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# INTEGRATION WITH RESPECT TO A $\oplus$-MEASURE

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In paper [5] the extension of $\sigma$-additive and $\sigma$-maxitive measures is performed simultaneously by help of some $\oplus$-measure. In this paper we show that one can perform simultaneously the integration theory as well as the product of $\sigma$-additive and $\sigma$-maxitive measures. Both $\sigma$-additive and $\sigma$-maxitive measures are so-called strong submeasures. For submeasures some more integrals are defined in literature (see [1], [3], [4], [7]). But none of these integrals fulfils the very natural requirement of $\sigma$-maxitive measures, that is $\int \sup(f, g) = \sup\{\int f, \int g\}$ for all non-negative functions $f$, $g$.

## Preliminary definitions and results

Let $\oplus$ be some binary operation on $\langle 0, \infty\rangle$ with the following properties:

1. $a \oplus b = b \oplus a$ for all $a, b \in \langle 0, \infty\rangle$
2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in \langle 0, \infty\rangle$
3. $k(a \oplus b) = ka \oplus kb$ for all $k > 0$, $a, b \in \langle 0, \infty\rangle$
4. $a \oplus 0 = a$, $a \oplus \infty = \infty$ for each $a \in \langle 0, \infty\rangle$
5. $a \leqq b \Rightarrow a \oplus c \leqq b \oplus c$ for all $a, b, c \in \langle 0, \infty\rangle$
6. $(a+b) \oplus (c+d) \leqq (a \oplus c) + (b \oplus d)$ for all $a, b, c, d \in \langle 0, \infty\rangle$
7. $a_n \to a$, $b_n \to b \Rightarrow a_n \oplus b_n \to a \oplus b$
   for all $a, b, a_n, b_n \in \langle 0, \infty\rangle$ $(n = 1, 2, \ldots)$.

We shall write briefly $\bigoplus_{i=1}^{n} a_i$ instead of $a_1 \oplus a_2 \oplus \ldots \oplus a_n$ and $\bigoplus_{i=1}^{\infty} a_i$ instead of $\sup_n \left(\bigoplus_{i=1}^{n} a_i\right)$.

Clearly the usual addition as well as the maximum of two real numbers fulfil the properties 1.—7.

**Definition 1.** *Let $(X, \mathcal{S})$ be a measurable space. A set function $m: \mathcal{S} \to \langle 0, \infty\rangle$ will be called a $\oplus$-measure if $m(\emptyset) = 0$ and $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigoplus_{i=1}^{\infty} m(E_i)$ for each sequence $\{E_i\}_{i=1}^{\infty}$ of mutually disjoint sets from $\mathcal{S}$.*

Clearly if $a \oplus b = a + b$ for all $a$, $b \in \langle 0, \infty \rangle$, the $\oplus$-measure becomes a $\sigma$-additive measure. If $a \oplus b = \max\{a, b\}$ for all $a$, $b \in \langle 0, \infty \rangle$, the $\oplus$-measure becomes a $\sigma$-maxitive measure (i.e. such a function $m: \mathscr{S} \to \langle 0, \infty \rangle$ that $m(\emptyset) = 0$ and $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_i m(E_i)$ for each sequence $\{E_i\}_{i=1}^{\infty}$ of mutually disjoint sets in $\mathscr{S}$).

It is easy to see that a $\oplus$-measure is $\oplus$-additive (i.e. $m(A \cup B) = m(A) \oplus m(B)$ for all $A, B \in \mathscr{S}$, $A \cap B = \emptyset$), monotone, $\oplus$-subadditive (i.e. $m(A \cup B) \leqq m(A) \oplus m(B)$ for all $A$, $B \in \mathscr{S}$) and continuous from below.

Let $m$ be a fixed $\oplus$-measure. First we define an integral with respect to $m$ for a non-negative simple function. Briefly for a *NSF*.

**Definition 2.** *Let* $(X, \mathscr{S}, m)$ *be a* $\oplus$*-measure space and let* $f$ *be a NSF,* $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ *where* $E_i \cap E_k = \emptyset$ for $i \neq k$, $0 < \alpha_i < \infty$. *We define* $\int f \, \mathrm{d}m = \bigoplus_{i=1}^{n} \alpha_i m(E_i)$ *and we say that* $f$ *is integrable iff* $\int f \, \mathrm{d}m < \infty$.

Clearly a *NSF* $f$ is integrable iff $m(N(f)) < \infty$ where $N(f) = \{x, f(x) \neq 0\}$.

We shall write $\int f$ in place of $\int f \, \mathrm{d}m$ since $m$ is fixed.

Remark. The definition 2 is correct by the distributivity of $\oplus$ and the $\oplus$-additivity of $m$.

**Proposition 1.** *Let* $f$, $g$ *be NSF-s on* $(X, \mathscr{S}, m)$ *such that* $f \leqq g$. *Then* $\int f \leqq \int g$.

Proof. $f$, $g$ are *NSF*-s, thus such mutually disjoint sets $E_i \in \mathscr{S}$ and numbers $0 \leqq \gamma_i \leqq \delta_i$ $(i = 1, 2, \ldots, k)$ exist that

$$f = \sum_{i=1}^{k} \gamma_i \chi_{E_i}, \quad g = \sum_{i=1}^{k} \delta_i \chi_{E_i} .$$

Then $\int f = \bigoplus_{i=1}^{k} \gamma_i m(E_i) \leqq \bigoplus_{i=1}^{k} \delta_i m(E_i) = \int g$.

**Proposition 2.** *Let* $f$, $g$ *be NSF-s on* $(X, \mathscr{S}, m)$. *Then* $\int f + g \leqq \int f + \int g$.

Proof. Take mutually disjoint sets $E_i \in \mathscr{S}$ and numbers $\gamma_i, \delta_i \geqq 0$ $(i = 1, 2, \ldots, k)$ such that $f = \sum_{i=1}^{k} \gamma_i \chi_{E_i}$, $g = \sum_{i=1}^{k} \delta_i \chi_{E_i}$. Then $\int f + g = \int \sum_{i=1}^{k} (\gamma_i + \delta_i) \chi_{E_i}$ $= \bigoplus_{i=1}^{k} (\gamma_i + \delta_i) m(E_i) \leqq \bigoplus_{i=1}^{k} \gamma_i m(E_i) + \bigoplus_{i=1}^{k} \delta_i m(E_i) = \int f + \int g$.

**Corollary.** *Let* $f$, $g$ *be integrable NSF-s on* $(X, \mathscr{S}, m)$. *Then* $|\int f - \int g| \leqq \int |f - g|$.

**Proposition 3.** *Let* $f$, $g$ *be such NSF-s that* $f \cdot g = 0$. *Then* $\int f + g = \int f \oplus \int g$.

Let $f$, $g$ be non-negative real functions on $X$. Let us define a function $f \oplus g$ as follows: $(f \oplus g)(x) = f(x) \oplus g(x)$ for all $x \in X$.

**Proposition 4.** *Let* $f$, $g$ *be NSF-s. Then the function* $f \oplus g$ *is a NSF and* $\int f \oplus g = \int f \oplus \int g$.

Proof. We can write $f = \sum_{i=1}^{k} \gamma_i \chi_{E_i}$, $g = \sum_{i=1}^{k} \delta_i \chi_{E_i}$ for suitable numbers $\gamma_i$, $\delta_i \geqq 0$ and mutually disjoint sets $E_i \in \mathcal{S}$ $(i=1, 2, \ldots, k)$. Then the function $f \oplus g = \sum_{i=1}^{k} (\gamma_i \oplus \delta_i)\chi_{E_i}$ is a *NSF*. $\int f \oplus g = \bigoplus_{i=1}^{k} (\gamma_i \oplus \delta_i) m(E_i) = \left(\bigoplus_{i=1}^{k} \gamma_i m(E_i)\right) \oplus \left(\bigoplus_{i=1}^{k} \delta_i m(E_i)\right) = \int f \oplus \int g$.

**Definition 3.** *Let $(X, \mathcal{S}, m)$ be a $\oplus$-measure space.*

*A) If $f: X \to \langle 0, \infty)$ is a measurable function, we put $\int f = \sup \{\int g: g \leqq f, g$ is a NSF$\}$ and we say that $f$ is integrable iff $\int f < \infty$.*

*B) If $f: X \to (-\infty, \infty)$ is measurable and at least one of the functions $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$ is integrable, we put $\int f = \int f^+ - \int f^-$ and we say that $f$ is integrable iff $-\infty < \int f < \infty$.*

Remarks. 1) A measurable function $f: X \to (-\infty, \infty)$ is integrable iff both $f^+, f^-$ are integrable.

2) For a *NSF* the definitions 2 and 3 do not differ.

3) If $m$ is a $\sigma$-additive measure, then integral from the definition 3 does not differ from the classical one (for definition see e.g. [2]).

4) For $\sigma$-maxitive measures N. Shilkret in [6] defined the integral of a non-negative measurable function as follows: $\int_{Sh} f \, dm = \sup_{a>0} a m\{x, f(x) \geqq a\}$. If a $\oplus$-measure $m$ is a $\sigma$-maxitive measure, we assert that $\int f = \int_{Sh} f$ for each non-negative measurable function $f$. Proof: Clearly $\int g = \int_{Sh} g$ for each *NSF* $g$. Let $f \geqq 0$ be measurable and denote $E_a = \{x, f(x) \geqq a\}$. Then $\int f = \sup \{\int g, g \leqq f$, g is a *NSF*$\} \geqq \sup_{a>0} \{\int a\chi_{E_a}\} = \int_{Sh} f$. On the other hand if $g \leqq f$, $g$ is a *NSF*, then $\int g = \int_{Sh} g \leqq \int_{Sh} f$, hence $\int f = \sup \{\int g, g \leqq f, g$ is a *NSF*$\} \leqq \int_{Sh} f$.

We leave the easy proof of the following theorem to the reader.

**Theorem 1.** *Let $f$, $g$, $h$ be measurable functions such that $\int f$, $\int g$, $\int h$ have a sense.* Then

1. $f \geqq 0 \Rightarrow \int f \geqq 0$
2. $f \leqq g \Rightarrow \int f \leqq \int g$
3. $f \leqq h \leqq g$, *$f$, $g$ are integrable $\Rightarrow$ $h$ is integrable*
4. *$f$ is integrable iff $|f|$ is integrable*
5. *Let $c \in (-\infty, \infty)$, $c \neq 0$. Then $f$ is integrable iff $cf$ is integrable and $\int cf = c \int f$.*

**Theorem 2.** *Let $f$ be a non-negative integrable function on $(X, \mathcal{S}, m)$. Let us define a set function $\nu_f: \mathcal{S} \to \langle 0, \infty)$ as follows: $\nu_f(E) = \int_E f = \int f\chi_E$ for each $E \in \mathcal{S}$. Then $\nu_f$ is a $\oplus$-measure on $\mathcal{S}$.*

Proof. It suffices to show that $\nu_f$ is $\oplus$-additive and continuous from below. First we show the $\oplus$-additivity. Let $A, B \in \mathcal{S}$, $A \cap B = \emptyset$ and $\varepsilon > 0$ be arbitrary. Then the

*NSF* $g \leqq f$ exists such that $v_f(A \cup B) - \varepsilon < v_g(A) \oplus v_g(B) \leqq v_f(A) \oplus v_f(B)$. Since $\varepsilon$ was arbitrary $v_f(A \cup B) \leqq v_f(A) \oplus v_f(B)$. On the other hand, for each $\varepsilon > 0$ the *NSF* $h \leqq f$ exists such that $v_f(A) \oplus v_f(B) \leqq \left(v_h(A) + \frac{\varepsilon}{2}\right) \oplus \left(v_h(B) + \frac{\varepsilon}{2}\right) \leqq (v_h(A) \oplus v_h(B)) + \left(\frac{\varepsilon}{2} \oplus \frac{\varepsilon}{2}\right) \leqq v_h(A \cup B) + \varepsilon \leqq v_f(A \cup B) + \varepsilon$. Since $\varepsilon$ was arbitrary the inequality $v_f(A) \oplus v_f(B) \leqq v_f(A \cup B)$ holds.

The proof of the continuity from below is realized in three steps. Let $E_i \in \mathscr{S}$ $(i = 1, 2, \ldots)$ be mutually disjoint.

1. First let $f = \alpha\chi_A$ for some $\alpha > 0$ and $A \in \mathscr{S}$. $v_f\left(\bigcup_{i=1}^{\infty} E_i\right) = \int \alpha\chi_{\bigcup_{i=1}^{\infty}(A \cap E_i)} = \alpha m\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) = \alpha \sup_n m\left(\bigcup_{i=1}^{n}(A \cap E_i)\right) = \sup_n \int \alpha\chi_{\bigcup_{i=1}^{n}(A \cap E_i)} = \sup_n v_f\left(\bigcup_{i=1}^{n} E_i\right)$.

2. Let $f = \sum_{i=1}^{k} \alpha_i\chi_{A_i}$ where $\alpha_i > 0$, $A_i \in \mathscr{S}$ are mutually disjoint $(i = 1, 2, \ldots, k)$. Let us denote $f_i = \alpha_i\chi_{A_i}$ $(i = 1, 2, \ldots, k)$. Then by the proposition 3 $v_f\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigoplus_{j=1}^{k} v_{f_j}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_n \left\{\bigoplus_{j=1}^{k} v_{f_j}\left(\bigcup_{i=1}^{n} E_i\right)\right\} = \sup_n \left\{v_f\left(\bigcup_{i=1}^{n} E_i\right)\right\}$.

3. Let $f$ be a non-negative integrable function and $\varepsilon > 0$ be arbitrary. Then the *NSF* $g \leqq f$ exists such that $v_f\left(\bigcup_{i=1}^{\infty} E_i\right) - \varepsilon < v_g\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_k v_g\left(\bigcup_{i=1}^{k} E_i\right) \leqq \sup_k v_f\left(\bigcup_{i=1}^{k} E_i\right) \leqq v_f\left(\bigcup_{i=1}^{\infty} E_i\right)$. Since $\varepsilon$ was arbitrary one has $v_f\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_k v_f\left(\bigcup_{i=1}^{k} E_i\right)$.

## Integration with respect to a continuous $\oplus$-measure

In this section we consider a fixed continuous $\oplus$-measure $m$ on a $\sigma$-ring $\mathscr{S}$ of subsets of $X \neq \emptyset$ (i.e. if $E_n$ is a decreasing sequence of sets in $\mathscr{S}$ with empty intersection and $m(E_k) < \infty$ for some $k$, then $\lim_{n \to \infty} m(E_n) = 0$).

**Theorem 3.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of bounded measurable functions such that $f_n \downarrow 0$. Let such an index $k$ exist that $m(N(f_k)) < \infty$. Then $\lim_{n \to \infty} \int f_n = 0$.*

Proof. Let $\varepsilon > 0$ be arbitrary. We put $E = N(f_k)$ and we assume $m(E) > 0$ (for $m(E) = 0$ the theorem is obvious). Let us denote $\varepsilon' = \frac{\varepsilon}{m(E)}$ and $E_n = \{x, f_n(x) \geqq \varepsilon'\}$ $(n = 1, 2, \ldots)$. $f_n \downarrow 0$ implies $E_n \downarrow \emptyset$ and by continuity of $m$ one has

$\lim_{n\to\infty} m(E_n)=0$. Let us denote $b=\max f_k$. Then for $n\geqq k$, $0\leqq\int f_n \leqq \int_{E_n} f_n + \int_{E-E_n} f_n \leqq bm(E_n) + \varepsilon' m(E-E_n) \leqq bm(E_n)+\varepsilon$. Hence $0\leqq\lim_{n\to\infty}\int f_n \leqq \lim_{n\to\infty}(bm(E_n)+\varepsilon)=\varepsilon$. $\varepsilon$ was arbitrary, thus $\lim_{n\to\infty}\int f_n=0$.

**Theorem 4.** *Let $f_n$, $f$ $(n=1, 2, \ldots)$ be integrable NSF-s such that $f_n\uparrow f$. Then $\lim_{n\to\infty}\int f_n=\int f$.*

Proof. The functions $f-f_n$ $(n=1, 2, \ldots)$ are bounded and $f-f_n\downarrow 0$. Since $m(N(f))<\infty$ one can apply the theorem 3. Hence $\lim_{n\to\infty}\int(f-f_n)=0$ and since $0\leqq\int f-\int f_n \leqq \int(f-f_n)$ for $n=1,2,\ldots$ one has $\lim_{n\to\infty}\int f_n=\int f$.

**Theorem 5.** *Let $f_n$, $f$ $(n=1, 2, \ldots)$ be NSF-s such that $f_n\uparrow f$ and $\lim_{n\to\infty}\int f_n<\infty$. Then $f$ is integrable.*

Proog. 1) First we assume $f=\chi_A$ for some $A\in\mathcal{S}$. We can suppose $f_1\neq 0$. Let us denote $\beta_n=\min f_n/N(f_n)$ for $n=1, 2, \ldots$. Then $\int f_n\geqq\beta_n m(N(f_n))\geqq\beta_1 m(N(f_n))$ and one has $m(N(f_n))\leqq\frac{1}{\beta_1}\int f_n$. Hence $m(A)=\lim_{n\to\infty} m(N(f_n))\leqq\frac{1}{\beta_1}\lim_{n\to\infty}\int f_n<\infty$.

2) Let $f=\sum_{i=1}^{k}\alpha_i\chi_{A_i}$ for some $\alpha_i\in(0, \infty)$, $A_i\in\mathcal{S}$ $(i=1, 2, \ldots, k)$ $A_i\cap A_j=\emptyset$ for $i\neq j$. Then $f_n\chi_{A_i}\uparrow\alpha_i\chi_{A_i}$ implies $0\leqq\frac{1}{\alpha_i}f_n\chi_{A_i}\uparrow\chi_{A_i}$ and $\lim_{n\to\infty}\int\frac{1}{\alpha_i}f_n\chi_{A_i}\leqq\lim_{n\to\infty}\frac{1}{\alpha_i}\int f_n<\infty$. Hence $m(A_i)<\infty$ for $i\in\{1, 2, \ldots, k\}$ and this implies $m(N(f))<\infty$, i.e. $f$ is an integrable function. Notice that we did not use the continuity of $m$.

**Theorem 6.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions such that $f_n\uparrow f$. Then $\int f=\lim_{n\to\infty}\int f_n$.*

Proof. If the $\lim_{n\to\infty}\int f_n=\infty$, the assertion is clear. Let the $\lim_{n\to\infty}\int f_n<\infty$ and for $n=1, 2, \ldots$ take a sequence $\{g_m^{(n)}\}_{m=1}^{\infty}$ of NSF-s such that $g_m^{(n)}\uparrow f_n$. Denote $h_n=\max\{g_n^{(1)}, g_n^{(2)}, \ldots, g_n^{(n)}\}$ for $n=1, 2, \ldots$. Then $h_n$ are NSF-s, $h_n\uparrow f$ and $\lim_{n\to\infty}\int h_n<\infty$. Let $g$ be NSF, $g\leqq f$. Denote $r_n=\min(h_n, g)\uparrow\min(f, g)=g$. Then $\int r_n\leqq\int h_n$ for $n=1, 2, \ldots$ thus $\lim_{n\to\infty}\int r_n\leqq\lim_{n\to\infty}\int h_n<\infty$. Hence $g$ is integrable by the theorem 5. Suppose $\int f=\infty$. Then NSF-s $p_m$ $(m=1, 2, \ldots)$ exist such that $p_m\leqq f$ and $\int p_m>m$. $p_m$ is integrable for $m=1, 2, \ldots$ and the $\lim_{m\to\infty}\int p_m=\infty$. Then

$s_n = \min(h_n, p_m) \uparrow \min(f, p_m) = p_m$ and by the theorem 4 $\int p_m = \lim_{n\to\infty} \int s_n \leqq \lim_{n\to\infty} \int h_n$.

Then also the $\lim_{m\to\infty} \int p_m \leqq \lim_{n\to\infty} \int h_n < \infty$, which is a contradiction. Thus $\int f < \infty$. Let $\varepsilon > 0$ be arbitrary. Then the *NSF* $t \leqq f$ exists such that $\int f - \varepsilon < \int t \leqq \int f$. Denote $t_n = \min(h_n, t) \uparrow \min(\mathrm{f}, \mathrm{t}) = \mathrm{t}$. Thus $\lim_{n\to\infty} \int t_n = \int t$ by the theorem 4. Hence $\int f - \varepsilon <$

$\int t = \lim_{n\to\infty} \int t_n \leqq \lim_{n\to\infty} \int h_n \leqq \lim_{n\to\infty} \int f_n \leqq \int f$. Since $\varepsilon$ was arbitrary $\int f = \lim_{n\to\infty} \int f_n$.

**Theorem 7.** *Let $f$, $g$ be non-negative measurable functions on $(X, \mathcal{S}, m)$. Then $\int f \oplus g = \int f \oplus \int g$.*

Proof. Take *NSF*-s $f_n$, $g_n$ $(n = 1, 2, \ldots)$ such that $f_n \uparrow f$, $g_n \uparrow g$. Then $f_n \oplus g_n \uparrow f \oplus g$ and by the theorem 6 and the proposition 4 $\int f \oplus g = \sup_n \int f_n \oplus g_n$

$= \sup_n \int f_n \oplus \int g_n = \sup_n \int f_n \oplus \sup_n \int g_n = \int f \oplus \int g$.

## Product of $\oplus$-measures

Let $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \nu)$ be measurable spaces with finite and continuous $\oplus$-measures $\mu$, resp. $\nu$. Let $\mathcal{R}$ be a ring of all finite disjoint unions $M = \bigcup_{i=1}^{n} (A_i \times B_i)$ where $A_i \in \mathcal{S}$, $B_i \in \mathcal{T}$ $(i = 1, 2, \ldots, n)$ and denote by $\mathcal{S} \times \mathcal{T}$ the $\sigma$-ring generated by $\mathcal{R}$. Let $M \in \mathcal{S} \times \mathcal{T}$. For each $x \in X$, $y \in Y$ define sections $M_x = \{y \in Y, (x, y) \in M\}$, $M^y = \{x \in X, (x, y) \in M\}$. Then $M_x \in \mathcal{T}$, $M^y \in \mathcal{S}$. Further define functions $f_M$: $X \to \langle 0, \infty)$, $g^M$: $Y \to \langle 0, \infty)$ as follows: $f_M(x) = \nu(M_x)$, $g^M(y) = \mu(M^y)$.

**Lemma.** *Let $M \in \mathcal{S} \times \mathcal{T}$. Then the functions $f_M$, $g^M$ are non-negative measurable.*

Proof. Let $M \in \mathcal{R}$, $M = \bigcup_{i=1}^{n} (A_i \times B_i)$ where $A_i \in \mathcal{S}$, $B_i \in \mathcal{T}$ and $A_i \times B_i$ are mutually disjoint $(i = 1, 2, \ldots, n)$. For all $x \in X$ $f_M(x) = \nu\left(\bigcup_{i=1}^{n} (A_i \times B_i)_x\right)$

$= \bigoplus_{i=1}^{n} \nu(A_i \times B_i)_x$. Hence $f_M = \bigoplus_{i=1}^{n} \nu(B_i)\chi_{A_i}$. By the propositon 4 $f_M$ is a *NSF* and hence is measurable. Similary $g^M$ is a *NSF*. Let $\mathcal{M}$ be a class of all $M \in \mathcal{S} \times \mathcal{T}$ such that both $f_M$, $g^M$ are measurable. Then $\mathcal{R} \subset \mathcal{M}$. By continuity of $\mu$ and $\nu$, $\mathcal{M}$ is a monotone class and hence $\mathcal{S} \times \mathcal{T} \subset \mathcal{M}$.

Remark. It is not difficult to see that for $M \in \mathcal{S} \times \mathcal{T}$ the functions $f_M$, $g^M$ are integrable.

Let us define real functions $\varphi$, $\psi$ on $\mathcal{S} \times \mathcal{T}$ as follows: $\varphi(M) = \int f_M d\mu$, $\psi(M) = \int g^M d\nu$ for all $M \in \mathcal{S} \times \mathcal{T}$.

**Theorem 8.** *The functions $\varphi$, $\psi$ are finite and continuous $\oplus$-measures.*

Proof. Clearly $\varphi$ is finite and $\varphi(\emptyset)=0$. Let $M, N \in \mathcal{S} \times \mathcal{T}$, $M \cap N = \emptyset$. Then $\varphi(M \cup N) = \int f_{M \cup N} \, d\mu = \int (f_M \oplus f_N) \, d\mu = \int f_M \, d\mu \oplus \int f_N \, d\mu = \varphi(M) \oplus \varphi(N)$. Let $M_n \downarrow \emptyset$, $M_n \in \mathcal{S} \times \mathcal{T}$ $(n=1, 2, \ldots)$. For all $x \in X$ $(M_n)_x \downarrow \emptyset$ and by continuity of $\nu$ $\lim_{n\to\infty} \nu((M_n)_x) = 0$. Hence $f_{M_n} \downarrow 0$ and by the theorem 3 $\lim_{n\to\infty} \varphi(M_n) = 0$. Thus $\varphi$ is continuous. Let $E_n \in \mathcal{S} \times \mathcal{T}$ $(n=1, 2, \ldots)$ are mutually disjoint. Put $E = \bigcup_{n=1}^{\infty} E_n$ and $F_n = E - \bigcup_{i=1}^{n} E_i$ $(n=1, 2, \ldots)$. Then $F_n \downarrow \emptyset$ and hence $\lim_{n\to\infty} \varphi(F_n) = 0$. $\varphi(E) = \varphi\left(\bigcup_{i=1}^{n} E_i\right) \oplus \varphi(F_n)$. Hence $\varphi(E) = \lim_{n\to\infty} \varphi\left(\bigcup_{i=1}^{n} E_i\right) = \bigoplus_{n=1}^{\infty} \varphi(E_n)$. Hence $\varphi$ is a $\oplus$-measure. For $\psi$ the proof is dual.

**Theorem 9.** *Let $M \in \mathcal{S} \times \mathcal{T}$. Then $\varphi(M) = \psi(M)$.*

Proof. Let $M \in \mathcal{R}$, $M = \bigcup_{i=1}^{n} (A_i \times B_i)$ where $A_i \in \mathcal{S}$, $B_i \in \mathcal{T}$, $A_i \times B_i$ are mutually disjoint $(i=1, 2, \ldots, n)$. Then $\int f_M \, d\mu = \int \bigoplus_{i=1}^{n} \nu(B_i) \chi_{A_i} \, d\mu = \bigoplus_{i=1}^{n} \int \nu(B_i) \chi_{A_i} = \bigoplus_{i=1}^{n} \mu(A_i) \nu(B_i) = \int \bigoplus_{i=1}^{n} \mu(A_i) \chi_{B_i} \, d\nu = \int g^M \, d\nu$. Thus $\varphi(M) = \psi(M)$ on $\mathcal{R}$. Let $\mathcal{M}$ be a class of all sets $M \in \mathcal{S} \times \mathcal{T}$ such that $\varphi(M) = \psi(M)$. Then $\mathcal{M}$ is a monotone class by the continuity of $\varphi$, resp. $\psi$, and $\mathcal{R} \subset \mathcal{M}$. Thus $\mathcal{S} \times \mathcal{T} \subset \mathcal{M}$.

We shall write $\mu \times \nu$ for a function $\varphi$ and we shall call it a product of $\oplus$-measures $\mu$, $\nu$.

Let $h$ be a real function on $X \times Y$. For all $x \in X$, $y \in Y$ let us define real functions $h_x$, $h^y$ on $Y$, resp. $X$, in the following way: $h_x(y) = h(x, y)$, $h^y(x) = h(x, y)$.

**Theorem 10.** *Let $h\colon X \times Y \to \langle 0, \infty)$ be an integrable function. Then the functions $f\colon X \to \langle 0, \infty)$, $g\colon Y \to \langle 0, \infty)$ defined as follows: $f(x) = \int h_x \, d\nu$, $g(y) = \int h^y \, d\mu$ are integrable and moreover $\int h \, d\mu \times \nu = \int f \, d\mu = \int g \, d\nu$.*

Proof. 1) First let $h = \chi_E$, $E \in \mathcal{S} \times \mathcal{T}$. Then $h_x = \chi_{E_x}$ and $f(x) = \int h_x \, d\nu = \nu(E_x) = f_E(x)$. Thus $\int h \, d\mu \times \nu = \mu \times \nu(E) = \int f_E \, d\mu = \int f \, d\mu$.

2) Let $h$ be a *NSF* on $X \times Y$. Then $\int h \, d\mu \times \nu = \bigoplus_{i=1}^{n} \alpha_i \mu \times \nu(E_i) = \bigoplus_{i=1}^{n} \alpha_i \int \chi_{E_i} \, d\mu \times \nu = \bigoplus_{i=1}^{n} \alpha_i \int (\int \chi_{(E_i)_x} \, d\nu) \, d\mu = \bigoplus_{i=1}^{n} \int \alpha_i \nu((E_i)_x) \, d\mu = \int \bigoplus_{i=1}^{n} \alpha_i \nu((E_i)_x) \, d\mu = \int (\int h_x \, d\nu) \, d\mu = \int f \, d\mu$.

3) Let $h$ be an arbitrary non-negative function on $X \times Y$. Take *NSF*-s $h_n$ $(n=1, 2, \ldots)$ such that $h_n \uparrow h$ and denote $f_n(x) = \int (h_n)_x \, d\nu$ for all $x \in X$, $(n = 1, 2, \ldots)$. The functions $f_n$ $(n=1, 2, \ldots)$ are $\mu$-measurable, thus the $\lim_{n\to\infty} f_n$ is $\mu$-measurable. By the theorem 6 $\int h \, d\mu \times \nu = \lim_{n\to\infty} \int h_n \, d\mu \times \nu = \lim_{n\to\infty} \int f_n \, d\mu$

$= \int \lim_{n\to\infty} f_n \, d\mu = \int \left( \lim_{n\to\infty} \int (h_n)_x \, d\nu \right) d\mu = \int \left( \int \lim_{n\to\infty} (h_n)_x \, d\nu \right) d\mu = \int (\int h_x \, d\nu) \, d\mu$
$= \int f \, d\mu.$

The function $f$ is integrable since $h$ is integrable. By the same arguments one can prove that $g$ is integrable and $\int g \, d\nu = \int h \, d\mu \times \nu$.

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## ИНТЕГРИРОВАНИЕ ПО ⊕-МЕРЕ

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Резюме

В статье показано, что как интегрирование $\sigma$-аддитивных мер и $\sigma$-макситивных мер, так и произведение этих мер можно рассматривать одновременно.