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FROM TRANSVERSE HETEROCLINIC CYCLES TO TRANSVERSE HOMOCLINIC ORBITS

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ABSTRACT. We show the existence of transversal homoclinic orbits for a sequence of periodic ordinary differential equations which has a limiting periodic ordinary differential equation with a transversal heteroclinic cycle.

1. Introduction

Let us consider the family of second order equations

$$\ddot{x} = 4x\big(2x^2 - 3x\coth(2m\pi) + 1\big) + \varepsilon q(t) \tag{1}$$

where $m \in \mathbb{N}$, ε is a small parameter and $q(t)$ is a 2π-periodic C^1-function. The unperturbed conservative first order equation

$$\dot{x} = y\,, \qquad \dot{y} = 4x\big(2x^2 - 3x\coth(2m\pi) + 1\big) \tag{2}$$

has a homoclinic solution $\big(p_m(t), \dot{p}_m(t)\big)$ of the form

$$p_m(t) = \frac{\mathrm{e}^{2t}(\mathrm{e}^{4m\pi} - 1)}{(\mathrm{e}^{2t} + \mathrm{e}^{2m\pi})(\mathrm{e}^{2t+2m\pi} + 1)}$$

to a hyperbolic equilibrium $(0,0)$. Associated to (1) and to the homoclinic orbit $p_m(t)$ there is the Melnikov function ([5])

$$M_m(\alpha) := \int_{-\infty}^{\infty} \dot{p}_m(s) q(s+\alpha)\,\mathrm{d}s\,. \tag{3}$$

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It is a classical result ([4]) that if $M_m(\alpha)$ has a simple zero α_0, i.e. $M_m(\alpha_0) = 0$ and $M'_m(\alpha_0) \neq 0$, then the dynamics of (1) is chaotic for $\varepsilon \neq 0$ small in the sense that (1) possesses, for $\varepsilon \neq 0$ small, a transversal homoclinic orbit with the associated S m a l e horseshoe ([4], [5], [7]). But we have proved in [1] that for any 2π-periodic C^1-function $q(t)$ and any $m \in \mathbb{N}$, the Melnikov function (3) is identically zero. The geometrical meaning of this is that, in spite of the fact that the perturbation (1) of the equation (2) is of the order $O(\varepsilon)$, the distance between the stable and unstable manifolds to a small periodic orbit of the perturbed equation, along a transverse direction, is of the order (at least) $O(\varepsilon^2)$. This means that in order to study the intersection of the stable and the unstable manifolds, we have to look at the second order Melnikov function. This was successfully done in [1] by showing the existence of a large class of 2π-periodic C^1-functions $q(t)$ for which the second order Melnikov function is nonzero.

Now, the limiting equation of (1), for $m \to \infty$, is

$$\ddot{x} = 4x(2x^2 - 3x + 1) + \varepsilon q(t)\,, \tag{4}$$

whose unperturbed equation

$$\ddot{x} = 4x(2x^2 - 3x + 1) \tag{5}$$

has two *heteroclinic* connections to the equilibria $x = 0$ and $x = 1$ with the heteroclinic solution going from $x = 0$ to $x = 1$:

$$p_\infty(t) = \frac{e^{2t}}{e^{2t} + 1}$$

and the one from $x = 1$ to $x = 0$:

$$p_\infty(-t) = \frac{1}{e^{2t} + 1}\,.$$

Since the Melnikov function (3) of equation (1) is identically zero for any 2π-periodic C^1-perturbation of the equation, one might wonder whether this fact holds for the Melnikov functions associated to the heteroclinic orbits $p_\infty(\pm t)$ of equation (4). The answer to this question is negative as it has been proved in [1]. Geometrically, this strange behaviour depends on the fact that the homoclinic solution of (2) gets orbitally closer and closer (as $m \to \infty$) to the *heteroclinic cycle* of (5) and not to any of the heteroclinic orbits $p_\infty(\pm t)$. Moreover, it has been proved in [1], that the Melnikov functions associated to the heteroclinic solutions $p_\infty(\pm t)$ of the limiting equation (4) have transverse zeroes at least for infinitely many 2π-periodic C^1-functions $q(t)$. Thus for any ε sufficiently small and such $q(t)$, (4) has a transverse heteroclinic cycle. The purpose of this note is to extend and study this relationship between (1) and (4) for more general systems. To this end, and to help the reader in understanding the assumptions we make, we observe that the difference between the r.h.s. of equations (1) and (5), given by:

$$12x^2\big(1-\coth(2m\pi)\big) = -\frac{24x^2}{\mathrm{e}^{4m\pi}-1}$$

tends to zero as $m \to \infty$ uniformly on compact sets and the same holds for its derivative with respect to x.

2. Bifurcation to homoclinic orbits

We consider a family of T-periodic differential equations

$$\dot{x} = f_m(t,x) = f_m(t+T,x) \tag{6}$$

where either $m \in \mathbb{N}$ or $m = \infty$, $t \in \mathbb{R}$, and $x \in \Omega$, an open and bounded subset of $\mathbb{R}^n$. We assume that $f_m(t,x)$ are C^2 functions in $(t,x) \in \mathbb{R}\times\Omega$, and that the following conditions hold:

(a) $\dot{x} = f_\infty(t,x)$ has a transversal heteroclinic cycle in Ω made of two hyperbolic periodic solutions $p_i(t)$, $i = 0,1$, and two heteroclinic orbits $p_\infty^{(01)}(t)$ and $p_\infty^{(10)}(t)$ connecting them, that is such that

$$\begin{aligned}\lim_{t\to-\infty}\big[p_\infty^{(01)}(t)-p_0(t)\big] &= \lim_{t\to\infty}\big[p_\infty^{(10)}(t)-p_0(t)\big] = 0\,,\\ \lim_{t\to\infty}\big[p_\infty^{(01)}(t)-p_1(t)\big] &= \lim_{t\to-\infty}\big[p_\infty^{(10)}(t)-p_1(t)\big] = 0\,.\end{aligned} \tag{7}$$

(b) $f_\infty(t,x)$ is a regular perturbation of $f_m(t,x)$, that is

$$\begin{aligned}\sup_{(t,x)\in\mathbb{R}\times\Omega}|f_m(t,x)-f_\infty(t,x)| \to 0 \qquad &\text{as} \quad m\to\infty\,,\\ \sup_{(t,x)\in\mathbb{R}\times\Omega}|\,\mathrm{D}_2 f_m(t,x)-\mathrm{D}_2 f_\infty(t,x)| \to 0 \qquad &\text{as} \quad m\to\infty\,.\end{aligned} \tag{8}$$

Note that by transversality of the heteroclinic cycle we mean that the stable and unstable manifolds of the periodic orbits $p_i(t)$ intersect transversally along both $p_\infty^{(01)}(t)$ and $p_\infty^{(10)}(t)$.

By using the implicit function theorem, it is not difficult to show that the conditions (a) and (b) imply that for any m sufficiently large, the T-periodic nonlinear system

$$\dot{x} = f_m(t,x)$$

has unique T-periodic solutions $q_m(t)$ and $r_m(t)$ such that $\sup_{t\in\mathbb{R}}|q_m(t)-p_0(t)| \to 0$ and $\sup_{t\in\mathbb{R}}|r_m(t)-p_1(t)| \to 0$ as $m\to\infty$. Moreover, both $q_m(t)$ and $r_m(t)$ are hyperbolic.

The purpose of this note is to prove the following theorem:

THEOREM 1. *There exists $m_0 \in \mathbb{N}$ such that for any $m \in \mathbb{N}$, $m > m_0$, system* (6) *has an orbit $p_m(t)$ homoclinic to $q_m(t)$ and such that*

$$\begin{aligned} \sup_{t\le 0} \left|p_m(t) - p_\infty^{(01)}(t+mT)\right| \to 0 \qquad & as \quad m\to\infty\,,\\ \sup_{t\ge 0} \left|p_m(t) - p_\infty^{(10)}(t-mT)\right| \to 0 \qquad & as \quad m\to\infty\,. \end{aligned} \tag{9}$$

Moreover, the stable and unstable manifolds of the periodic solution $q_m(t)$ of system (6) *intersect transversely along the homoclinic solution $p_m(t)$.*

Theorem 1 can be related to the following result (see [6]):

> *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeomorphism possessing hyperbolic fixed points p_1 and p_0. If $W^u_{p_1}$ transversally intersects $W^s_{p_2}$ and $W^s_{p_1}$ transversally intersects $W^u_{p_2}$, then $W^u_{p_i}$ transversally intersects $W^s_{p_i}$ for $i = 1,2$.*

This result is proved by using the λ-lemma. Our proof, instead, is based on such notions as exponential dichotomies and roughness. We emphasize the fact that our proof is more constructive than the one given in [6] and also leads to equation (9), which allow to locate, within some small error, the homoclinic orbit.

P r o o f o f T h e o r e m 1 . To simplify the proof, we first replace x with $y = x + p_0(t) - q_m(t)$. We obtain the family of equations

$$\dot y = \hat f_m(t,y)\,,$$

where

$$\hat f_m(t,y) = f_m\big(t, y - p_0(t) + q_m(t)\big) + f_\infty\big(t, p_0(t)\big) - f_m\big(t, q_m(t)\big)$$

and

$$\hat f_\infty(t,y) = f_\infty(t,y)\,.$$

Note that the family $\hat f_m(t,y)$ satisfies assumptions (a) and (b). Hence, without loss of generality, we suppose in this proof that for any $m \in \mathbb{N}$ system (6) has the periodic solution $p_0(t)$.

From the transversality assumption of the heteroclinic cycle (see [5]) it follows that the linear system

$$\dot x = \mathrm{D}_2 f_\infty\big(t, p_\infty^{(01)}(t)\big)x \tag{10}$$

has an exponential dichotomy on $\mathbb{R}$ with projection, say Q_∞, that is the fundamental matrix $X_\infty(t)$ of (10) satisfies

$$\begin{aligned} \|X_\infty(t)Q_\infty X_\infty^{-1}(s)\| \le K\,\mathrm{e}^{-\delta(t-s)}\,, \qquad & s \le t\,,\\ \|X_\infty(t)[\mathbb{I} - Q_\infty]X_\infty^{-1}(s)\| \le K\,\mathrm{e}^{\delta(t-s)}\,, \qquad & t \le s\,, \end{aligned}$$

for a constant $\delta > 0$. Similarly there exists a projection P_∞ such that the fundamental matrix $Y_\infty(t)$ of the linear system $\dot{x} = \mathrm{D}_2 f_\infty\big(t, p_\infty^{(10)}(t)\big)x$ satisfies

$$\begin{aligned} \|Y_\infty(t)P_\infty Y_\infty^{-1}(s)\| &\le K\,\mathrm{e}^{-\delta(t-s)}\,, \qquad s \le t\,, \\ \|Y_\infty(t)[\mathbb{I} - P_\infty]Y_\infty^{-1}(s)\| &\le K\,\mathrm{e}^{\delta(t-s)}\,, \qquad t \le s\,. \end{aligned}$$

Now, the the fundamental matrix $X_\infty(t+mT)X_\infty^{-1}(mT)$ of the linear system $\dot{x} = \mathrm{D}_2 f_\infty\big(t, p_\infty^{(01)}(t+mT)\big)x$ has also an exponential dichotomy on $\mathbb{R}$ with projection matrix

$$Q_\infty(mT) = X_\infty(mT)Q_\infty X_\infty^{-1}(mT)\,,$$

and similarly, the fundamental matrix $Y_\infty(t-mT)Y_\infty^{-1}(-mT)$ of the linear system $\dot{x} = \mathrm{D}_2 f_\infty\big(t, p_\infty^{(10)}(t-mT)\big)x$ has an exponential dichotomy on $\mathbb{R}$ with projection matrix

$$P_\infty(-mT) = Y_\infty(-mT)P_\infty Y_\infty^{-1}(-mT)\,.$$

We seek for a solution $p_m(t)$ of the nonlinear system (6) with $m \in \mathbb{N}$ sufficiently large such that (9) holds. Hence, setting $x_1(t) = p_m(t) - p_\infty^{(01)}(t+mT)$ and $x_2(t) = p_m(t) - p_\infty^{(10)}(t-mT)$, we see that we look for a pair of functions $\big(x_1(t), x_2(t)\big)$ such that

$$\sup_{t\le 0}|x_1(t)| \qquad \text{and} \qquad \sup_{t\ge 0}|x_2(t)|$$

are small, and actually tend to zero as $m \to +\infty$, and satisfying:

$$\begin{aligned} \dot{x}_1 - \mathrm{D}_2 f_\infty\big(t, p_\infty^{(01)}(t+mT)\big)x_1 &= h_m^-(t, x_1) \qquad \text{for} \quad t \le 0\,, \\ \dot{x}_2 - \mathrm{D}_2 f_\infty\big(t, p_\infty^{(10)}(t-mT)\big)x_2 &= h_m^+(t, x_2) \qquad \text{for} \quad t \ge 0\,, \\ x_2(0) - x_1(0) = b_m &:= p_\infty^{(01)}(mT) - p_\infty^{(10)}(-mT)\,, \end{aligned} \tag{11}$$

where

$$\begin{aligned} h_m^-(t,x) = {} & f_\infty\big(t, x + p_\infty^{(01)}(t+mT)\big) - f_\infty\big(t, p_\infty^{(01)}(t+mT)\big) \\ & - \mathrm{D}_2 f_\infty\big(t, p_\infty^{(01)}(t+mT)\big)x \\ & + f_m\big(t, x + p_\infty^{(01)}(t+mT)\big) - f_\infty\big(t, x + p_\infty^{(01)}(t+mT)\big)\,, \\ h_m^+(t,x) = {} & f_\infty\big(t, x + p_\infty^{(10)}(t-mT)\big) - f_\infty\big(t, p_\infty^{(10)}(t-mT)\big) \\ & - \mathrm{D}_2 f_\infty\big(t, p_\infty^{(10)}(t-mT)\big)x \\ & + f_m\big(t, x + p_\infty^{(10)}(t-mT)\big) - f_\infty\big(t, x + p_\infty^{(10)}(t-mT)\big)\,. \end{aligned}$$

Note that $b_m = o(1)$ as $m \to \infty$. Let $\rho > 0$ be a fixed positive number such that the closure of the sets:

$$\{x + p_\infty^{(01)}(t) : \ t \in \mathbb{R}, \ \text{ and } |x| \le \rho\}$$

and

$$\{x + p_\infty^{(10)}(t) : \ t \in \mathbb{R}, \ \text{ and } |x| \le \rho\}$$

are both contained in Ω. Then note that

$$\begin{aligned} \sup_{t\in\mathbb{R}_\pm,\, |x|\le\rho} |h_m^\pm(t,x)| &= \Delta^\pm\big(|x|\big)|x| + o_1^\pm(1)\,, \\ \sup_{t\in\mathbb{R}_\pm,\, |x|\le\rho} |\,\mathrm{D}_2\, h_m^\pm(t,x)| &= \Delta^\pm\big(|x|\big) + o_2^\pm(1) \end{aligned} \tag{12}$$

where

$$\Delta^-(r) = \sup_{t\in\mathbb{R},\, |x|\le r} \big|\mathrm{D}_2\, f_\infty\big(t, x + p_\infty^{(01)}(t+mT)\big) - \mathrm{D}_2\, f_\infty\big(t, p_\infty^{(01)}(t+mT)\big)\big|$$

and

$$\Delta^+(r) = \sup_{t\in\mathbb{R},\, |x|\le r} \big|\mathrm{D}_2\, f_\infty\big(t, x + p_\infty^{(10)}(t-mT)\big) - \mathrm{D}_2\, f_\infty\big(t, p_\infty^{(10)}(t-mT)\big)\big|$$

are positive increasing functions such that $\Delta^\pm(r) \to 0$ as $r \to 0$ uniformly with respect to $m \in \mathbb{N}$ (see (8)) and, for example,

$$o_1^-(1) = \sup_{t\in\mathbb{R},\, |x|\le r} \big|f_m\big(t, x + p_\infty^{(01)}(t+mT)\big) - f_\infty\big(t, x + p_\infty^{(01)}(t+mT)\big)\big| \to 0$$

as $m \to \infty$, uniformly with respect to $t \in \mathbb{R}$ and $|x| \le \rho$, because of assumption (b). Of course, a similar conclusion holds as far as $o_1^+(1)$ and $o_2^\pm(1)$ are concerned.

Owing to the exponential dichotomy, any solution of the first two equations in (11) whose sup-norm in $(-\infty, 0]$ is less than a given $r > 0$ satisfies

$$\begin{aligned} x_1(t) = {} & X_\infty(t+mT)[\mathbb{I} - Q_\infty]X_\infty^{-1}(mT)\xi \\ & + \int_{-\infty}^{t} X_\infty(t+mT)Q_\infty X_\infty^{-1}(s+mT)h_m^-\big(s, x_1(s)\big)\, \mathrm{d}s \\ & - \int_{t}^{0} X_\infty(t+mT)[\mathbb{I} - Q_\infty]X_\infty^{-1}(s+mT)h_m^-\big(s, x_1(s)\big)\, \mathrm{d}s \end{aligned} \tag{13}$$

and similarly

$$\begin{aligned} x_2(t) = {} & Y_\infty(t-mT)P_\infty Y_\infty^{-1}(-mT)\eta \\ & + \int_{0}^{t} Y_\infty(t-mT)P_\infty Y_\infty^{-1}(s-mT)h_m^+\big(s, x_2(s)\big)\, \mathrm{d}s \\ & - \int_{t}^{\infty} Y_\infty(t-mT)[\mathbb{I} - P_\infty]Y_\infty^{-1}(s-mT)h_m^+\big(s, x_2(s)\big)\, \mathrm{d}s\,. \end{aligned} \tag{14}$$

A classical argument shows (see [2]) that the maps defined by the right-hand sides define contractions on the appropriate spaces $C_b^0(\mathbb{R}_-, n)$ and $C_b^0(\mathbb{R}_+, n)$ of bounded continuous functions on $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively, provided $m \geq m_0$ is sufficiently large, $6K|\xi| < r$, $6K|\eta| < r$, and $\|x_i\| < r$ where $r > 0$ is such that

$$3K\delta^{-1}\Delta^{\pm}(r) < 1 .$$

Let $x_1(t, \xi, m)$, $x_2(t, \eta, m)$ be the solutions of the above fixed point equations. From equation (13) and the properties of the functions $h_m^{\pm}(s, x)$ we easily obtain

$$\sup_{t<0} |x_1(t, \xi, m)| \leq K|\xi| + 2K\delta^{-1}\Big[\Delta^-(r) \sup_{t\leq 0} |x_1(t, \xi, m)| + o_1^-(1)\Big]$$

and then

$$\sup_{t\leq 0} |x_1(t, \xi, m)| \leq 3K|\xi| + o_1(1) < r \tag{15}$$

where $o_1(1) \to 0$ as $m \to +\infty$ uniformly with respect to ξ, η. Similarly:

$$\sup_{t\geq 0} |x_2(t, \eta, m)| \leq 3K|\eta| + o_2(1) < r \tag{16}$$

where $o_2(1) \to 0$ as $m \to +\infty$ uniformly with respect to ξ, η. In order to find $p_m(t)$, we have to solve the equation:

$$\begin{aligned} P_\infty(-mT)\eta - \big[\mathbb{I} - Q_\infty(mT)\big]\xi & \\ = \int_0^\infty Y_\infty(-mT)[\mathbb{I} - P_\infty]Y_\infty^{-1}(s - mT)h_m^+\big(s, x_2(s, \eta, m)\big)\ \mathrm{d}s & \\ + \int_{-\infty}^0 X_\infty(mT)Q_\infty X_\infty^{-1}(s + mT)h_m^-\big(s, x_1(s, \xi, m)\big)\ \mathrm{d}s + b_m . & \end{aligned} \tag{17}$$

Now, according to (7) and (8), $\mathrm{D}_2 f_\infty\big(t, p_\infty^{(01)}(t+mT)\big)$ and $\mathrm{D}_2 f_\infty\big(t, p_\infty^{(10)}(t-mT)\big)$ tend to $\mathrm{D}_2 f_\infty\big(t, p_1(t)\big)$, uniformly on compact intervals in $\mathbb{R}$ as $m \to \infty$. Hence from [3; p. 70, Lemma 1] the projections $P_\infty(-mT)$ and $Q_\infty(mT)$ tend, as $m \to \infty$, to the projection $\mathcal{P}$ of the dichotomy of $\mathbb{R}$ of the linear system along $p\ (t)$:

$$\dot{x} = \mathrm{D}_2 f_\infty\big(t, p_1(t)\big)x .$$

Thus for any $m \in \mathbb{N}$ sufficiently large $\|P_\infty(-mT)\|$ and $\|\mathbb{I} - Q_\infty(mT)\|$ are bounded below by a positive constant. Then, using (12), (15) and (16), we see that the right-hand side of (17) is bounded by a term like

$$3K^2\delta^{-1}\big[\Delta^-\big(3K|\xi| + o_1(1)\big)|\xi| + \Delta^+\big(3K|\eta| + o_2(1)\big)|\eta|\big] + o(1) \tag{18}$$

where $o(1) \to 0$ as $m \to \infty$ uniformly with respect to $|\xi|$, $|\eta|$. Thus by using the implicit function theorem, we see that (17) can be uniquely solved for $\xi = \xi_m$

and $\eta = \eta_m$. Moreover, since the expression in (18) tends to zero as $\xi \to 0$, $\eta \to 0$ and $m \to +\infty$, we easily see, from the uniqueness, that ξ_m and η_m tend to zero as $m \to \infty$. We set

$$p_m(t) = \begin{cases} x_1(t, \xi_m, m) + p_\infty^{(01)}(t + mT) & \text{if } t \le 0\,, \\ x_2(t, \eta_m, m) + p_\infty^{(10)}(t - mT) & \text{if } t \ge 0\,. \end{cases}$$

Then $p_m(t)$ satisfies (9) (because of (15), (16) and the fact that $|\xi_m|, |\eta_m| \to 0$ as $m \to +\infty$) and hence, for $|t|$ sufficiently large, remains in a small neighbourhood of the periodic orbit $p_0(t)$. Thus, because of the saddle node property of hyperbolic periodic solutions, $p_m(t)$ is homoclinic to $p_0(t)$.

To complete the proof of the theorem, we have to show that the stable and unstable manifolds $\mathcal{W}_m^s(p_0)$ and $\mathcal{W}_m^u(p_0)$ of the solution $p_0(t)$ of (6) with $m \ge m_0$ intersect transversely along $p_m(t)$. From the hyperbolicity of the periodic solution $p_0(t)$ and the roughness of exponential dichotomies, for any $m \ge m_0$, the linear systems

$$\dot{x} = \mathrm{D}_2 f_m\bigl(t, p_m(t)\bigr)x \tag{19}$$

have an exponential dichotomy on $\mathbb{R}_-$ with projections Q_m, that is the fundamental matrix $X_m(t)$ of (19) satisfies:

$$\begin{aligned} \|X_m(t) Q_m X_m^{-1}(s)\| &\le k\, \mathrm{e}^{-\delta(t-s)}\,, \qquad & s \le t \le 0\,, \\ \|X_m(t)[\mathbb{I} - Q_m] X_m^{-1}(s)\| &\le k\, \mathrm{e}^{\delta(t-s)}\,, \qquad & t \le s \le 0\,. \end{aligned}$$

Moreover, from (7), (8), (9) it follows (see also [5]) that the projections Q_m can be chosen so that

$$\lim_{m\to\infty} |Q_m - Q_\infty(mT)| = 0\,. \tag{20}$$

On the other hand equation (19) has also an exponential dichotomy on $\mathbb{R}_+$ with projection, say, P_m and we can similarly assume that

$$\lim_{m\to\infty} |P_m - P_\infty(-mT)| = 0\,. \tag{21}$$

We now describe the unstable manifold $\mathcal{W}_m^u(p_0)$ of $p_0(t)$. Let $x_m(t, \xi)$ be the solution of (6) such that $x(0) = \xi$. We have

$$\mathcal{W}_m^u(p_0) = \Bigl\{ \widetilde{\xi} \in \mathbb{R} : \lim_{t\to-\infty} \bigl|x_m(t, \widetilde{\xi}\,) - p_0(t)\bigr| = 0 \Bigr\}\,.$$

Because of the exponential dichotomy, the solutions occurring in the definition of $\mathcal{W}_m^u(p_0)$ can be written as $x_m\bigl(t, \widetilde{\xi}\,\bigr) = z_m(t) + p_m(t)$, where $z_m(t) = z_m(t, \xi)$ is the unique solution of the implicit equation:

$$\begin{aligned} z_m(t) = X_m(t)[\mathbb{I} - Q_m]\xi &+ \int_{-\infty}^{t} X_m(t) Q_m X_m^{-1}(s) h_m\bigl(s, z_m(s)\bigr)\, \mathrm{d}s \\ &- \int_{t}^{0} X_m(t)[\mathbb{I} - Q_m] X_m^{-1}(s) h_m\bigl(s, z_m(s)\bigr)\, \mathrm{d}s\,, \end{aligned} \tag{22}$$

where $\xi = \widetilde{\xi} - p_m(0)$ and

$$h_m(t,z) = f_m\big(t, z + p_m(t)\big) - f_m\big(t, p_m(t)\big) - \mathrm{D}_2 f_m\big(t, p_m(t)\big)z \,. \qquad (23)$$

Note that (22) defines $z_m(t)$ for $t \le 0$, however $z_m(t)$ can be extended up to any finite time mT and satisfies the same formula. Moreover, because of the uniqueness, we have $z_m(t,0) = 0$. Thus the tangent space of $\mathcal{W}^u_\infty(p_0)$ at the point $p^{(01)}_\infty(mT)$ is spanned by the vectors $X_\infty(mT)[\mathbb{I} - Q_\infty]\xi = [\mathbb{I} - Q_\infty(mT)]X_\infty(mT)\xi$ while the tangent space of $\mathcal{W}^u_m(p_0)$ at the point $p_m(0)$ is spanned by the vectors like $[\mathbb{I} - Q_m]\xi$ (where we used the identity $X_m(0) = \mathbb{I}$). Thus

$$T_{p_m(0)}\mathcal{W}^u_m(p_0) = \mathcal{N}Q_m \,,$$
$$T_{p^{(01)}_\infty(mT)}\mathcal{W}^u_\infty(p_0) = \mathcal{N}Q_\infty(mT) \,.$$

Similarly

$$T_{p_m(0)}\mathcal{W}^s_m(p_0) = \mathcal{R}P_m \,,$$
$$T_{p^{(10)}_\infty(-mT)}\mathcal{W}^s_\infty(p_0) = \mathcal{R}P_\infty(-mT) \,.$$

Thus, in order to show the transversality of the intersection of $\mathcal{W}^s_m(p_0)$ and $\mathcal{W}^u_m(p_0)$ along $p_m(t)$, we have to show that $\mathbb{R}^n = \mathcal{R}P_m \oplus \mathcal{N}Q_m$.

But, we have already seen that $Q_\infty(mT)$ and $P_\infty(-mT)$ tend, as $m \to \infty$, to the projection $\mathcal{P}$ of the dichotomy on $\mathbb{R}$ of the linear system along $p_1(t)$:

$$\dot{x} = \mathrm{D}_2 f_\infty\big(t, p_1(t)\big)x \,.$$

So, using also (20), (21), we have

$$\lim_{m\to\infty} \|Q_m - \mathcal{P}\| = 0$$

and similarly

$$\lim_{m\to\infty} \|P_m - \mathcal{P}\| = 0 \,.$$

Thus we can assume $m_0 \in \mathbb{N}$ is so large that

$$\mathbb{R}^n = \mathcal{R}P_m \oplus \mathcal{N}Q_m$$

for any $m \ge m_0$ and then the stable and unstable manifolds $\mathcal{W}^s_m(p_0)$ and $\mathcal{W}^u_m(p_0)$ intersect transversally along $p_m(t)$. The proof is finished. □

Remark. Theorem 1 holds also for the periodic solutions $r_m(t)$.

As an application of this result we can consider the family of second order equations (1) whose limiting equation, for $m \to \infty$, is (4). We have already mentioned in the Introduction that the unperturbed limit equation (5) has the heteroclinic cycle made of the two heteroclinic connections $p_\infty(t)$, and

$p_\infty(-t)$. Moreover, it has been proved in [1] that the Melnikov functions associated to both heteroclinic orbits has a transverse zero at least for infinitely many 2π-periodic C^1-functions $q(t)$. Hence, for any ε sufficiently small, $\alpha_\pm(\varepsilon)$ exist such that equation (4) has hyperbolic periodic solutions $p_0(t,\varepsilon)$, $p_1(t,\varepsilon)$ and bounded solutions $p_\pm(t,\varepsilon)$ such that

$$\sup_{t\in\mathbb{R}} |p_0(t,\varepsilon)| \to 0\,, \qquad \sup_{t\in\mathbb{R}} |p_1(t,\varepsilon) - 1| \to 0\,,$$
$$\sup_{t\in\mathbb{R}} \big|p_+(t,\varepsilon) - p_\infty(t-\alpha_+(\varepsilon))\big| \to 0\,,$$
$$\sup_{t\in\mathbb{R}} \big|p_-(t,\varepsilon) - p_\infty(-t-\alpha_-(\varepsilon))\big| \to 0$$

as $\varepsilon \to 0$ (and the same holds for the t-derivative). Moreover the variational equations of (4) along $p_\pm(t,\varepsilon)$ have an exponential dichotomy on $\mathbb{R}$. Since $p_\infty(t)$ tends to $x=0$ when $t\to-\infty$ and to $x=1$ when $t\to+\infty$, and the periodic solutions $p_0(t,\varepsilon)$, $p_1(t,\varepsilon)$ have the saddle point property, we easily obtain that the solutions $\{p_0(t,\varepsilon), p_+(t,\varepsilon), p_1(t,\varepsilon), p_-(t,\varepsilon)\}$ form a heteroclinic cycle which is transverse thanks to the exponential dichotomy of the linear systems. Thus, the result of this note applies, and we obtain the following theorem:

THEOREM 2. *If the function $q(t)$ in system (1) is such that the Melnikov functions associated to $p_\infty(\pm t)$ of the limiting equation (4) have transverse zeroes, then for any given $|\varepsilon| < \varepsilon_0$, sufficiently small, there exists $m(\varepsilon)$ such that, for any $m > m(\varepsilon)$, system (1) has a transversal homoclinic orbit $p_{m,\varepsilon}(t)$ with such a $q(t)$ and such that*

$$\sup_{t\le 0} |p_{m,\varepsilon}(t) - p_+(t+mT,\varepsilon)| \to 0 \qquad as \quad m\to\infty\,,$$
$$\sup_{t>0} |p_{m,\varepsilon}(t) - p_-(t-mT,\varepsilon)| \to 0 \qquad as \quad m\to\infty\,.$$

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