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THE REMARKABLE GENERALIZED PETERSEN GRAPH $G(8,3)$

DRAGAN MARUŠIČ — TOMAŽ PISANSKI

(*Communicated by Martin Škoviera*)

ABSTRACT. Some properties of $G(8,3)$ are presented showing its uniqueness among generalized Petersen graphs.

For a positive integer $n \geq 3$ and $1 \leq r < n/2$, the *generalized Petersen graph* $G(n,r)$ has vertex set $\{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$ and edges of the form $u_i v_i$, $u_i u_{i+1}$, $v_i v_{i+r}$, $i \in \{0, 1, \dots, n-1\}$ with arithmetic modulo n.

In [6] the automorphism group of $G(n,r)$ was determined for each n and r. With the exception of the dodecahedron $G(10,2)$, the generalized Petersen graph $G(n,r)$ is vertex-transitive, if and only if $r^2 \equiv \pm 1 \pmod{n}$. Furthermore, $G(n,r)$ is a *Cayley graph* if and only if $r^2 \equiv 1 \pmod{n}$; see [9], [10]. Finally it was also shown in [6] that $G(n,r)$ is arc-transitive if and only if

$$(n,r) \in \{(4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)\}.$$

Note that $G(4,1)$ is the cube, and that $G(8,3)$, $G(12,5)$ and $G(24,5)$ are its covers ([3]). On the other hand, $G(5,2)$ is the Petersen graph whose canonical double cover is $G(10,3)$, while $G(10,2)$ arises as a double cover of its pentagonal embedding in the projective plane. $G(8,3)$ is known as the *Möbius-Kantor graph* ([3]), since it is the Levi graph of the unique 8_3-configuration. Similarly,

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$G(10,3)$ is the *Levi graph* of the Desargues 10_3-configuration and $G(12,5)$ is the Levi graph of one of the 229 12_3-configurations (see [8]). The number of n_3-configurations was recently computed up to $n \leq 18$ in [1].

We support our claim from the title by the following facts. $G(8,3)$ is the only generalized Petersen graph except for the trivial examples $G(n,1)$, $n \geq 3$, that is a Cayley graph of a dihedral group. More precisely, it is a Cayley graph Γ for the dihedral group

$$D_8 = \langle x, y \mid x^8 = y^2 = 1\,,\ x^{-1} = yxy \rangle$$

of order 16 with respect to the generating set $\{y, xy, x^3y\}$ which clearly identifies the two bipartition sets. This fact is not mentioned in [4] where $G(8,3)$ is given as an example of the Cayley graph for the group $\langle 2,2,2 \rangle_2$.

Note that any bipartite Cayley graph of a dihedral group D_n with respect to a generating set consisting solely of reflections $x^t y$, where $t \in T \subset Z_n$ and $0 \in T$, can be described by its symbol T. This, in turn, can be put in one-to-one correspondence with a positive integer N via its binary notation:

$$N = b_0 2^{n-1} + \cdots + b_{n-2} 2 + b_{n-1}$$

by letting $t \in T$ if and only if $b_t = 1$. In this way we get a graph $H(N)$ for each integer N called the *Haar graph* of N (see [7]). Clearly, $G(2m+1,1)$ does not have a Haar graph representation, whereas $G(2m,1) = H(2^{2m-1}+3)$ and $G(8,3)$, the only other generalized Petersen graph that is a Haar graph, is isomorphic to $H(133)$.

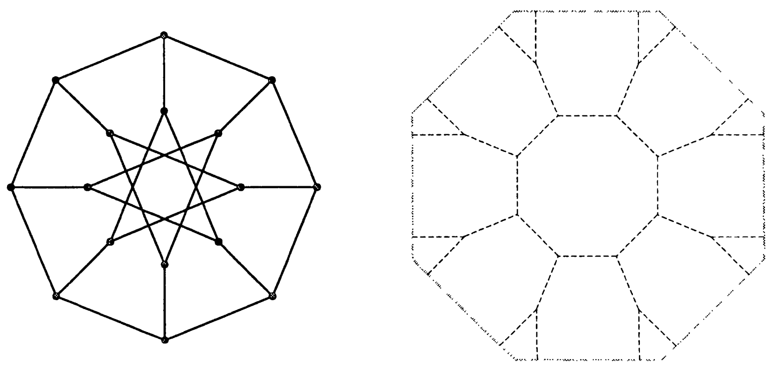

FIGURE 1. Two views of the Möbius-Kantor graph $G(8,3) = H(133)$.

To continue with special properties of $G(8,3)$ we turn to Z_k-covers, $k \geq 2$, of complete graphs. It is proved in [5] that 2-arc-transitive connected Z_k-covers of K_n exist only for $k = 2,4$. The case $k = 2$ gives rise to the canonical double cover $K_{n,n} - nK_2$, whereas in the case $k = 4$ such a graph exists and is unique if and only if $n = p^{2s+1} + 1$, where p is a prime congruent to 3 modulo 4. $G(8,3)$ is the smallest member in this family and corresponds to the pair $(p,s) = (3,0)$. It is obtained from K_4 with vertices $0,1,2,3$, by assigning voltage 1 to the three arcs 12, 23, 31 and voltage 0 to all other arcs. The next case exists for $(p,s) = (7,0)$ and yields a 7-valent graph on 32 vertices, a 4-fold cover of K_8.

Like all cubic Haar graphs, $G(8,3)$ embeds in a torus with hexagonal faces only (see [11]), which implies that it has the infinite hexagonal lattice graph H_∞ among its covers. A toroidal hexagonal embedding of $G(8,3)$ can be obtained by taking the Cayley map for the dihedral group D_8 with an arbitrary cyclic permutation of the generating set $\{y, xy, x^3y\}$. By [12; Theorem 2] one can prove that the resulting embedding is not regular.

Also, let us mention that the automorphism group $\mathrm{Aut}\, G(8,3)$ has order 96 and is the group Γ of Thomas Tucker, the only group of genus 2 (see [14]). The Cayley graph for Γ that embeds in double torus is depicted in Figure 2.

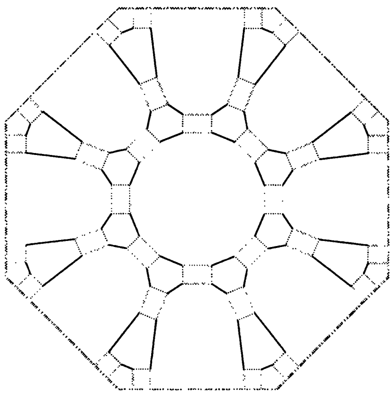

FIGURE 2. A Cayley graph for the automorphism group Γ of $G(8,3)$ embedded in double torus. One can easily read off the presentation $\Gamma = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^3 = (bc)^8 = 1\rangle$. The map is dual to the barycentric subdivision of the map of $G(8,3)$ in Figure 1.

The graph $G(8,3)$ also has a regular octagonal embedding in the double torus shown in [4; Figure 3.6.c]. This embedding can be constructed from the presentation $\Gamma = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^3 = (bc)^8 = 1\rangle$ by taking the orbits of $\langle a,c\rangle$ as vertices, the orbits of $\langle a,b\rangle$ as edges and the orbits of $\langle b,c\rangle$ as faces. Since the map is reflexible and bipartite its Petrie dual is also orientable, regular (and reflexible). Hence $G(8,3)$ admits a regular 12-gonal map in the triple torus.

As a final remark, we would like to point out the reference [13] that came to our attention during the revision of this manuscript, in which the author studies the map in Figure 1 and other regular maps that result from branched covering of the standard Q_3 in the sphere.

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