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THE INTERSECTIONS OF RANDOM FINITE SETS

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1. Introduction

The present paper deals with a particular scheme of the capture-recapture procedure. Suppose we have a finite set S of n elements numbered $1, \ldots, n$. The sample procedure is performed in k independent stages in the following way. At each stage we draw randomly m elements from S without replacement. Each combination of m elements is thus drawn with probability $\binom{n}{m}^{-1}$. Let M_j be the set of elements drawn at stage j, $j=1, \ldots, k$ and let M be the set of elements which have been drawn at all k stages, i.e. $M=\bigcap_{j=1}^{k} M_j$. Let C_{nk} denote the cardinality of M. The aim of this paper is to derive some basic properties of C_{nk}. As it will be seen later, the exact distribution of C_{nk} is not easy to handle and hence some asymptotic results would be helpful. We shall state conditions under which C_{nk} is asymptotically normally distributed or possesses an asymptotically Poisson distribution. The entrance time into zero will be investigated and some numerical results given. For some other aspects of the capture-recapture theory see [4].

2. Exact distribution and auxiliary results

Let I_i denote the indicator of M, i.e. $I_i=1$ if $i \in M$, $I_i=0$ otherwise, $i=1, \ldots, m$. Obviously, $C_{nk}=\sum_{i=1}^{n} I_i$. To derive the exact distribution of C_{nk} we make use of Jordan's identity. Let $A_1, \ldots, A_n$ be random events on the same probability space, and let $W_{n,r}$ denote the probability of the event that exactly r events from among $A_1, \ldots, A_n$ occur. Then we have

$$W_{n,r}=\sum_{j=0}^{n-r}(-1)^j \binom{r+j}{j} S_{r+j},$$

where

$$S_j = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \mathrm{P}(A_{i_1} \cap \ldots \cap A_{i_j}), \quad j = 1, \ldots, n, \; S_0 = 1.$$

The event $\{C_{nk} = r\}$ occurs iff just r of $I_1, \ldots, I_n$ equal to 1. In our case we identify A_i with $\{I_i = 1\}$, $i = 1, \ldots, n$, and obtain

$$\begin{aligned} \mathrm{P}(A_{i_1} \cap \ldots \cap A_{i_j}) &= \left[\binom{n-j}{m-j} \Big/ \binom{n}{m}\right]^k, \quad j = 0, \ldots, m, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Therefore

$$\begin{aligned} S_0 &= 1, \\ S_j &= \binom{n}{j} \left[\binom{n-j}{m-j} \Big/ \binom{n}{m}\right]^k, \quad j = 0, \ldots, m, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} \mathrm{P}(C_{nk} = r) &= \sum_{j=0}^{m-r} (-1)^j \binom{r+j}{j} \binom{n}{r+j} \left[\binom{n-r-j}{m-r-j} \Big/ \binom{n}{m}\right]^k, \\ &= 0 \quad \text{otherwise,} \quad r = 0, \ldots, m, \end{aligned} \tag{2.1}$$

or, using the identity $\binom{r+j}{j}\binom{n}{r+j} = \binom{n}{r}\binom{n-r}{j}$,

$$\begin{aligned} \mathrm{P}(C_{nk} = r) &= \binom{n}{r} \binom{n}{m}^{-k} \sum_{j=0}^{m-r} (-1)^j \binom{n-r}{j} \binom{n-r-j}{m-r-j}^k, \\ &= 0 \quad \text{otherwise.} \qquad r = 0, \ldots, m, \end{aligned} \tag{2.2}$$

It should be noted that for $k = 2$ this becomes a hypergeometrical distribution. To the best of our knowledge last formulas cannot be substantially simplified.

Sometimes the following Markovian property may be useful. The intersection $\bigcap_{j=1}^{t} M_j$, $t = 0, 1, 2, \ldots$, may be considered as a system observed at time points t. The system will be said to be in the state i iff $\bigcap_{j=1}^{t} M_j$ contains exactly i elements, $i = 0, \ldots, m$. The transition probabilities are

$$\begin{aligned} p_{ij} &= \binom{i}{j} \binom{n-i}{m-j} \Big/ \binom{n}{m}, \quad j = 0, \ldots, i, \; j = i+1, \ldots, m, \\ &= 0, \end{aligned} \tag{2.3}$$

and the initial distribution is $a_m = 1$, $a_i = 0$, $0 \leqq i \leqq m-1$. This completely defines a discrete homogeneous Markov chain with states $0, \ldots, m$.

For further investigation the moments of C_{nk} are of fundamental importance. To simplify the notation we put

$$p=\frac{m}{n}, \quad p_j=\left(\frac{m-j}{n-j}\right)^k, \quad j=0, \ldots, m.$$

Then, $\mathrm{E}\,I_i^r=p_0$, $\mathrm{E}\,I_i^s I_j^r=p_0p_1$ for $i\neq j$, $\mathrm{E}\,I_i^s I_j^r I_h^t=p_0p_1p_2$ for all i, j, h different, $\mathrm{E}\,I_iI_jI_hI_l=p_0p_1p_2p_3$ for all i, j, h, l different and s, r, t natural. We immediately obtain

$$\mathrm{E}\,C_{nk}=np_0=np^k. \tag{2.4}$$

Further,

$$\mathrm{E}\,C_{nk}^2=\sum \mathrm{E}\,I_i^2+\sum_{i\neq j}\mathrm{E}\,I_iI_j=np_0+n(n-1)p_0p_1,$$

and hence

$$\operatorname{var} C_{nk}=n^2p_0(p_1-p_0)+np_0(1-p_1). \tag{2.5}$$

After a straightforward but tedious algebra we can obtain higher moments. We just put the formulas for two higher moments

$$\begin{gathered}\mathrm{E}\,C_{nk}^3=n^3p_0p_1p_2+3n^2p_0p_1(1-p_2)+np_0(1-3p_1+2p_1p_2),\\ \mathrm{E}\,C_{nk}^4=n^4p_0p_1p_2p_3+6n^3p_0p_1p_2(1-p_3)+n^2p_0p_1(7-18p_2+11p_2p_3)+\\ +np_0(1-7p_1+12p_1p_2-6p_1p_2p_3)\end{gathered}$$

from which we get

$$\begin{gathered}\mu_3=n^3p_0(p_1p_2-3p_0p_1+2p_0^2)+3n^2p_0[p_1(1-p_2)-p_0(1-p_1)]+\\ +np_0(1-3p_1+2p_1p_2)\end{gathered}$$

and

$$\begin{gathered}\mu_4=n^4p_0(p_1p_2p_3-4p_0p_1p_2+6p_0^2p_1-3p_0^3)+\\ +6n^3p_0[p_1p_2(1-p_3)-2p_0p_1(1-p_2)+p_0^2(1-p_1)]+\\ +n^2p_0[p_1(7-18p_2+11p_2p_3)-4p_0(1-3p_1+2p_1p_2)]+\\ +np_0(1-7p_1+12p_1p_2-6p_1p_2p_3)\end{gathered}$$

for the third and fourth central moment, respectively.

Using the obvious fact that the conditional distribution of C_{nk} given $C_{n,k-1}$ is hypergeometrical, we can evaluate the factorial moments of C_{nk} which will be helpful for deriving the asymptotic-Poisson-distribution of C_{nk} in Section 3.

We have

$$\begin{gathered}\mathrm{E}(C_{nk}^{(r)}\mid C_{n,k-1}=c)=\mathrm{E}[C_{nk}(C_{nk}-1)\ldots(C_{nk}-(r-1))\mid C_{n,k-1}=c]=\\ =\frac{1}{\binom{n}{m}}\sum_j\frac{c!}{(j-r)!\,(c-j)!}\,\frac{(n-c)!}{(m-j)!\,(n-c-m+j)!}=\frac{c^{(r)}m^{(r)}}{n^{(r)}}.\end{gathered}$$

Then,

$$\mathrm{E}(C_{nk}^{(r)})=\mathrm{E}[\mathrm{E}(C_{nk}^{(r)}\mid C_{n,k-1})]=\frac{m^{(r)}}{n^{(r)}}\,\mathrm{E}(C_{n,k-1}^{(r)}), \tag{2.6}$$

which is the recurrent formula for the evaluation of $E(C_{nk}^{(r)})$. Applying (2.6) repeatedly we get

$$E(C_{nk}^{(r)}) = \left(\frac{m^{(r)}}{n^{(r)}}\right)^{k-1} \cdot E(C_{n1}^{(r)}).$$

However, $C_1 = m$ with probability 1, so that

$$E(C_{nk}^{(r)}) = \left(\frac{m^{(r)}}{n^{(r)}}\right)^{k-1} m^{(r)} = \frac{[m(m-1)\dots(m-(r-1))]^k}{[n(n-1)\dots(n-(r-1))]^{k-1}} \tag{2.7}$$

Lemma 2.1. *If $n \to \infty$ and $m \to \infty$ in such a way that p remains fixed then*

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{var} C_{nk} = \varrho_k = p^k[1 - kp^{k-1} + (k-1)p^k]. \tag{2.8}$$

Proof. First note that

$$\left(1-\frac{1}{m}\right)^k \left(1-\frac{1}{n}\right)^{-k} = 1 + \frac{k}{np}(p-1) + o(n^{-1}).$$

Then

$$\frac{1}{n} \operatorname{var} C_{nk} = np^k \left\{p^k \left[1 + \frac{k}{np}(p-1) + o(n^{-1})\right] - p^k\right\} +$$

$$+ p^k \left\{1 - p^k \left[1 - \frac{k}{np}(p-1) + o(n^{-1})\right]\right\} \to p^k[1 - kp^{k-1} + (k-1)p^k],$$

Q.E.D.

Remark. Denote the expression in (2.5) as $\varrho(k)$. Using standard methodes of calculus we can find that $\varrho(k) > 0$ for $0 < p < 1$.

Lemma 2.2. *Under the assumptions of Lemma 2.1.*

$$\frac{1}{n} C_{nk} \xrightarrow{P} p^k \quad \text{as} \quad n \to \infty.$$

Proof. This follows from Chebyshev inequality and from lemma 2.1. Q.E.D.

Now let us leave the assumption of a fixed number k of stages and study the case when the sampling is repeated until at some stage the set M is found empty for the first time. Denote by τ the first entrance time into zero, i.e., $\tau = \min\{s: C_{ns} = 0\}$. Using the facts that

$$P(C_{ns} = 0) = \sum_{0}^{m} (-1)^j \binom{n}{j} \left[\binom{n-j}{m} \Big/ \binom{n}{m}\right]$$

and

$P(\tau = s) = P(\tau \leq s, \tau > s-1) = P(\tau \leq s) - P(\tau \leq s-1) = P(C_{ns} = 0) - P(C_{n,s-1} = 0)$

we find that the distribution of τ is

$$P(\tau=s)=\sum_{j=1}^{m}(-1)^{j}\binom{n}{j}\left[\binom{n-j}{m-j}\Big/\binom{n}{m}\right]^{s-1}\left[\binom{n-j}{m-j}\Big/\binom{n}{m}-1\right], \quad s=2,3,\ldots. \tag{2.9}$$

Next we give a formula for $E\tau$. Since

$$E\tau=\sum_{s=1}^{\infty}P(\tau\geqq s)=\sum_{s=1}^{\infty}[1-P(\tau\leqq s)]+\sum_{s=1}^{\infty}P(\tau=s)=\sum_{s=1}^{\infty}[1-P(C_{ns}=0)]+1$$

we have

$$E\tau=1+\sum_{s=1}^{\infty}\sum_{j=1}^{m}(-1)^{j+1}\binom{n}{j}\left[\binom{n-j}{m-j}\Big/\binom{n}{m}\right]^{s}=$$

$$=1+\sum_{j=1}^{m}(-1)^{j+1}\frac{\binom{n}{j}\binom{n-j}{m-j}}{\binom{n}{m}-\binom{n-j}{m-j}}$$

or

$$E\tau=1+\sum_{j=1}^{m}(-1)^{j+1}\frac{\binom{m}{j}\binom{n}{j}}{\binom{n}{j}-\binom{m}{j}}.$$

The numerical illustration for $m=\frac{n}{2}$.

n	2	4	6	8	10	12	20	30	50
$E\tau$	3	3,8	4,3	4,67	4,93	5,16	5,82	6,37	7,06

Finally, we derive the characteristic function φ_{nk} of C_{nk}. First remark that

$$\exp\{itI_j\}=1+\sum_{s=1}^{\infty}\frac{1}{s!}(itI_j)^s=1+I_j\,(e^{it}-1).$$

Then

$$\varphi_{nk}(t)=E\exp\left\{it\sum_{j=1}^{n}I_j\right\}=E\prod_{j=1}^{n}\,[1+I_j(e^{it}-1)]=E\sum_{j=0}^{n}\binom{n}{j}\,I_j^*\,(e^{it}-1)^j$$

where I_j^* are such random variables that

$$\begin{aligned} E I_0^* &= 1, \\ E I_j^* &= p_0\ldots p_{j-1}, && j=1,\ldots,m, \\ &= 0, && j=m+1,\ldots,n. \end{aligned}$$

Hence,

$$\varphi_{nk}(t)=1+\sum_{j=1}^{m}\binom{n}{j}\,(e^{it}-1)^j\prod_{s=0}^{j-1}p_s. \tag{2.10}$$

3. Convergence to the Poisson distribution

Now we shall study conditions under which the asymptotic distribution of C_{nk} for $n \to \infty$ is the Poisson one. In this situation $m = m_n$, $k = k_n$, so that $C_{nk} - C_{nk_n m_n} - C_n$. The ratio $m_n\ n = p$ is fixed as in the preceding sections. First, we shall investigate the behaviour of the factorial moments $\mathrm{E}(C_{nk}^{(r)})$, given by (2.7), when $n \to \infty$.

Lemma 3.1. *Let us suppose that $n \to \infty$, $m_n \to \infty$, $k = k_n \to \infty$ in such a way that for fixed p $np^k = m_n^k\ n^{k\ 1} \to \lambda$, $0 < \lambda < \infty$. Then,*

$$\mathrm{E}(C_n^{(r)}) \to \lambda^r. \tag{3.1}$$

Proof. According to (2.7), we have

$$\mathrm{E}(C_n^{(r)}) = (np^{k_n})^r \frac{\prod_{i=1}^{r-1}\left(1 - \frac{i}{np}\right)^{k_n}}{\prod_{i=1}^{r-1}\left(1 - \frac{i}{n}\right)^{k_n}}.$$

The rest is obvious Q.E.D.

Remark. The sufficient condition for fulfilling $np^{k_n} \to \lambda$ is $k_n = 0(n)$, e.g.

Having calculated the factorial moments of the Poisson distribution $\mathscr{P}(\lambda)$ with the parameter λ we can see that the limiting values in (3.1) are the same. This fact enables us to prove easily that the limiting distribution of C_n is Poisson. It is interesting to note that the same procedure applied to (not factorial) moments of C_n would involve much more difficulties.

Theorem 3.1. *Suppose that $n \to \infty$, $m_n \to \infty$, $k = k_n \to \infty$ in such a way that for a fixed p, $np^k = m^k\ n^{k\ 1} \to \lambda$, $0 < \lambda < \infty$. Then the asymptotic distribution of C_n is Poisson with the parameter λ.*

Proof. As it follows from Lemma 3.1, the limiting values of the factorial moments $\mathrm{E}(C_n^{(r)})$ are equal to the corresponding moments of $\mathscr{P}(\lambda)$. Obviously, the same relation holds for moments. The Poisson distribution (having an analytic characteristic function) is completely characterized by its moments. The assertion follows from Theorem B in [2] p. 198. Q.E.D.

Now we shall return to the case of fixed k. Investigating the asymptotic distribution we shall see that to get the Poisson distribution we shall not be able to continue to fix $p - p(m)\quad m_n\ n$.

Theorem 3.2. *Let us assume that $n \to \infty$, $m_n \to \infty$ in such a way that $n[p(n)]^k \to \lambda$, $0 < \lambda < \infty$, for fixed k. Then, C_n has the asymptotically Poisson distribution with the parameter λ.*

Proof. Having proved that $\mathrm{E}[C_{nk}^{(r)}] \to \lambda^r$ for $\mathrm{E}[C_{nk}^{(r)}]$ given by (2.7) (which is obvious) we use the same argument as in the proof of Theorem 3.1. Q E.D.

4. Asymptotic normality of C_{nk}

Consider the model of random intersections introduced in Section 1 with a fixed number of repetitions $k>1$ and a fixed ratio $\frac{m}{n}=\left(\frac{m_n}{n}\right)=p$, $p\in[0,1]$. Recall that $\mathrm{E}\,C_{nk}=np^k$ and, according to Lemma 2.1, $\frac{1}{n}\operatorname{var} C_{nk}\to\varrho_k$, $n\to\infty$, where ϱ_k is given by (2.8). According to Lemma 2.2, $C_{nn}/n\to p^k$, $n\to\infty$, in probability.

The asymptotic normality of C_{nk} (for $n\to\infty$) with the parameters np^k and $n\varrho_k$ seems to be an abvious conjecture but its analytical treatment is definitely uncomfortable even in the relatively simple case $k=2$. This fact may be easily seen when looking over the proof of the limit theorem for the hypergeometrical distribution (see [3], p. 398).

We suggest to utilize the invariance principle for exchangeable random variables and the possibility to represent C_{nk}'s on the corresponding empirical processes to get a simple verification of our conjecture which may be formally stated as follows:

Theorem. *Assume that* $\frac{m_n}{n}=p$ *for each* n. *Then, for each* $k>1$ *and* $p\in[0,1]$

$$\frac{C_{nk}-np^k}{\sqrt{n}}\xrightarrow{\mathcal{D}}N(0,\varrho_k)\quad\text{as}\quad n\to\infty,\tag{4.1}$$

where $\varrho_k=\varrho_k(p)$ *is defined by (2.8) and the symbol* $\mathcal{D}$ *denotes the convergence in distribution.*

Proof. Without loss of generality assume that $p\in(0,1)$ and put for $n\geqq 1$

$$y_{ni}=\frac{1-p}{\sqrt{n(1-p)p}},\quad 1\leqq i\leqq m_n,$$

$$=-\frac{p}{\sqrt{n(1-p)p}},\quad m_n+1\leqq i\leqq n.$$

Obviously,

$$\sum_{i=1}^{n}y_{ni}=0,\quad \sum_{i=1}^{n}y_{ni}^2=1\quad\text{and}\quad \max_{1\leqq i\leqq n}|y_{ni}|\to 0,\quad n\to\infty.\tag{4.2}$$

Further, let $\xi_{n1},\xi_{n2},\dots,\xi_{n,n}$ be a random permutation of these numbers (on a suitable probability space $(\Omega_n,\mathcal{A}_n,\mathrm{P}_n)$) each permutation having probability $\frac{1}{n!}$ and define a stochastic process Y_n with trajectories in Skorochod space $D[0,1]$ by

$$Y_n(t,\omega)=\sum_{i=1}^{[nt]}\xi_{ni}(\omega),\quad n^{-1}\leqq t\leqq 1,\tag{4.3}$$

$$=0,\qquad 0\leqq t<n^{-1},\quad n\in N,\ \omega\in\Omega.$$

It follows directly from Theorem 24.1 in [1] and relations (4.2) that

$$Y_n \xrightarrow{\mathscr{D}} W^0, \quad n \to \infty, \tag{4.4}$$

where W^0 denotes a Brownian bridge on $D[0, 1]$. On the other hand a simple probabilistic argument shows that

$$Y_n(p) = \sum_{i=1}^{m_n} \xi_{ni} = \frac{(1-p)D_{n2} - p(m_n - D_{n2})}{\sqrt{n(1-p)p}} = \frac{D_{n2} - np^2}{\sqrt{n(1-p)p}}, \tag{4.5}$$

where D_{n2} is a random variable with the same distribution as C_{n2}.

Now, since $\mathscr{L}(W^0(p)) = N(0, p(1-p))$ and $\varrho_2(p) = [p(1-p)]^2$ it follows from (4.4) and (4.5) that our assertion (4.1) works in the special case $k=2$.

To proceed by mathematical induction assume the validity of (4.1) for some $k \geqq 2$ and choose a probability space $(\Omega_n, \mathscr{A}_n, \mathrm{P}_n)$ which permits to define the stochastic process Y_n in the way of (4.3) together with a random variable D_{nk} such that for each $n \geqq 1$

$$\mathscr{L}(D_{nk}) = \mathscr{L}(C_{nk}) \quad \text{and} \quad D_{nk},\ Y_n \quad \text{are independent.} \tag{4.6}$$

It follows from (4.6) using (4.1) and (4.4)

$$\frac{p(D_{nk} - np^k)}{\sqrt{np(1-p)}} + Y_n \xrightarrow{\mathscr{D}} N + W^0 \quad \text{as} \quad n \to \infty, \tag{4.7}$$

where

$$W^0 \quad \text{is a Brownian bridge,} \quad \mathscr{L}(N) = N\left(0, \frac{p\varrho_k}{1-p}\right), \tag{4.8}$$

$$n,\ W^0 \quad \text{are independent and defined on a suitable probability space.}$$

Now, employing the random change of time $t \to \dfrac{D_{nk}(\omega)}{n}$ in (4.7) and using Lemma 2.2, we may argue by (17.7), (17.8) and (17.9) in [1], p. 145, to conclude that

$$Z_{nk} = \frac{p(D_{nk} - np^k)}{\sqrt{np(1-p)}} + Y_n\left(\frac{D_{nk}}{n}\right) \xrightarrow{\mathscr{D}} N + W^0(p^k) \quad \text{as} \quad n \to \infty. \tag{4.9}$$

Obviously, by (4.8) and (4.9) we have

$$Z_{nk} \xrightarrow{\mathscr{D}} N\left(0, \frac{p\varrho_k}{1-p} + p^k(1-p^k)\right) = N\left(0, \frac{\varrho_{k+1}}{p(1-p)}\right). \tag{4.10}$$

On the other hand we may easily see that

$$Z_{nk} = \frac{p(D_{nk} - np^k)}{\sqrt{np(1-p)}} + \sum_{i=1}^{D_{nk}} \xi_{ni} = \frac{p(D_{nk} - np^k)}{\sqrt{np(1-p)}} + \tag{4.11}$$

$$+\frac{(1-p)D_{n,k+1}-p(D_{nk}-D_{n,k+1})}{\sqrt{np(1-p)}}=\frac{D_{n,k+1}-np^{k+1}}{\sqrt{np(1-p)}}, \quad n \geqq 1,$$

where $D_{n,k+1}$ is a random variable with the same distribution as $C_{n,k+1}$. Thus, combining (4.10) and (4.11) we verify the validity of our theorem for $k+1$ and hence, by mathematical induction, (4.1) holds for arbitrary $k \geqq 2$. Q.E.D.

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ПЕРЕСЕЧЕНИЯ СЛУЧАЙНЫХ КОНЕЧНЫХ МНОЖЕСТВ

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Резюме

Мы предполагаем, что множество S состоит из n элементов $1, \ldots, n$. Из него выбирается k раз независимо m элементов при помощи простого случайного выбора. Изучаются свойства числа C_{nk} элементов, которые находятся в пересечении всех k выборочных совокупностей.

Кроме точного распределения C_{nk} и его основных характеристик были получены асимптотические свойства C_{nk} при $n \to \infty$. Асимптотическое распределение C_{nk} является или Пуассоновым или нормальным по условиям, которые выполнены для $m = m_n$ и $k = k_n$ при $n \to \infty$.