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ASYMPTOTIC ANALYSIS OF nTH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to deduce oscillatory and asymptotic behaviour of the solutions of the linear differential equations

$$\left(\frac{1}{r(t)}u'(t)\right)^{(m)} + p(t)u(t) = 0\,.$$

We consider the nth $(n = m + 1)$ order differential equation

$$\left(\frac{1}{r(t)}u'(t)\right)^{(m)} + p(t)u(t) = 0\,, \tag{1}$$

where $m \geqslant 2$, and the functions $p(t)$ and $r(t)$ are continuous and positive on some ray (t_0, ∞). We always assume that

$$\int^{\infty} r(s)\,\mathrm{d}s = \infty\,.$$

We consider only nontrivial solutions of (1). Such a solution is called *oscillatory* if it has an infinite sequence of zeros tending to infinity. Otherwise, it is called *nonoscillatory*. An equation is itself said to be oscillatory if all its solutions are oscillatory.

Let us write

$$\begin{aligned} L_0u(t) &= u(t)\,,\\ L_1u(t) &= \frac{1}{r(t)}\big(L_0u(t)\big)'\,,\\ L_iu(t) &= \big(L_{i-1}u(t)\big)'\,, \qquad i = 2,3,\dots,n\,. \end{aligned}$$

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Equation (1) can be rewritten as

$$L_n u(t) + p(t)u(t) = 0 \qquad \text{with } n = m+1\,.$$

The purpose of this paper is to study asymptotic properties of the solutions of (1). The following generalization of a classical lemma of Kiguradze can be found in [4].

LEMMA 1. *Let $u(t)$ be a nonoscillatory solution of (1), then there exist an integer ℓ, $\ell \in \{0,1,\dots,n-1\}$, and a $t_1 \geqslant t_0$ with $n+\ell$ odd such that*

$$\begin{aligned} u(t)L_i u(t) &> 0\,, \qquad 0 \leqslant i \leqslant \ell\,,\\ (-1)^{i-\ell}u(t)L_i u(t) &> 0\,, \qquad \ell \leqslant i \leqslant n\,, \end{aligned} \tag{2}$$

for all $t \geqslant t_1$.

A function $u(t)$ satisfying (2) is said to be a *function of degree* ℓ (see Foster and Grimmer [5]). The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then, by Lemma 1,

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_{n-1} \qquad \text{for } n \text{ odd},\\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \dots \cup \mathcal{N}_{n-1} \qquad \text{for } n \text{ even}. \end{aligned}$$

One should remark that (1) with n odd always has an decreasing solution, i.e., $\mathcal{N}_0 \neq \emptyset$ (see, e.g., [6]). Therefore we are interested in the particular situation described in the following definition.

DEFINITION 1. Equation (1) is said to have *property* (A) if for n even (1) is oscillatory (i.e., $\mathcal{N} = \emptyset$), and for n odd $\mathcal{N} = \mathcal{N}_0$.

It is known that (1) has property (A) provided that $\int^{\infty} p(s)\,\mathrm{d}s$ is divergent, see, e.g., [8], and so, in what follows, we may assume that this integral is convergent.

In [8], Tanaka has discussed property (A) of a special case of (1), namely the odd order differential equation

$$\left(\frac{1}{r(t)}u'(t)\right)^{(2m)} + p(t)u(t) = 0\,, \qquad m \geqslant 1\,, \tag{3}$$

by comparing (3) with the second order differential equation

$$z''(t) + q(t)z(t) = 0 \tag{4}$$

with

$$q(t) = \frac{1}{(2m-3)!}\left\{\int_t^{\infty}\left(\int_t^{s}(\sigma-t)^{2m-3}r(\sigma)\,\mathrm{d}\sigma\right)p(s)\,\mathrm{d}s\right\}. \tag{5}$$

Tanaka has shown that if (4) is oscillatory, then (3) has property (A) (i.e., $\mathcal{N} = \mathcal{N}_0$ for (3)). Using, e.g., Hille's oscillation criterion for (4), we have:

THEOREM A. *Let $q(t)$ be defined by* (5). *If*

$$\liminf_{t\to\infty} t \int_t^\infty q(s)\,\mathrm{d}s > \frac{1}{4}\,, \tag{6}$$

then (3) *has property* (A).

Our aim in this paper is to improve Tanaka's result and extend it to (1). We shall show that it is more conveniently to compare (1) with an mth order differential equation.

LEMMA 2. *If the differential inequality*

$$\left\{y^{(m)} + a(t)y\right\} \operatorname{sgn} y \leqslant 0 \qquad \text{with continuous and positive } a(t)$$

has an increasing nonoscillatory solution $y(t)$ (i.e., $y \notin \mathcal{N}_0$), then so does the corresponding differential equation

$$y^{(m)} + a(t)y = 0\,.$$

For the proof, see Čanturija [1] or Kusano and Naito [7].

THEOREM 1. *Let m be even. If the mth order differential equation*

$$y^{(m)} + \left(r(t)\int_t^\infty p(s)\,\mathrm{d}s\right)y = 0 \tag{7}$$

has property (A), *then so does* (1).

Proof. Assume that (1) possesses a nonoscillatory solution $u(t)$, which is eventually positive. Then $u(t)$ satisfies (2) for all $t \geqslant t_1$ with even integer $\ell \in \{0, 2, \dots, m\}$. Assume that $\ell \geqslant 2$.

From (2), we observe that

$$L_1 u(t) > 0 \quad \text{and} \quad L_m u(t) > 0 \qquad \text{for all} \quad t \geqslant t_1\,.$$

Integrating (1) from t to ∞ we get

$$L_m u(t) \geqslant \int_t^\infty p(s)u(s)\,\mathrm{d}s\,, \qquad t \geqslant t_1\,. \tag{8}$$

Integrating the identity $L_1 u(t) = L_1 u(t)$ from t_1 to t leads to

$$u(t) = u(t_1) + \int_{t_1}^t r(x) L_1 u(x)\,\mathrm{d}x\,, \qquad t \geqslant t_1\,. \tag{9}$$

In what follows and for convenience, let us denote $q(t) = r(t) \int_t^\infty p(s)\, \mathrm{d}s$. Combining (8) and (9) one gets

$$\begin{aligned} L_m u(t) &\geqslant \int_t^\infty p(s) \int_{t_1}^s r(x) L_1 u(x)\, \mathrm{d}x\, \mathrm{d}s \\ &\geqslant \int_t^\infty p(s) \int_t^s r(x) L_1 u(x)\, \mathrm{d}x\, \mathrm{d}s\,. \end{aligned}$$

Changing the order of integration leads to

$$L_m u(t) \geqslant \int_t^\infty q(x) L_1 u(x)\, \mathrm{d}x\,. \tag{10}$$

Let $\ell = m$. Then integrating $(m-1)$-times the relation (10) from t_1 to t we see that $w(t) = L_1 u(t) > 0$ satisfies

$$w(t) \geqslant w(t_1) + \int_{t_1}^t \int_{t_1}^{s_1} \dots \int_{t_1}^{s_{m-2}} \int_{s_{m-1}}^\infty q(x) L_1 u(x)\, \mathrm{d}x\, \mathrm{d}s_{m-1} \dots \mathrm{d}s_2\, \mathrm{d}s_1 \tag{11}$$

for $t \geqslant t_1$. Denoting the right hand side of (11) by $y(t)$, it is easy to see that $y(t) > 0$ is of degree $m-1$ and

$$y^{(m)}(t) + q(t) w(t) = 0\,.$$

Hence $y(t)$ is a nonoscillatory solution of the differential inequality

$$y^{(m)}(t) + q(t) y(t) \leqslant 0\,, \qquad t \geqslant t_1\,. \tag{12}$$

Lemma 2 implies that (7) has an increasing solution. But this contradicts property (A) of (7).

Now let $2 \leqslant \ell < m$. By successive integration of (10) from t to ∞ and then from t_1 to t, we get

$$L_1 u(t) \geqslant L_1 u(t_1) + \int_{t_1}^t \int_{t_1}^{s_1} \dots \int_{s_\ell}^\infty \dots \int_{s_{m-1}}^\infty q(x) L_1 u(x)\, \mathrm{d}x\, \mathrm{d}s_{m-1} \dots \mathrm{d}s_\ell \dots \mathrm{d}s_2\, \mathrm{d}s_1\,. \tag{13}$$

Let us denote the right hand side of (13) by $y(t)$. Then $y(t) > 0$ is of degree $\ell - 1$ (i.e., $y(t)$ is increasing), and $y(t)$ is a nonoscillatory solution of (12), and Lemma 2 now implies that (7) has an increasing solution, which again contradicts property (A) of (7). The proof is complete. □

In the previous theorem, we have compared (1) with the simpler equation (7). Using Čanturija's sufficient condition for property (A) of (7) (see [2] or [3]) we obtain:

COROLLARY 1. *Let m be even. Assume that*

$$\liminf_{t\to\infty} t^{m-1} \int_t^\infty r(x) \int_x^\infty p(s)\,\mathrm{d}s\,\mathrm{d}x > \frac{M^*}{(m-1)}\,, \tag{14}$$

where M^* is the maximum of all local maxima of the polynomial

$$Q_m(k) = -k(k-1)(k-2)\dots(k-m+1)\,.$$

Then (1) *has property* (A).

P r o o f. As condition (14) ensures property (A) of (7) (see, e.g., [2] or [3]), this corollary follows from Theorem 1. □

In the following illustrative example, we show that we have indeed improved T a n a k a's result.

EXAMPLE 1. Consider the fifth order differential equation

$$\left(\frac{1}{t}u'(t)\right)^{(\mathrm{IV})} + \frac{a}{t^6}u(t) = 0\,, \qquad a > 0\,,\ \ t > 1\,. \tag{15}$$

By Corollary 2 in [8], (15) has property (A) provided $a > 7.5$. On the other hand, by Corollary 1, it is sufficient to require $a > 4$.

Now we turn to equation (1) with m odd.

THEOREM 2. *Let m be odd. Let* (7) *has property* (A). *Further assume that the second order differential equation*

$$\left(\frac{1}{r(t)}z'\right)' + \left(\int_t^\infty \frac{(x-t)^{m-2}}{(m-2)!}p(x)\,\mathrm{d}x\right) z = 0 \tag{16}$$

is oscillatory. Then (1) *has property* (A) *provided that so does* (7).

P r o o f. Assume that (1) possesses a nonoscillatory solution $u(t)$, which is eventually positive. Then $u(t)$ satisfies (2) for all $t \geqslant t_1$ with odd integer $\ell \in \{1, 3, \dots, m\}$.

If $\ell > 1$, then, exactly as in the proof of Theorem 1, it can be shown that (7) has an increasing solution, which contradicts property (A) of (7).

Let $\ell = 1$. Then successive integration of (8) from t to ∞ provides

$$-L_2u(t) \geqslant \int_t^\infty \int_{s_2}^\infty \dots \int_{s_{m-2}}^\infty \int_{s_{m-1}}^\infty p(x)u(x)\,\mathrm{d}x\,\mathrm{d}s_{m-1}\dots\mathrm{d}s_3\,\mathrm{d}s_2\,.$$

Using the fact that $u(t)$ is increasing and changing the order of integration leads to

$$-L_2 u(t) \geqslant u(t) \int_t^\infty \frac{(x-t)^{m-2}}{(m-2)!} p(x)\, dx\,.$$

Therefore $u(t)$ is an increasing solution of

$$\left(\frac{1}{r(t)} z'\right)' + \left(\int_t^\infty \frac{(x-t)^{m-2}}{(m-2)!} p(x)\, dx\right) z \leqslant 0\,,$$

and, by Corollary 1 in [7], (16) has also an increasing solution, which contradicts the hypothesis. The proof is now complete. □

COROLLARY 2. *Let m be odd. Assume that* (14) *is satisfied. Further assume that*

$$\liminf_{t\to\infty} \left(\int_{t_0}^t r(x)\, dx\right) \left(\int_t^\infty \int_s^\infty \frac{(x-s)^{m-2}}{(m-2)} p(x)\, dx\, ds\right) > \frac{1}{4}\,. \tag{17}$$

Then (1) *has property* (A)

P r o o f . Noting that (17) is sufficient for (16) to be oscillatory (see [3]), this corollary can be proved exactly as Corollary 1. □

Remark 1. It remains an open problem how to relax condition (17) (if possible) in Theorem 2 and Corollary 2.

Remark 2. The method we have used in this paper can be applied to more general differential equations with deviating arguments of the form

$$\left(\frac{1}{r(t)} u'(t)\right)^{(m)} + p(t) u\big(\tau(t)\big) = 0\,,$$

where $\tau \in C\big((t_0,\infty)\big)$ and $\lim_{t\to\infty} \tau(t) = \infty$.

REFERENCES

[1] ČANTURIJA, T. A.: *Some comparison theorems for higher order ordinary differential equations*, Bull. Acad. Polon. Sci. Sér. Sci. Tech. **25** (1977), 745–756.

[2] ČANTURIJA, T. A.—KIGURADZE, I. T.: *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Nauka, Moscow, 1990. (Russian)

[3] DŽURINA, J.: *Comparison theorems for nonlinear ODE'*, Math. Slovaca **42** (1992), 299–315.

[4] ELIAS, U.: *Generalizations of an inequality of Kiguradze*, J. Math. Anal. Appl. **97** (1983), 277–290.

[5] FOSTER, K. E.—GRIMMER, R. C.: *Nonoscillatory solutions of higher order differential equations*, J. Math. Anal. Appl. **71** (1979), 1–17.

[6] HARTMAN, P.—WINTNER, A.: *Linear differential and difference equations with monotone solutions*, Amer. J. Math. **75** (1953), 731–743.

[7] KUSANO, T.—NAITO, M.: *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. Japan **3** (1981), 509–532.

[8] TANAKA, K.: *Asymptotic analysis of odd order ordinary differential equations*, Hiroshima Math. J. **10** (1980), 391–408.

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