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THE REMARKABLE GENERALIZED PETERSEN GRAPH $G(8,3)$

Dragan Marušič — Tomaz Pisanski

(Communicated by Martin Škoviera)

ABSTRACT. Some properties of $G(8,3)$ are presented showing its uniqueness among generalized Petersen graphs.

For a positive integer $n \geq 3$ and $1 \leq r < n/2$, the generalized Petersen graph $G(n, r)$ has vertex set \{$u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}$\} and edges of the form $u_iu_{i+1}$, $v_iv_{i+r}$, $i \in \{0, 1, \ldots, n-1\}$ with arithmetic modulo $n$.

In [6] the automorphism group of $G(n, r)$ was determined for each $n$ and $r$. With the exception of the dodecahedron $G(10,2)$, the generalized Petersen graph $G(n, r)$ is vertex-transitive, if and only if $r^2 \equiv \pm 1 \pmod n$. Furthermore, $G(n, r)$ is a Cayley graph if and only if $r^2 \equiv 1 \pmod n$; see [9], [10]. Finally it was also shown in [6] that $G(n, r)$ is arc-transitive if and only if

$$(n, r) \in \{(4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)\}.$$ 

Note that $G(4,1)$ is the cube, and that $G(8,3)$, $G(12,5)$ and $G(24,5)$ are its covers ([3]). On the other hand, $G(5,2)$ is the Petersen graph whose canonical double cover is $G(10,3)$, while $G(10,2)$ arises as a double cover of its pentagonal embedding in the projective plane. $G(8,3)$ is known as the Möbius-Kantor graph ([3]), since it is the Levi graph of the unique $8_3$-configuration. Similarly,

1991 Mathematics Subject Classification: generalized Petersen graph, arc-transitive graph, Cayley graph, Möbius-Kantor graph, regular map.

Key words: Primary 05C25.

Supported in part by Ministrstvo za znanost in tehnologijo Slovenije, proj. No. J1-7035-0101-97. (1st author)

$G(10,3)$ is the Levi graph of the Desargues $10_3$-configuration and $G(12,5)$ is the Levi graph of one of the 229 $12_3$-configurations (see [8]). The number of $n_3$-configurations was recently computed up to $n \leq 18$ in [1].

We support our claim from the title by the following facts. $G(8,3)$ is the only generalized Petersen graph except for the trivial examples $G(n,1)$, $n \geq 3$, that is a Cayley graph of a dihedral group. More precisely, it is a Cayley graph $\Gamma$ for the dihedral group

$$D_8 = \langle x, y \mid x^8 = y^2 = 1, x^{-1} = yxy \rangle$$

of order 16 with respect to the generating set $\{y, xy, x^3y\}$ which clearly identifies the two bipartition sets. This fact is not mentioned in [4] where $G(8,3)$ is given as an example of the Cayley graph for the group $(2,2,2)_2$.

Note that any bipartite Cayley graph of a dihedral group $D_n$ with respect to a generating set consisting solely of reflections $x^ty$, where $t \in T \subset \mathbb{Z}_n$ and $0 \in T$, can be described by its symbol $T$. This, in turn, can be put in one-to-one correspondence with a positive integer $N$ via its binary notation:

$$N = b_02^{n-1} + \cdots + b_{n-2}2 + b_{n-1}$$

by letting $t \in T$ if and only if $b_t = 1$. In this way we get a graph $H(N)$ for each integer $N$ called the Haar graph of $N$ (see [7]). Clearly, $G(2m + 1,1)$ does not have a Haar graph representation, whereas $G(2m,1) = H(2^{2m-1} + 3)$ and $G(8,3)$, the only other generalized Petersen graph that is a Haar graph, is isomorphic to $H(133)$.

![Two views of the Möbius-Kantor graph $G(8,3) = H(133)$](image)

To continue with special properties of $G(8,3)$ we turn to $Z_k$-covers, $k \geq 2$, of complete graphs. It is proved in [5] that 2-arc-transitive connected $Z_k$-covers of $K_n$ exist only for $k = 2, 4$. The case $k = 2$ gives rise to the canonical double...
cover $K_{n,n} - nK_2$, whereas in the case $k = 4$ such a graph exists and is unique if and only if $n = p^{2s+1} + 1$, where $p$ is a prime congruent to 3 modulo 4. $G(8, 3)$ is the smallest member in this family and corresponds to the pair $(p, s) = (3, 0)$. It is obtained from $K_4$ with vertices 0,1,2,3, by assigning voltage 1 to the three arcs 12, 23, 31 and voltage 0 to all other arcs. The next case exists for $(p, s) = (7, 0)$ and yields a 7-valent graph on 32 vertices, a 4-fold cover of $K_8$.

Like all cubic Haar graphs, $G(8, 3)$ embeds in a torus with hexagonal faces only (see [11]), which implies that it has the infinite hexagonal lattice graph $H_\infty$ among its covers. A toroidal hexagonal embedding of $G(8, 3)$ can be obtained by taking the Cayley map for the dihedral group $D_8$ with an arbitrary cyclic permutation of the generating set \{$y, xy, x^3y$\}. By [12; Theorem 2] one can prove that the resulting embedding is not regular.

Also, let us mention that the automorphism group $\text{Aut} G(8, 3)$ has order 96 and is the group $\Gamma$ of Thomas Tucker, the only group of genus 2 (see [14]). The Cayley graph for $\Gamma$ that embeds in double torus is depicted in Figure 2.

![Figure 2](image-url)

**Figure 2.** A Cayley graph for the automorphism group $\Gamma$ of $G(8, 3)$ embedded in double torus. One can easily read off the presentation $\Gamma = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^3 = (bc)^8 = 1 \rangle$. The map is dual to the barycentric subdivision of the map of $G(8, 3)$ in Figure 1.

The graph $G(8, 3)$ also has a regular octagonal embedding in the double torus shown in [4; Figure 3.6.c]. This embedding can be constructed from the presentation $\Gamma = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^3 = (bc)^8 = 1 \rangle$ by taking the orbits of $\langle a, c \rangle$ as vertices, the orbits of $\langle a, b \rangle$ as edges and the orbits of $\langle b, c \rangle$ as faces. Since the map is reflexible and bipartite its Petrie dual is also orientable, regular (and reflexible). Hence $G(8, 3)$ admits a regular 12-gonal map in the triple torus.

As a final remark, we would like to point out the reference [13] that came to
our attention during the revision of this manuscript, in which the author studies the map in Figure 1 and other regular maps that result from branched covering of the standard $Q_3$ in the sphere.

Acknowledgement

We would like to express our thanks to Martin Škoviera who brought the regularity issues of $G(8, 3)$ to our attention and helped us dealing with them.

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Received October 7, 1998
Revised November 26, 1998