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INTEGRATION WITH RESPECT TO A ⊕-MEASURE

IVICA MARINOVÁ

In paper [5] the extension of σ-additive and σ-maxitive measures is performed simultaneously by help of some ⊕-measure. In this paper we show that one can perform simultaneously the integration theory as well as the product of σ-additive and σ-maxitive measures. Both σ-additive and σ-maxitive measures are so-called strong submeasures. For submeasures some more integrals are defined in literature (see [1], [3], [4], [7]). But none of these integrals fulfils the very natural requirement of σ-maxitive measures, that is \( \sup (\int f, \int g) = \int (\sup f, \sup g) \) for all non-negative functions \( f, g \).

Preliminary definitions and results

Let \( \oplus \) be some binary operation on \( (0, \infty) \) with the following properties:

1. \( a \oplus b = b \oplus a \) for all \( a, b \in (0, \infty) \)
2. \( (a \oplus b) \oplus c = a \oplus (b \oplus c) \) for all \( a, b, c \in (0, \infty) \)
3. \( k(a \oplus b) = ka \oplus kb \) for all \( k > 0, a, b \in (0, \infty) \)
4. \( a \oplus 0 = a, a \oplus \infty = \infty \) for each \( a \in (0, \infty) \)
5. \( a \leq b \Rightarrow a \oplus c \leq b \oplus c \) for all \( a, b, c \in (0, \infty) \)
6. \( (a + b) \oplus (c + d) \leq (a \oplus c) + (b \oplus d) \) for all \( a, b, c, d \in (0, \infty) \)
7. \( a_n \to a, b_n \to b \Rightarrow a_n \oplus b_n \to a \oplus b \)

for all \( a, b, a_n, b_n \in (0, \infty) \) \( (n = 1, 2, \ldots) \).

We shall write briefly \( \bigoplus_{i=1}^n a_i \) instead of \( a_1 \oplus a_2 \oplus \ldots \oplus a_n \) and \( \bigoplus a_i \) instead of \( \sup_{n} \left( \bigoplus_{i=1}^n a_i \right) \).

Clearly the usual addition as well as the maximum of two real numbers fulfil the properties 1.—7.

Definition 1. Let \( (X, \mathcal{F}) \) be a measurable space. A set function \( m: \mathcal{F} \to (0, \infty) \) will be called a ⊕-measure if \( m(\emptyset) = 0 \) and \( m \left( \bigcup_{i=1}^n E_i \right) = \bigoplus m(E_i) \) for each sequence \( \{E_i\}_{i=1}^n \) of mutually disjoint sets from \( \mathcal{F} \).
Clearly if $a \oplus b = a + b$ for all $a, b \in (0, \infty)$, the $\oplus$-measure becomes a $\sigma$-additive measure. If $a \oplus b = \max\{a, b\}$ for all $a, b \in (0, \infty)$, the $\oplus$-measure becomes a $\sigma$-maxitive measure (i.e. such a function $m: \mathcal{F} \to (0, \infty)$ that $m(\emptyset) = 0$ and $m\left(\bigcup_{i=1}^{n} E_i\right) = \sup_{i=1}^{n} m(E_i)$ for each sequence $\{E_i\}_{i=1}^{n}$ of mutually disjoint sets in $\mathcal{F}$).

It is easy to see that a $\oplus$-measure is $\oplus$-additive (i.e. $m(A \cup B) = m(A) \oplus m(B)$ for all $A, B \in \mathcal{F}$, $A \cap B = \emptyset$), monotone, $\oplus$-subadditive (i.e. $m(A \cup B) \leq m(A) \oplus m(B)$ for all $A, B \in \mathcal{F}$) and continuous from below.

Let $m$ be a fixed $\oplus$-measure. First we define an integral with respect to $m$ for a non-negative simple function. Briefly for a NSF.

**Definition 2.** Let $(X, \mathcal{F}, m)$ be a $\oplus$-measure space and let $f$ be a NSF, $f = \sum_{i=1}^{n} a_i \chi_{E_i}$, where $E_i \cap E_k = \emptyset$ for $i \neq k$, $0 < a_i < \infty$. We define $\int f \, dm = \sum_{i=1}^{n} a_i m(E_i)$ and we say that $f$ is integrable iff $\int f \, dm < \infty$.

Clearly a NSF $f$ is integrable iff $m(N(f)) < \infty$ where $N(f) = \{x, f(x) \neq 0\}$.

We shall write $\int f$ in place of $\int f \, dm$ since $m$ is fixed.

**Remark.** The definition 2 is correct by the distributivity of $\oplus$ and the $\oplus$-additivity of $m$.

**Proposition 1.** Let $f, g$ be NSF-s on $(X, \mathcal{F}, m)$ such that $f \leq g$. Then $\int f \leq \int g$.

**Proof.** $f, g$ are NSF-s, thus such mutually disjoint sets $E_i \in \mathcal{F}$ and numbers $0 \leq \gamma_i \leq \delta_i$ ($i = 1, 2, \ldots, k$) exist that

$$f = \sum_{i=1}^{k} \gamma_i \chi_{E_i}, \quad g = \sum_{i=1}^{k} \delta_i \chi_{E_i}.$$  

Then $\int f = \sum_{i=1}^{k} \gamma_i m(E_i) \leq \sum_{i=1}^{k} \delta_i m(E_i) = \int g$.

**Proposition 2.** Let $f, g$ be NSF-s on $(X, \mathcal{F}, m)$. Then $\int f + g \leq \int f + \int g$.

**Proof.** Take mutually disjoint sets $E_i \in \mathcal{F}$ and numbers $\gamma_i, \delta_i \geq 0$ ($i = 1, 2, \ldots, k$) such that $f = \sum_{i=1}^{k} \gamma_i \chi_{E_i}, \quad g = \sum_{i=1}^{k} \delta_i \chi_{E_i}$. Then $\int f + g = \sum_{i=1}^{k} (\gamma_i + \delta_i) \chi_{E_i}$

$$= \sum_{i=1}^{k} (\gamma_i + \delta_i) m(E_i) \leq \sum_{i=1}^{k} \gamma_i m(E_i) + \sum_{i=1}^{k} \delta_i m(E_i) = \int f + \int g.$$

**Corollary.** Let $f, g$ be integrable NSF-s on $(X, \mathcal{F}, m)$. Then $|\int f - \int g| \leq \int |f - g|$.

**Proposition 3.** Let $f, g$ be such NSF-s that $f \cdot g = 0$. Then $\int f + g = \int f \oplus \int g$.

Let $f, g$ be non-negative real functions on $X$. Let us define a function $f \oplus g$ as follows: $(f \oplus g)(x) = f(x) \oplus g(x)$ for all $x \in X$.

**Proposition 4.** Let $f, g$ be NSF-s. Then the function $f \oplus g$ is a NSF and $\int f \oplus g = \int f \oplus \int g$.

**Proof.** We can write $f = \sum_{i=1}^{k} \gamma_i \chi_{E_i}, \quad g = \sum_{i=1}^{k} \delta_i \chi_{E_i}$ for suitable numbers $\gamma_i, \delta_i \geq 0$ and
mutually disjoint sets $E_i \in \mathcal{S}$ ($i = 1, 2, \ldots, k$). Then the function $f \oplus g = \sum_{i=1}^{k} (\gamma_i \oplus \delta_i) \chi_{E_i}$ is a NSF. 

$$\int f \oplus g = \sum_{i=1}^{k} (\gamma_i \oplus \delta_i) m(E_i) = \left( \sum_{i=1}^{k} \gamma_i m(E_i) \right) \oplus \left( \sum_{i=1}^{k} \delta_i m(E_i) \right) = \int f \oplus \int g.$$ 

**Definition 3.** Let $(X, \mathcal{S}, m)$ be a $\oplus$-measure space.

A) If $f: X \to (0, \infty)$ is a measurable function, we put $\int f = \sup \{ \int g : g \leq f, g$ is a NSF$\}$ and we say that $f$ is integrable iff $\int f < \infty$.

B) If $f: X \to (-\infty, \infty)$ is measurable and at least one of the functions $f^+ = \max(f, 0), f^- = -\min(f, 0)$ is integrable, we put $\int f = \int f^+ - \int f^-$ and we say that $f$ is integrable iff $-\infty < \int f < \infty$.

Remarks. 1) A measurable function $f: X \to (-\infty, \infty)$ is integrable iff both $f^+, f^-$ are integrable.

2) For a NSF the definitions 2 and 3 do not differ.

3) If $m$ is a $\sigma$-additive measure, then integral from the definition 3 does not differ from the classical one (for definition see e.g. [2]).

4) For $\sigma$-maxitive measures N. Shiokret in [6] defined the integral of a non-negative measurable function as follows: $\int f \, dm = \sup_{a>0} am \{ x, f(x) \geq a \}$. If a $\oplus$-measure $m$ is a $\sigma$-maxitive measure, we assert that $\int f = \int f \, dm$ for each non-negative measurable function $f$. Proof: Clearly $\int g = \int f \, dm$ for each NSF $g$.

Let $f \geq 0$ be measurable and denote $E_a = \{ x, f(x) \geq a \}$. Then $\int f = \sup \{ \int g, g \leq f, g$ is a NSF$\} = \sup_{a>0} (\int a \chi_{E_a}) = \int f$. On the other hand if $g \leq f, g$ is a NSF, then

$$\int g = \int_g \leq \int f, \text{ hence } \int f = \sup \{ \int g, g \leq f, g \text{ is a NSF} \} \leq \int f.$$ 

We leave the easy proof of the following theorem to the reader.

**Theorem 1.** Let $f$, $g$, $h$ be measurable functions such that $\int f$, $\int g$, $\int h$ have a sense. Then

1. $f \geq 0 \implies \int f \geq 0$
2. $f \leq g \implies \int f \leq \int g$
3. $f \leq h \leq g, f, g$ are integrable $\implies h$ is integrable
4. $f$ is integrable iff $|f|$ is integrable
5. Let $c \in (-\infty, \infty), c \neq 0$. Then $f$ is integrable iff $cf$ is integrable and $\int cf = c \int f$.

**Theorem 2.** Let $f$ be a non-negative integrable function on $(X, \mathcal{S}, m)$. Let us define a set function $\nu_f: \mathcal{S} \to (0, \infty)$ as follows: $\nu_f(E) = \int f \chi_E$ for each $E \in \mathcal{S}$.

Then $\nu_f$ is a $\oplus$-measure on $\mathcal{S}$.

Proof. It suffices to show that $\nu_f$ is $\oplus$-additive and continuous from below. First we show the $\oplus$-additivity. Let $A, B \in \mathcal{S}, A \cap B = \emptyset$ and $\varepsilon > 0$ be arbitrary. Then the
NSF $g \leq f$ exists such that $v_f(A \cup B) - \varepsilon < v_g(A) \oplus v_g(B) \leq v_f(A) \oplus v_f(B)$. Since $\varepsilon$ was arbitrary $v_f(A \cup B) \leq v_f(A) \oplus v_f(B)$. On the other hand, for each $\varepsilon > 0$ the NSF $h \leq f$ exists such that $v_f(A) \oplus v_f(B) \leq \left(v_h(A) + \frac{\varepsilon}{2}\right) \oplus \left(v_h(B) + \frac{\varepsilon}{2}\right) \leq \left(v_h(A) \oplus v_h(B)\right) + \left(\frac{\varepsilon}{2} \oplus \frac{\varepsilon}{2}\right) \leq v_f(A \cup B) + \varepsilon \leq v_f(A \cup B) + \varepsilon$. Since $\varepsilon$ was arbitrary the inequality $v_f(A) \oplus v_f(B) \leq v_f(A \cup B)$ holds.

The proof of the continuity from below is realized in three steps. Let $E_i \in \mathcal{F}$ ($i = 1, 2, \ldots$) be mutually disjoint.

1. First let $f = \alpha \chi_A$ for some $\alpha > 0$ and $A \in \mathcal{F}$. $v_f\left(\bigcup_{i=1}^{\infty} E_i\right) = \int \alpha \chi_{A \cap E_i} = \alpha m\left(\bigcup_{i=1}^{\infty} (A \cap E_i)\right) = \sup_n \int \alpha \chi_{\bigcup_{i=1}^{n} (A \cap E_i)} = \sup_n v_f\left(\bigcup_{i=1}^{n} E_i\right)$.

2. Let $f = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$, where $\alpha_i > 0$, $A_i \in \mathcal{F}$ are mutually disjoint ($i = 1, 2, \ldots, k$). Let us denote $f_i = \alpha_i \chi_{A_i}$, ($i = 1, 2, \ldots, k$). Then by the proposition 3 $v_f\left(\bigcup_{i=1}^{k} E_i\right) = \bigoplus_{i=1}^{k} v_{f_i}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{n} \left\{ \bigoplus_{i=1}^{k} v_{f_i}\left(\bigcup_{i=1}^{n} E_i\right) \right\} = \sup_{n} \left\{ v_f\left(\bigcup_{i=1}^{n} E_i\right) \right\}$.

3. Let $f$ be a non-negative integrable function and $\varepsilon > 0$ be arbitrary. Then the NSF $g \leq f$ exists such that $v_f\left(\bigcup_{i=1}^{k} E_i\right) - \varepsilon < v_g\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{k} v_{f_k}\left(\bigcup_{i=1}^{k} E_i\right) \leq \sup_{k} v_{f_k}\left(\bigcup_{i=1}^{k} E_i\right) \leq v_f\left(\bigcup_{i=1}^{k} E_i\right)$. Since $\varepsilon$ was arbitrary one has $v_f\left(\bigcup_{i=1}^{k} E_i\right) = \sup_{k} v_{f_k}\left(\bigcup_{i=1}^{k} E_i\right)$.

**Integration with respect to a continuous $\oplus$-measure**

In this section we consider a fixed continuous $\oplus$-measure $m$ on a $\sigma$-ring $\mathcal{F}$ of subsets of $X \neq \emptyset$ (i.e. if $E_n$ is a decreasing sequence of sets in $\mathcal{F}$ with empty intersection and $m(E_k) < \infty$ for some $k$, then $\lim_{n \to \infty} m(E_n) = 0$).

**Theorem 3.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of bounded measurable functions such that $f_n \downarrow 0$. Let such an index $k$ exist that $m(N(f_k)) < \infty$. Then $\lim_{n \to \infty} \int f_n = 0$.

**Proof.** Let $\varepsilon > 0$ be arbitrary. We put $E = N(f_k)$ and we assume $m(E) > 0$ (for $m(E) = 0$ the theorem is obvious). Let us denote $\varepsilon' = \frac{\varepsilon}{m(E)}$ and $E_n = \{x, f_n(x) \geq \varepsilon'\}$ ($n = 1, 2, \ldots$). $f_n \downarrow 0$ implies $E_n \downarrow \emptyset$ and by continuity of $m$ one has
\[ \lim_{n \to \infty} m(E_n) = 0. \] Let us denote \( b = \max_k f_k \). Then for \( n \geq k \), \( 0 \leq f_n \leq \int_{E_n} f_n + \int_{E-E_n} f_n \leq bm(E_n) + \varepsilon' m(E-E_n) \leq bm(E_n) + \varepsilon. \) Hence \( 0 \leq \lim_{n \to \infty} \int f_n \leq \lim_{n \to \infty} (bm(E_n) + \varepsilon) = \varepsilon. \) \( \varepsilon \) was arbitrary, thus \( \lim_{n \to \infty} \int f_n = 0. \)

**Theorem 4.** Let \( f_n, f \) \( (n = 1, 2, ...) \) be integrable NSF-s such that \( f_n \uparrow f \). Then \( \lim_{n \to \infty} \int f_n = \int f. \)

**Proof.** The functions \( f - f_n \) \( (n = 1, 2, ...) \) are bounded and \( f - f_n \downarrow 0. \) Since \( m(N(f)) < \infty \) one can apply the theorem 3. Hence \( \lim_{n \to \infty} \int (f - f_n) = 0 \) and since \( 0 \leq \int f - \int f_n \leq \int (f - f_n) \) for \( n = 1, 2, ... \) one has \( \lim_{n \to \infty} \int f_n = \int f. \)

**Theorem 5.** Let \( f_n, f \) \( (n = 1, 2, ...) \) be NSF-s such that \( f_n \uparrow f \) and \( \lim_{n \to \infty} \int f_n < \infty. \) Then \( f \) is integrable.

**Proof.** 1) First we assume \( f = \chi_A \) for some \( A \in \mathcal{F}. \) We can suppose \( f \neq 0. \) Let us denote \( \beta_n = \min f_n / N(f_n) \) for \( n = 1, 2, ... \). Then \( \int f_n \geq \beta_n m(N(f_n)) \geq \beta_1 m(N(f_n)) \) and one has \( m(N(f_n)) \leq \frac{1}{\beta_1} \int f_n. \) Hence \( m(A) = \lim_{n \to \infty} m(N(f_n)) \leq \frac{1}{\beta_1} \lim_{n \to \infty} \int f_n < \infty. \)

2) Let \( f = \sum_{i=1}^{k} \alpha_i \chi_{A_i} \) for some \( \alpha_i \in (0, \infty), \) \( A_i \in \mathcal{F} \) \( (i = 1, 2, ..., k) \) \( A_i \cap A_j = \emptyset \) for \( i \neq j. \) Then \( f_n \chi_{A_i} \uparrow \alpha_i \chi_{A_i} \) implies \( 0 \leq \frac{1}{\alpha_i} f_n \chi_{A_i} \uparrow \chi_{A_i} \) and \( \lim_{n \to \infty} \int \frac{1}{\alpha_i} f_n \chi_{A_i} \leq \lim_{n \to \infty} \frac{1}{\alpha_i} \int f_n < \infty. \) Hence \( m(A_i) < \infty \) for \( i \in \{1, 2, ..., k\} \) and this implies \( m(N(f)) < \infty, \) i.e. \( f \) is an integrable function. Notice that we did not use the continuity of \( m. \)

**Theorem 6.** Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of non-negative measurable functions such that \( f_n \uparrow f. \) Then \( \int f = \lim_{n \to \infty} \int f_n. \)

**Proof.** If the limit \( \lim_{n \to \infty} \int f_n = \infty, \) the assertion is clear. Let the limit \( \lim_{n \to \infty} \int f_n < \infty \) and for \( n = 1, 2, ... \) take a sequence \( \{g^{(n)}\}_{m=1}^{\infty} \) of NSF-s such that \( g^{(n)} \uparrow f_n. \) Denote \( h_n = \max \{g^{(n)}(1), g^{(n)}(2), ..., g^{(n)}(m)\} \) for \( n = 1, 2, ... \). Then \( h_n \) are NSF-s, \( h_n \uparrow f \) and \( \lim_{n \to \infty} \int h_n < \infty. \) Let \( g \) be NSF, \( g \leq f. \) Denote \( r_n = \min (h_n, g) \uparrow \min (f, g) = g. \) Then \( \int r_n \leq \int h_n \) for \( n = 1, 2, ... \) thus \( \lim_{n \to \infty} \int r_n \leq \lim_{n \to \infty} \int h_n < \infty. \) Hence \( g \) is integrable by the theorem 5. Suppose \( \int f = \infty. \) Then NSF-s \( p_m \) \( (m = 1, 2, ...) \) exist such that \( p_m \leq f \) and \( \int p_m > m. \) \( p_m \) is integrable for \( m = 1, 2, ... \) and the \( \lim_{m \to \infty} \int p_m = \infty. \) Then
Then also the \( \lim_{n \to \infty} \int p_m \leq \lim_{n \to \infty} \int h_n < \infty \), which is a contradiction. Thus \( \int f < \infty \). Let \( \varepsilon > 0 \) be arbitrary. Then the NSF \( t \leq f \) exists such that \( \int f - \varepsilon < \int t \leq \int f \). Denote \( t_n = \min \left( h_n, t \right) \). Then also the \( \lim_{n \to \infty} t_n = t \) by the theorem 4. Hence \( \int f - \varepsilon < \int t = \lim_{n \to \infty} \int t_n \leq \lim_{n \to \infty} \int h_n \leq \lim_{n \to \infty} \int f_n \leq \int f \). Since \( \varepsilon \) was arbitrary \( \int f = \lim_{n \to \infty} \int f_n \).

**Theorem 7.** Let \( f, g \) be non-negative measurable functions on \((X, \mathcal{F}, m)\). Then \( \int f \oplus g = \int f \oplus \int g \).

**Proof.** Take NSF-s \( f_n, g_n \) (\( n = 1, 2, \ldots \)) such that \( f_n \uparrow f, g_n \uparrow g \). Then \( f_n \oplus g_n \uparrow f \oplus g \) and by the theorem 6 and the proposition 4 \( \int f \oplus = \sup_n \int f_n \oplus g_n = \sup_n \int f_n \oplus \sup_n \int g_n = \int f \oplus \int g \).

### Product of \( \oplus \)-measures

Let \((X, \mathcal{F}, \mu), (Y, \mathcal{T}, \nu)\) be measurable spaces with finite and continuous \( \oplus \)-measures \( \mu, \nu \). Let \( \mathcal{R} \) be a ring of all finite disjoint unions \( M = \bigcup_{i=1}^{n} (A_i \times B_i) \) where \( A_i \in \mathcal{F}, B_i \in \mathcal{T} \) \( i = 1, 2, \ldots, n \) and denote by \( \mathcal{F} \times \mathcal{T} \) the \( \sigma \)-ring generated by \( \mathcal{R} \). Let \( M \in \mathcal{F} \times \mathcal{T} \). For each \( x \in X, y \in Y \) define sections \( M_x = \{ y \in Y, (x, y) \in M \} \), \( M'_y = \{ x \in X, (x, y) \in M \} \). Then \( M_x, M'_y \in \mathcal{F} \). Further define functions \( f_M : X \to (0, \infty), g_M : Y \to (0, \infty) \) as follows: \( f_M(x) = \nu(M_x), g_M(y) = \mu(M'_y) \).

**Lemma.** Let \( M \in \mathcal{F} \times \mathcal{T} \). Then the functions \( f_M, g_M \) are non-negative measurable.

**Proof.** Let \( M \in \mathcal{R}, M = \bigcup_{i=1}^{n} (A_i \times B_i) \) where \( A_i \in \mathcal{F}, B_i \in \mathcal{T} \) and \( A_i \times B_i \) are mutually disjoint \( i = 1, 2, \ldots, n \). For all \( x \in X \) \( f_M(x) = \nu \left( \bigcup_{i=1}^{n} (A_i \times B_i) \right) = \bigoplus_{i=1}^{n} \nu(A_i \times B_i) \). Hence \( f_M = \bigoplus_{i=1}^{n} \nu(B_i) \chi_{A_i} \). By the proposition 4 \( f_M \) is a NSF and hence is measurable. Similar \( g_M \) is a NSF. Let \( \mathcal{M} \) be a class of all \( M \in \mathcal{F} \times \mathcal{T} \) such that both \( f_M, g_M \) are measurable. Then \( \mathcal{R} \subset \mathcal{M} \). By continuity of \( \mu \) and \( \nu \), \( \mathcal{M} \) is a monotone class and hence \( \mathcal{F} \times \mathcal{T} \subset \mathcal{M} \).

**Remark.** It is not difficult to see that for \( M \in \mathcal{F} \times \mathcal{T} \) the functions \( f_M, g_M \) are integrable.

Let us define real functions \( \varphi, \psi \) on \( \mathcal{F} \times \mathcal{T} \) as follows: \( \varphi(M) = \int f_M d\mu, \psi(M) = \int g_M d\nu \) for all \( M \in \mathcal{F} \times \mathcal{T} \).
Theorem 8. The functions $\varphi, \psi$ are finite and continuous $\odot$-measures.

Proof. Clearly $\varphi$ is finite and $\varphi(\emptyset) = 0$. Let $M, N \in \mathcal{F} \times \mathcal{T}, M \cap N = \emptyset$. Then $\varphi(M \cup N) = \int f_{M\cup N} \, d\mu = \int (f_M \odot f_N) \, d\mu = \int f_M \, d\mu \odot \int f_N \, d\mu = \varphi(M) \odot \varphi(N)$. Let $M_n \downarrow \emptyset, M_n \in \mathcal{F} \times \mathcal{T}$ ($n = 1, 2, \ldots$). For all $x \in X, (M_n)_x \downarrow \emptyset$ and by continuity of $\nu$ \[ \lim_{n \to \infty} \nu((M_n)_x) = 0. \] Hence $f_{M_n} \downarrow 0$ and by the theorem 3 \[ \lim_{n \to \infty} \varphi(M_n) = 0. \] Thus $\varphi$ is continuous. Let $E_n \in \mathcal{F} \times \mathcal{T}$ ($n = 1, 2, \ldots$) are mutually disjoint. Put $E = \bigcup_{n=1}^{\infty} E_n$ and \[ F_n = E - \bigcup_{i=1}^{n} E_i \quad (n = 1, 2, \ldots). \] Then $F_n \downarrow \emptyset$ and hence \[ \lim_{n \to \infty} \varphi(F_n) = 0. \] $\varphi(E) = \varphi\left(\bigcup_{i=1}^{\infty} E_i\right) \odot \varphi(F_n)$. Hence $\varphi(E) = \lim_{n \to \infty} \varphi\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{\infty} \varphi(E_n)$. Hence $\varphi$ is a $\odot$-measure. For $\psi$ the proof is dual.

Theorem 9. Let $M \in \mathcal{F} \times \mathcal{T}$. Then $\varphi(M) = \psi(M)$.

Proof. Let $M \in \mathcal{R}, M = \bigcup_{i=1}^{n} (A_i \times B_i)$ where $A_i \in \mathcal{F}, B_i \in \mathcal{T}, A_i \times B_i$ are mutually disjoint $(i = 1, 2, \ldots, n)$. Then $\int f_{M} \, d\mu = \int \bigoplus_{i=1}^{n} v(B_i) \chi_{A_i} \, d\mu = \bigoplus_{i=1}^{n} \int v(B_i) \chi_{A_i} \, d\mu = \bigoplus_{i=1}^{n} \mu(A_i) v(B_i) = \int \bigoplus_{i=1}^{n} \mu(A_i) \chi_{B_i} \, dv = \int g^M \, dv.$ Thus $\varphi(M) = \psi(M)$ on $\mathcal{R}$. Let $M$ be a class of all sets $M \in \mathcal{F} \times \mathcal{T}$ such that $\varphi(M) = \psi(M)$. Then $\mathcal{M}$ is a monotone class by the continuity of $\varphi, \psi$, and $\mathcal{R} \subset \mathcal{M}$. Thus $\mathcal{F} \times \mathcal{T} \subset \mathcal{M}$.

We shall write $\mu \times \nu$ for a function $\varphi$ and we shall call it a product of $\odot$-measures $\mu, \nu$.

Let $h$ be a real function on $X \times Y$. For all $x \in X, y \in Y$ let us define real functions $h_x, h^y$ on $Y$, resp. $X$, in the following way: $h_x(y) = h(x, y), h^y(x) = h(x, y)$.

Theorem 10. Let $h: X \times Y \to (0, \infty)$ be an integrable function. Then the functions $f: X \to (0, \infty), g: Y \to (0, \infty)$ defined as follows: $f(x) = \int h_x \, dv, g(y) = \int h^y \, du$ are integrable and moreover $\int h \, d\mu \times \nu = \int f \, d\mu \times g \, dv$. Thus $\int h \, d\mu \times \nu = \mu \times \nu(E) = \int f \, d\mu.$

Proof. 1) First let $h = \chi_E, E \in \mathcal{F} \times \mathcal{T}$. Then $h_x = \chi_{E_x}$ and $f(x) = \int h_x \, dv = \nu(E_x) = f_E(x)$. Thus $\int h \, d\mu \times \nu = \mu \times \nu(E) = \int f \, d\mu.$

2) Let $h$ be a NSF on $X \times Y$. Then $\int h \, d\mu \times \nu = \bigoplus_{i=1}^{n} \alpha_i \mu \times \nu = \bigoplus_{i=1}^{n} \alpha_i \int (\chi_{(E_i)_x} \, dv) \, d\mu = \bigoplus_{i=1}^{n} \alpha_i \int (\chi_{(E_i)_x}) \, d\mu \times \nu = \bigoplus_{i=1}^{n} \alpha_i \int (\chi_{(E_i)_x} \, dv) \, d\mu = \bigoplus_{i=1}^{n} \alpha_i \int (\chi_{(E_i)_x}) \, d\mu \times \nu = \bigoplus_{i=1}^{n} \alpha_i \chi_{(E_i)_x} \, d\mu \times \nu.$

3) Let $h$ be an arbitrary non-negative function on $X \times Y$. Take NSF's $h_n$ ($n = 1, 2, \ldots$) such that $h_n \uparrow h$ and denote $f_n(x) = \int (h_n)_x \, dv$ for all $x \in X$, ($n = 1, 2, \ldots$). The functions $f_n$ ($n = 1, 2, \ldots$) are $\mu$-measurable, thus the lim $f_n$ is $\mu$-measurable. By the theorem 6 $\int h \, d\mu \times v = \lim_{n \to \infty} \int h_n \, d\mu \times v = \lim_{n \to \infty} \int f_n \, d\mu$. 21
\[
\lim_{n \to \infty} \int_{h_n(x)} f_n \, dv = \int \left( \lim_{n \to \infty} \int_{h_n(x)} f_n \, dv \right) \, d\mu = \int \left( \int \lim_{n \to \infty} (h_n(x)) \, dv \right) \, d\mu = \int h(x) \, dv \, d\mu = \int f \, d\mu.
\]

The function \( f \) is integrable since \( h \) is integrable. By the same arguments one can prove that \( g \) is integrable and \( \int g \, dv = \int h \, d\mu \times v. \)

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ИНТЕГРИРОВАНИЕ ПО \( \bigoplus \)-МЕРЕ

Ivica Marinová

Резюме

В статье показано, что как интегрирование \( \sigma \)-аддитивных мер и \( \sigma \)-макситивных мер, так и произведение этих мер можно рассматривать одновременно.