big((x \rightharpoonup y) \wedge (y \rightharpoonup x)\big) = \neg(x \vee y)$,
$\big((x \leftharpoondown y) \wedge (y \leftharpoondown x)\big) + {\sim}(x \vee y) = {\sim}(x \vee y)$.

Proof.
(1) $\implies$ (2): By Lemma 2(12) and Lemma 3(5), $\neg x \wedge \neg y = \neg{\sim}(\neg x \wedge \neg y) = \neg({\sim}\neg x \vee {\sim}\neg y) = \neg\big({\sim}\neg(x \vee y)\big) = \neg(x \vee y)$. Analogously ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$.

(2) $\implies$ (1): Using Lemma 2(12), we have $\neg{\sim}x \wedge \neg{\sim}y = \neg({\sim}x \vee {\sim}y) = \neg\big({\sim}(x \wedge y)\big) = \neg{\sim}(x \wedge y)$.

(2) $\implies$ (3): By Lemma 1, $\neg x = 1 \rightharpoonup x = \big(1 \rightharpoonup (x \vee y)\big) + \big((x \vee y) \rightharpoonup x\big) = \neg(x \vee y) + (y \rightharpoonup x)$. Analogously $\neg y = \neg(x \vee y) + (x \rightharpoonup y)$.

From this we get $\neg(x \vee y) = \neg x \wedge \neg y = \big(\neg(x \vee y) + (y \rightharpoonup x)\big) \wedge \big(\neg(x \vee y) + (x \rightharpoonup y)\big) = \neg(x \vee y) + \big((y \rightharpoonup x) \wedge (x \rightharpoonup y)\big)$.

Similarly, by Lemma 1, ${\sim}x = 1 \leftharpoondown x = \big((x \vee y) \leftharpoondown x\big) + \big(1 \leftharpoondown (x \vee y)\big)$ and ${\sim}y = 1 \leftharpoondown y = \big((x \vee y) \leftharpoondown y\big) + \big(1 \leftharpoondown (x \vee y)\big)$, hence ${\sim}(x \vee y) = \big((x \leftharpoondown y) \wedge (y \leftharpoondown x)\big) + {\sim}(x \vee y)$.

(3) $\implies$ (2): $\neg x \wedge \neg y = \big(\neg(x \vee y) + (y \rightharpoonup x)\big) \wedge \big(\neg(x \vee y) + (x \rightharpoonup y)\big) = \neg(x \vee y) + \big((y \rightharpoonup x) \wedge (x \rightharpoonup y)\big) = \neg(x \vee y)$.

Similarly ${\sim}x \wedge {\sim}y = {\sim}(x \vee y)$. $\square$

Let us recall that the duals of pseudo BL-algebras are exactly the bounded $DR\ell$-monoids satisfying the equalities

$$(x \rightharpoonup y) \wedge (y \rightharpoonup x) = 0, \qquad (x \leftharpoondown y) \wedge (y \leftharpoondown x) = 0.$$

COROLLARY 6. *Every good dual pseudo BL-algebra satisfies all the identities from the preceding proposition.*

3. Regular and dense elements

Let M be a bounded $DR\ell$-monoid and $x \in M$. Then x is called a *regular element* in M if $\neg\sim x = x = \sim\neg x$.

Denote by $R(M)$ the set of all regular elements in M.

PROPOSITION 7. *If a bounded $DR\ell$-monoid M is good, then $R(M)$ is a subalgebra of the reduct $(M; 0, 1, \vee, \rightharpoonup, \leftharpoondown)$.*

P r o o f. It follows from Lemma 2(1) and Lemma 3(3), (5). □

As a consequence of preceding propositions we get the following theorem.

THEOREM 8.

(a) *If M is a bounded good $DR\ell$-monoid satisfying the identity $\neg\sim(x + y) = \neg\sim x + \neg\sim y$, then $R(M)$ is a subalgebra of $(M; +, 0, 1, \vee, \rightharpoonup, \leftharpoondown)$ and the mapping $x \mapsto \neg\sim x$ is a retract of $(M; +, 0, 1, \vee, \rightharpoonup, \leftharpoondown)$ onto $\big(R(M); +, 0, 1, \vee, \rightharpoonup, \leftharpoondown\big)$.*

(b) *If M is a good dual BL-algebra, then $R(M)$ is a subalgebra of M.*

THEOREM 9. *If a bounded good $DR\ell$-monoid M satisfies the identity $\neg\sim(x+y) = \neg\sim x + \neg\sim y$, then $R(M) = \big(R(M); +, 0, 1, \vee, \wedge_{R(M)}, \rightharpoonup, \leftharpoondown\big)$, where $y \wedge_{R(M)} z = \neg\sim(y \wedge z)$ for any $y, z \in R(M)$, is a $DR\ell$-monoid induced by a GMV-algebra.*

P r o o f. From Lemma 2(2) and from the fact that operations $\rightharpoonup$ and $\leftharpoondown$ are antitone in the second variable it follows that $\neg\sim$ is an interior operator on the lattice $(M; \vee, \wedge)$. Hence $\neg\sim x$ is the greatest element in $R(M)$ which is contained in $x \in M$. Furthermore, $\big(R(M); \leq\big)$ is a lattice and for any $y, z \in R(M)$ it holds that

$$y \vee_{R(M)} z = y \vee z, \qquad y \wedge_{R(M)} z = \neg\sim(y \wedge z).$$

Let $w, y, z \in R(M)$. Then

$$\begin{aligned} w + (y \wedge_{R(M)} z) &= w + \neg\sim(y \wedge z) = \neg\sim w + \neg\sim(y \wedge z) = \neg\sim\big(w + (y \wedge z)\big) \\ &= \neg\sim\big((w + y) \wedge (w + z)\big) = (w + y) \wedge_{R(M)} (w + z). \end{aligned}$$

Similarly we can prove the distributivity from the right. Moreover, if $y, z \in R(M)$, then

$$y \rightharpoonup_{R(M)} z \qquad \text{and} \qquad y \leftharpoondown_{R(M)} z$$

exist and

$$y \rightharpoonup_{R(M)} z = y \rightharpoonup z \qquad \text{and} \qquad y \leftharpoondown_{R(M)} z = y \leftharpoondown z\,.$$

Thus $\big(R(M); +, 0, 1, \vee, \wedge_{R(M)}, \rightharpoonup, \leftharpoondown\big)$ is a bounded $DR\ell$-monoid. Since it is regular, it is induced by a GMV-algebra. □

Let M be a bounded $DR\ell$-monoid. Then an element $x \in M$ is called *dense* if $\neg{\sim}x = {\sim}\neg x = 0$. Denote by $D(M)$ the set of all dense elements in M.

Let us recall the notions of an ideal and a normal ideal of M. Let again M be a bounded $DR\ell$-monoid and $\emptyset \neq I \subseteq M$. Then I is called an *ideal* of M if

(a) $x, y \in I \implies x + y \in I$;

(b) $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

An ideal I is called *normal* if for any $x, y \in M$,

(c) $x \rightharpoonup y \in I \iff x \leftharpoondown y \in I$.

By [13], normal ideals of M are in a one-to-one correspondence with congruences on M. Namely, let I be a normal ideal of M. Then $\Theta(I)$, the congruence on M induced by I, is determined as follows: If $x, y \in M$, then

$$\langle x, y\rangle \in \Theta(I) \iff (x \rightharpoonup y) \vee (y \rightharpoonup x) \in I$$

(which is equivalent to $(x \leftharpoondown y) \vee (y \leftharpoondown x) \in I$).

Conversely, let Θ be a congruence on M. Then $I(\Theta) = [0]_\Theta = \{x \in M : \langle x, 0\rangle \in \Theta\}$ is the normal ideal of M corresponding to Θ.

THEOREM 10. *If M is a bounded good $DR\ell$-monoid, then $D(M)$ is a normal ideal of M and $M/D(M) \cong R(M)$.*

Proof. Let $x, y \in D(M)$. Then by Lemma 3(4), $\neg{\sim}(x+y) \leq \neg{\sim}x + \neg{\sim}y = 0$, thus $x + y \in D(M)$. If $x \in D(M)$, $z \in M$ and $z \leq x$, then $\neg{\sim}z \leq \neg{\sim}x = 0$, hence $z \in D(M)$. Therefore $D(M)$ is an ideal of M.

Further, if $x, y \in M$, then $x \rightharpoonup y \in D(M)$ iff $\neg{\sim}(x \rightharpoonup y) = 0$ iff (by Lemmas 3(3) and 1) $\neg{\sim}x \leftharpoondown \neg{\sim}y = 0$, hence again by Lemma 3(3) iff $\neg{\sim}(x \leftharpoondown y) = 0$, i.e. iff $x \leftharpoondown y \in D(M)$. Therefore the ideal $D(M)$ is normal.

Let us consider the congruence $\Theta\big(D(M)\big)$ induced by $D(M)$. That means, if $x, y \in M$, then $\langle x, y\rangle \in \Theta\big(D(M)\big)$ iff $(x \rightharpoonup y) \vee (y \rightharpoonup x) \in D(M)$, hence iff $\neg{\sim}\big((x \rightharpoonup y) \vee (y \rightharpoonup x)\big) = 0$, hence by Lemma 3(5) iff $\neg{\sim}(x \rightharpoonup y) \vee \neg{\sim}(y \rightharpoonup x) = 0$, and by Lemma 3(3) iff $(\neg{\sim}x \rightharpoonup \neg{\sim}y) \vee (\neg{\sim}y \rightharpoonup \neg{\sim}x) = 0$, and this holds iff $\neg{\sim}x \rightharpoonup \neg{\sim}y = 0 = \neg{\sim}y \rightharpoonup \neg{\sim}x$. By Lemma 1 it is equivalent to $\neg{\sim}x \leq \neg{\sim}y \leq \neg{\sim}x$, i.e. with $\neg{\sim}x = \neg{\sim}y$.

Therefore $M/D(M) \cong R(M)$. □

Remark. In an analogous theorem in [17], for a commutative bounded $DR\ell$-monoid M it was, moreover, supposed that M satisfies the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$. The proof of Theorem 10 shows that the mentioned assumption was superfluous.

A $DR\ell$-monoid M is called (*congruence*) *simple* if M is non-trivial and has no proper congruence different from the identity.

THEOREM 11. *If M is a bounded good $DR\ell$-monoid satisfying the identity $\neg{\sim}(x+y) = \neg{\sim}x + \neg{\sim}y$, then M is simple if and only if it is induced by a simple GMV-algebra.*

Proof. By Theorem 10, $D(M)$ is a normal ideal in M for any bounded good $DR\ell$-monoid M. Let M satisfy the identity $\neg{\sim}(x+y) = \neg{\sim}x + \neg{\sim}y$ and let M be simple. Then M has a unique proper normal ideal, hence $D(M) = \{0\}$. Therefore, by Theorem 9, M is induced by a GMV-algebra. □

Let M be a bounded $DR\ell$-monoid and I be a normal ideal in M. Then I is called a *GMV-ideal* if the $DR\ell$-monoid $M/\Theta(I)$ is induced by a GMV-algebra.

THEOREM 12. *Let M be a bounded good $DR\ell$-monoid satisfying the identity $\neg{\sim}(x+y) = \neg{\sim}x + \neg{\sim}y$ and I be a normal ideal in M. Then the following conditions are equivalent.*

(1) *I is a GMV-ideal.*
(2) *$x \rightharpoonup \neg{\sim}x \in I$ for each $x \in M$.*
(3) *$\neg{\sim}x \in I \implies x \in I$ for each $x \in M$.*
(4) *$D(M) \subseteq I$.*

Proof.

(1) $\iff$ (2): Since M is good, $M/\Theta(I)$ is induced by a GMV-algebra if and only if $\langle x, \neg{\sim}x\rangle \in \Theta(I)$ for each $x \in M$, i.e. if and only if $(x \rightharpoonup \neg{\sim}x) \vee (\neg{\sim}x \rightharpoonup x) \in I$ for each $x \in M$, and this is equivalent to $x \rightharpoonup \neg{\sim}x \in I$ for each $x \in M$.

(2) $\implies$ (3): Let $x \rightharpoonup \neg{\sim}x \in I$ and $\neg{\sim}x \in I$. Then $x = x \vee \neg{\sim}x = (x \rightharpoonup \neg{\sim}x) + \neg{\sim}x \in I$.

(3) $\implies$ (4): Let $y \in D(M)$. Then $\neg{\sim}y = 0 \in I$, hence $y \in I$. Therefore $D(M) \subseteq I$.

(4) $\implies$ (1): Let $D(M) \subseteq I$. Then $M/\Theta(I)$ is isomorphic to a subalgebra of $M/\Theta\big(D(M)\big)$ which is induced by a GMV-algebra. □

THEOREM 13. *Let M be a bounded good $DR\ell$-monoid satisfying the identity $\neg{\sim}(x+y) = \neg{\sim}x + \neg{\sim}y$. If I is a maximal ideal in M and is normal, then I is a GMV-ideal.*

Proof. Let I be a maximal and normal ideal in M. Then M/I is a simple $DR\ell$-monoid, thus by Theorem 11 we have that I is a GMV-ideal. □

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